Majority Dom-Chromatic Number of a Bipartite Graph

J. Joseline Manora[1], R. Mekala[2]

[1]Department of mathematics, Tranquebar Bishop Manickam Lutheran College
(Affiliated to Bharathidasan university, Tiruchirappalli), Porayar.

[2]Department of Mathematics, E.G.S Pillay Arts and Science College
(Affiliated to Bharathidasan university, Tiruchirappalli), Nagapattinam.

Abstract

In this article, the majority dom-chromatic sets of a bipartite graphs are studied. The characterization theorems on the majority dom-chromatic number $\gamma_{M\chi}(G)$ for bipartite graphs are determined. Also its relationship with other graph theoretic parameters and the majority dom-chromatic number for complement of a bipartite graphs are investigated.

Keywords: Majority dom-chromatic set, Majority dom-chromatic number.

1. Introduction

All the graphs $G = (V, E)$ considered here are simple, finite and undirected. The concept of domination is early discussed by Ore and Berge in 1962. Then Haynes et.al [2] defined the domination number $\gamma(G)$. The majority domination number $\gamma_M(G)$ was introduced by Swaminathan and Joseline Manora [6] is the smallest cardinality of a minimal majority dominating set $S \subseteq V(G)$ of vertices and satisfies $|N[S]| \geq \left|\left\lceil\frac{V(G)}{2}\right\rceil\right|$. Janakiraman and Poobalaranjani [3] defined the dom-chromatic set as a dominating set $S \subseteq V(G)$ such that the induced subgraph $< S >$ satisfies the property $\chi(< S >) = \chi(G)$, where $\chi(G)$ is the chromatic number of G. The minimum cardinality of a dom-chromatic set S is called dom-chromatic number and is denoted by $\gamma_{ch}(G)$.

Definition : 1.1 [4] The set $S \subseteq V(G)$ is called the Majority Dominating Chromatic set (MDC- set) of a graph G if the set S is a majority dominating set and satisfies the property $\chi(< S >) = \chi(G)$ where $< S >$ is a induced subgraph of G. It is also called a majority dom-chromatic set of a graph.A majority dom-chromatic number (MDC-number) $\gamma_{M\chi}(G)$ is defined as the smallest cardinality of the majority dom-chromatic set of a graph G.

Results on $\gamma_M(G)$ and $\gamma_{M\chi}(G)$: 1.2 [4] and [6]

(i) For a path P_p and cycle C_p, $\gamma_M(G) = \left\lceil\frac{p}{6}\right\rceil$, $p \geq 3$.

(ii) If a graph $G = \overline{K_p}$ then $\gamma_{M\chi}(G) = \left\lceil\frac{p}{2}\right\rceil$.

(iii) Let $G = mK_2$, $m \geq 1$. Then $\gamma_{M\chi}(G) = \left\lceil \frac{p}{4} \right\rceil + 1, p \geq 2$.

(iv) Let G be any graph of order p. Then $\gamma_{M\chi}(G) = p$ if and only if G is vertex color critical.

(v) For a graph $G = K_{m,n}$, $\gamma_{M\chi}(G) = 2$.

(vi) For any cycle C_p, $\gamma_{M\chi}(G) = \begin{cases} \left\lceil \frac{p}{6} \right\rceil & , \quad if\ p \equiv 2\ (mod\ 6) \\ \left\lceil \frac{p}{6} \right\rceil + 1 & , \quad if\ p \equiv 0,4\ (mod\ 6) \\ p & , \quad if\ p\ is\ odd\ . \end{cases}$

(vii) If G is a path then $\gamma_{M\chi}(G) = \begin{cases} \left\lceil \frac{p}{6} \right\rceil & , if\ p \equiv 1,2\ (mod\ 6) \\ \left\lceil \frac{p}{6} \right\rceil + 1 & , if\ p \equiv 0,3,4,5\ (mod\ 6). \end{cases}$

(viii) For a graph $G = D_{r,s}$, $\gamma_{M\chi}(G) = 2$.

(ix) Let $G = K_{1,p-1}$ be a star graph. Then $\gamma_{M\chi}(G) = 2$.

(x) [3] Let G be a tree of diameter 3. Then $\gamma_{\chi}(G) \leq p - \Delta(G)$.

2. Characterisation Theorems for Bipartite Graph

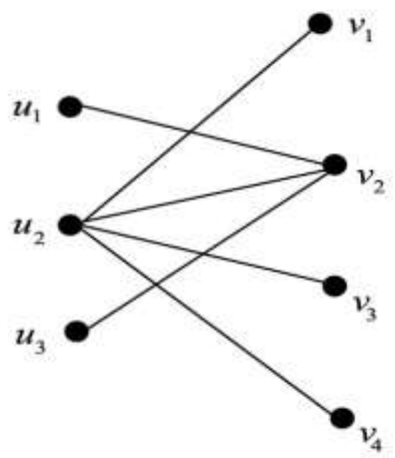
Theorem: 2.1 Let G be a connected bipartite graph with p vertices. Then $\gamma_{M\chi}(G) = 2$ if and only if $G_1 = K_{m,n}$, $m \leq n$, a path $G_1 = P_i$, $i \leq 8$ and $G_3 = B_{X,Y}$ such that $|N[u_1] \cup N[v_1]| \geq \frac{p}{2}$ and $d(u_1, v_1) = 1$, where $u_1 \in V_1(G)\ and\ v_1 \in V_2(G)$.

Proof :

Let $\gamma_{M\chi}(G) = 2$. (1) Then $\chi(G) = 2 = \chi(< S >)$, where S is a majority dom-chromatic set of G with $|S| = 2$.

Case: (i) Suppose $diam(G) = 1$ then the graph $G = K_p$. Since K_p is vertex color critical, $\gamma_{M\chi}(G) = p$. By assumption (1), the only graph $G = K_2 = K_{1,1}$ is complete bipartite.

Case: (ii) Suppose $diam(G) = 2$ then the graph G becomes $K_{m,n}$, $m \leq n$, P_3 and $K_{1,p-1}$, a star. Since $\gamma_{M\chi}(G) = 2$, by the result(1.2) , we obtain the graphs which have a structures as $G_1 = C_4 = K_{2,2}$ and $G_1 = K_{1,p-1}$, $G_2 = P_3$ and also $G_3 = B_{X,Y}$ includes the following structure with $diam(G) = 2$.



G_3: Fig(i)

For G_3 , $S = \{u_2, v_2\} \subseteq V(G)$ such that $d(u_2, v_2) = 1, |N[S]| = |N[u_2] \cup N[v_2]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $\chi(< S >) = 2 = \chi(G)$. It implies that S is a majority dom-chromatic set of G_3. Hence $G_3 = B_{X,Y}$ with these properties.

Case: (iii) Suppose $diam(G) = 3$. The bipartite graph G becomes P_4 and $D_{r,s}$, a double star. Since $\gamma_{M\chi}(G) = 2$, by the result (1.2)(vii), $\gamma_{M\chi}(P_4) = 2$. Hence $G_2 = P_4$. In $D_{r,s}$, $r \leq s$, by assumption (1), $S = \{u_1, v_1\}$ is the subset of G such that $d(u_1) \leq \left\lceil \frac{p}{2} \right\rceil - 1$, and $d(v_1) \geq \left\lceil \frac{p}{2} \right\rceil - 1$ with $d(u_1, v_1) = 1$, where $u_1 \in V_1(G)$ and $v_1 \in V_2(G)$ and $|N[S]| = |N[u_2] \cup N[v_2]| \geq \left\lceil \frac{p}{2} \right\rceil$. Also $\chi(< S >) = 2 = \chi(G)$. Hence S is a majority dom-chromatic set of G. It implies that $G_2 = B_{X,Y} = D_{r,s}$, $r \leq s$.

Case: (iv) Suppose $diam(G) \geq 4$. Then the bipartite graphs are P_p , $p \geq 5$ and any bipartite graph $B_{X,Y}$. By the result (1.2)(vii), $\gamma_{M\chi}(P_p) = \left\lceil \frac{p}{6} \right\rceil = 2$, $p = 5,6,7,8$ and $\gamma_{M\chi}(P_p) > 2, if\ p \geq 9$. Since $\gamma_{M\chi}(G) = 2$, the only bipartite graph $G_2 = P_5$ to P_8 . For a bipartite graph $B_{X,Y}$, if $S = \{u_1, v_1\} u_1 \in V(G)$ such that $|N[u_1] \cup N[v_1]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $d(u_1, v_1) = 1$, where $u_1 \in V_1(G)\ and\ v_1 \in V_2(G)$ with $diam(G) = 4$, then S is a majority dom-chromatic set of $B_{X,Y}$. Also clearly $\chi(< S >) = 2 = \chi(G)$ and satisfies the assumption (1). Hence the bipartite graph $G_3 = B_{X,Y}$ with the above said properties and also the only bipartite graphs are $G_2 = P_5$ to P_8 .

Conversely, let $G = K_{m,n}$, $m \leq n$ which is complete bipartite with $p = m + n$. By the result (1.2)(v) and (vii), $\gamma_{M\chi}(G_1) = 2$ and for a path $\gamma_{M\chi}(P_i) = 2 , if\ i = 2, \dots, 8$. Let $G_3 = B_{X,Y}$ be a graph with bipartition $V_1(G)$ and $V_2(G)$. Let $u_1 \in V_1(G)$ and $v_1 \in V_2(G)$ such that $d(u_1, v_1) = 1$. Since $|N[u_1] \cup N[v_1]| \geq \frac{p}{2}$ and $\chi(< S >) = 2 = \chi(G)$. Hence $S = \{u_1, v_1\}$ is a majority dom-chromatic set of G and $\gamma_{M\chi}(G_3) = 2$.

Proposition: 2.2 Let G be any bipartite graph $B_{X,Y}$ with p vertices and without isolates. Then $\gamma_{M\chi}(G) \leq \left\lceil \frac{p}{4} \right\rceil + 1$ and $\gamma_{M\chi}(G) = \left\lceil \frac{p}{4} \right\rceil + 1$ if and only if $G = K_{1,j}$, $j = 1,2,3$, $K_{2,2}$, P_4 and $mK_2, m \geq 1$.

Proof : Let $G = B_{X,Y}$ be a bipartite graph with $\{u_1 , u_2 , \ldots , u_m\}$ and $\{v_1 , v_2 , \ldots , v_n\}$ and $|V(G)| = p = m + n$.

Case : (i) Suppose $G = K_{m,n}$, is a complete bipartite with $m \leq n$. Let $S = \{u_1 , v_1\}$, where $u_1 \in V(X)$ and $v_1 \in V(Y)$. Then $|N[S]| = |N[u_1]| + |N[v_1]|$

$= (n + 1) + (m + 1) \geq \left\lceil \frac{p}{2} \right\rceil$. Therefore S is a majority dominating set of G. Since G is complete bipartite, $\chi(G) = 2 = \chi(< S >)$. It implies that S is a majority dom-chromatic set of G. Hence $\gamma_{M\chi}(G) \leq |S| = 2 = \left\lceil \frac{p}{4} \right\rceil + 1$, where $p = 2,3,4$. Thus the graph becomes $G = K_{1,1}$, $K_{1,2}$, $K_{1,3}$ and $K_{2,2}$. When $p \geq 5$, for $G = K_{m,n}$, $m \leq n$, by the result (1.2)(v), $\gamma_{M\chi}(G) = 2 < \left\lceil \frac{p}{4} \right\rceil + 1$. Hence, $\gamma_{M\chi}(G) \leq \left\lceil \frac{p}{4} \right\rceil + 1$,for $G = K_{m,n}$, $m \leq n$.

Case: (ii) The graph G is not complete and connected bipartite.

Then the minimally connected bipartite graph is a path P_p , $p \geq 2$. By known result (1.2)(vii), $\gamma_{M\chi}(P_p) = \left\lceil \frac{p}{6} \right\rceil$ *or* $\left\lceil \frac{p}{6} \right\rceil + 1$. Hence in this structure, when $p = 2, 3, 4$, $\gamma_{M\chi}(G) = 2 = \left\lceil \frac{p}{6} \right\rceil + 1 = \left\lceil \frac{p}{4} \right\rceil + 1$. When $p \geq 5$, $\gamma_{M\chi}(G) = \left\lceil \frac{p}{6} \right\rceil$ or $\left\lceil \frac{p}{6} \right\rceil + 1 < \left\lceil \frac{p}{4} \right\rceil + 1$. Hence, $\gamma_{M\chi}(G) \leq \left\lceil \frac{p}{4} \right\rceil + 1$, if , $p \geq 2$.

Case: (iii) The graph G is not complete and disconnected bipartite.

Then the graph structure becomes mK_2 , mP_4 , mC_4 *and* mP_6. In such cases, by the result (1.2)(iii), $\gamma_{M\chi}(mK_2) = \left\lceil \frac{p}{4} \right\rceil + 1$ and all other graphs the majority dom-chromatic number is $\gamma_{M\chi}(G) < \left\lceil \frac{p}{4} \right\rceil + 1$. Hence $\gamma_{M\chi}(G) \leq \left\lceil \frac{p}{4} \right\rceil + 1$. From the above cases, we obtain $\gamma_{M\chi}(G) \leq \left\lceil \frac{p}{4} \right\rceil + 1$.

Conversely , let $\gamma_{M\chi}(G) = \left\lceil \frac{p}{4} \right\rceil + 1$. By case (i), if G is a complete bipartite graph, we obtain the graphs $G = K_{1,j}$, $j = 1,2,3$ *and* $K_{2,2}$. By case (ii), if G is not complete bipartite then the graphs are $G = P_2 , P_3 , P_4 = K_{1,1}, K_{1,2} , P_4$. Also by case (iii), if G is not complete and disconnected bipartite, the graph $G = mK_2$, $m \geq 1$. Hence $\gamma_{M\chi}(G) = \left\lceil \frac{p}{4} \right\rceil + 1$ if and only if $G = K_{1,j}$, $j = 1,2,3$, $K_{2,2}$, P_4 and mK_2, $m \geq 1$.

Proposition: 2.3 Let G be any connected bipartite graph with p vertices. Then $\gamma_{M\chi}(G) = \left\lceil \frac{p}{2} \right\rceil$ if and only if $G = P_3$, P_4 , C_4 and $K_{1,3}$.

Proof: Assume that $\gamma_{M\chi}(G) = \left\lceil \frac{p}{2} \right\rceil$. (1)

Since G is connected bipartite graph, $\chi(G) \geq 2$.

Case: (i) If $diam(G) = 1$, then $G = K_2$ and $\gamma_{M\chi}(G) = 2 = p$, which is a contradiction to the assumption (1). Hence $G \neq K_2$.

Case: (ii) If $diam(G) = 2$, then $G = P_3, C_4, K_{1,n}$. By the result (1.2)(vii), $\gamma_{M\chi}(P_3) = 2 = \left\lceil \frac{p}{2} \right\rceil$. By the result (1.2)(vi), $\gamma_{M\chi}(C_4) = \left\lceil \frac{p}{2} \right\rceil$. Suppose $G = K_{1,3}$, by the result(1.2)(ix) , $\gamma_{M\chi}(G) = 2 = \left\lceil \frac{p}{2} \right\rceil$.

Case: (iii) If $diam(G) = 3$, then $G = P_4$and $D_{r,s}$. By the result (1.2) (vii) , $\gamma_{M\chi}(G) = 2 = \left\lceil \frac{p}{2} \right\rceil$. In $D_{r,s}$, by the result (1.2)(viii), $\gamma_{M\chi}(G) = 2$. The condition (1) holds when $r = s = 1$.

Case: (iv) If $diam(G) \geq 4$, then $G = P_p, C_p$, $p \geq 5$ and any other graphs. By the result (1.2)(vii) , $\gamma_{M\chi}(G) = \left\lceil \frac{p}{6} \right\rceil + 1 = 2 < \left\lceil \frac{p}{2} \right\rceil$, which is a contradiction to the condition (1).

Thus from the above four cases, G must be P_3 , P_4 , C_4 and $K_{1,3}$.

The converse is obvious.

Proposition: 2.4 Suppose G is a disconnected bipartite graph. If the graph structures are $G_1 = K_{1,3} \cup mK_2$, m is even and $m \geq 2$, $G_2 = mP_p$, $m = 4$, $p = 3$ and $G_3 = mK_{1,3}$, $m = 3$ then $\gamma_{M\chi}(G) = \frac{p}{4}$.

Corollary: 2.5 Let G be a disconnected bipartite graph. . If the graph structure is $K_{1,3} \cup mK_2$, m is odd then $\gamma_{M\chi}(G) = \frac{p}{4} + 1$.

Proposition: 2.6 Let G be a disconnected bipartite graph without isolates. Then $\gamma_{M\chi}(G) = \frac{p}{2}$ if and only if $G = mK_2$, $1 < m \leq 3$.

Proof : Let $\gamma_{M\chi}(G) = \frac{p}{2}$. (1)

Since G be a disconnected bipartite graph, let $G_1, G_2, \dots, G_k$ are the components of G and $V(G) = V(G_1) \dots \cup V(G_k)$.

Case (i) : All components are of diameter 1. Then the graph $G = mK_2$. By the assumption (1), when $G = mK_2$ if $m = 2$ *and* 3 then $G = 2K_2$ *and* $3K_2$. It implies that $\gamma_{M\chi}(G) = 2$ and $3 = \frac{p}{2}$. Suppose $m \geq 4$, then by the result (1.2)(iii), $\gamma_{M\chi}(G) = \left\lceil \frac{p}{4} \right\rceil + 1 < \frac{p}{2}$. It is a contradiction to the assumption (1).

Case (ii) : Suppose G contains the components which are of diameter1 and 2.

Then $G = K_{1,t} \cup mK_2$, where $G_1 = K_{1,t}$, $G_2 = mK_2$ and $V(G) = \{u, u_1, \dots, u_t, v_1, \dots, v_{2m}\}$ with $p = 1 + t + 2m$.

subcase : (i) If $|t| \geq \left\lceil \frac{p}{2} \right\rceil - 1$ and $2m = p - \left(\left\lceil \frac{p}{2} \right\rceil - 1 \right)$ then the majority dom-chromatic set $S = \{u, u_1\}$ where $u, u_1 \in V(G_1)$ such that $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $\chi(G_1) = 2 = \chi(< S >)$. It implies that S is a majority dom-chromatic set of G and $\gamma_{M\chi}(G) = 2 < \frac{p}{2}$, if $|t| \geq \left\lceil \frac{p}{2} \right\rceil - 1$, which is a contradiction to (1). Therefore $G \neq K_{1,t} \cup mK_2$.

subcase : (ii) If $|t| \leq \left\lceil \frac{p}{2} \right\rceil - 2$ then the MDC-set $S = \{u, u_1, v_1, v_2, \dots, v_k\}$, where $|k| = \left\lceil \frac{p}{2} \right\rceil - (1 + t)$ such that $|N[S]| = 1 + t + 2k \geq \left\lceil \frac{p}{2} \right\rceil$. Also $\chi(G) = 2 = \chi(< S >)$. Hence $\gamma_{M\chi}(G) = |S| = (2 + k) < \frac{p}{2}$, it is a contradiction to (1). Hence the graph $G \neq K_{1,t} \cup mK_2$.

Case (iii) : If the components G_i of G with $diam(G_i) \geq 2, i = 1,2, \dots, k$ then $\gamma_{M\chi}(G) < \frac{p}{2}$. From the above cases, we get the graph structures become $G = mK_2$, $1 < m \leq 3$.

Conversely, let $G = mK_2, m \leq 3$. Then by the result (1.3)(iii) , $\gamma_{M\chi}(G) = \left\lceil \frac{p}{4} \right\rceil + 1 = \frac{p}{2}$.

Proposition: 2.7 Let G be a disconnected graph which is not bipartite with isolates. Then $\gamma_{M\chi}(G) \leq \left\lceil \frac{p}{2} \right\rceil$ and $\gamma_{M\chi}(G) = \left\lceil \frac{p}{2} \right\rceil$ if and only if $G = pK_1$.

Proposition : 2.8 For a disconnected graph with p vertices, $\gamma_{M\chi}(G) = \left\lceil \frac{p}{2} \right\rceil$ if and only if $G_1 = mK_2, m = 2,3$ and $G_2 = K_t \cup (p - t)K_1$, where K_t is a complete graph of t vertices with $|t| \leq \left\lceil \frac{p}{2} \right\rceil$.

Proof: Let G be a disconnected graph p vertices. Suppose $\gamma_{M\chi}(G) = \left\lceil \frac{p}{2} \right\rceil$, then S is a majority dom-chromatic set with $\left\lceil \frac{p}{2} \right\rceil$ vertices. Also the chromatic number of the induced subgraph $< S >$ and the graph G are equal.

Case (i) : The graph G without isolates. Then $G = mK_2, m \geq 2$. By the result (1.2)(iii), $\gamma_{M\chi}(G) = \left\lceil \frac{p}{4} \right\rceil + 1$. It implies that when $G = 2K_2, 3K_2$, $\gamma_{M\chi}(G) = \left\lceil \frac{p}{2} \right\rceil$. If $G = mK_3$ or $G = mP_3$ then $\gamma_{M\chi}(G) < \left\lceil \frac{p}{2} \right\rceil$. If each components of G such as mK_2, $m \geq 4$, mK_t , mP_t , $t \geq 3$ then $\gamma_{M\chi}(G) < \left\lceil \frac{p}{2} \right\rceil$. Hence the graph $G_1 = mK_2, m = 2,3$.

Case (ii): The graph G has isolates. Let $\gamma_{M\chi}(G) = \left\lceil \frac{p}{2} \right\rceil$. Then the majority dom-chromatic set S contains $\left\lceil \frac{p}{2} \right\rceil$ vertices. It implies that, by the result (1.2) (iii) , the graph $G = \overline{K_p} = K_1 \cup (p-1)K_1$.

Subcase: (i) If $diam(G) = 1$ then the components of the given disconnected graph becomes a complete graph with isolates. i.e) $G = K_t \cup (p-t)K_1$, $t \geq 2$. Since $\gamma_{M\chi}(G) = \left\lceil \frac{p}{2} \right\rceil$ and $|t| \leq \left\lceil \frac{p}{2} \right\rceil$, the graph structure is $G = K_t \cup (p-t)K_1$, where K_t is the complete graph of t vertices.

Subcase : (ii) If $diam(G) = 2$ then the components of the disconnected graph become $G_1 = P_3 \cup (p-3)K_1$ or $G_2 = K_{1,t} \cup (p-(t+1))K_1$ or $G_3 = C_4 \cup (p-4)K_1$. Then $\gamma_{M\chi}(G_1) < \left\lceil \frac{p}{2} \right\rceil$ and $\gamma_{M\chi}(G_2) < \left\lceil \frac{p}{2} \right\rceil$. In particular, $G_2 = K_{1,1} \cup (p-2)K_1 = K_2 \cup (p-2)K_1$ and $\gamma_{M\chi}(G_2) = \left\lceil \frac{p}{2} \right\rceil$. Since $\gamma_{M\chi}(G) = \left\lceil \frac{p}{2} \right\rceil$ and $|t| \leq \left\lceil \frac{p}{2} \right\rceil$, the majority dom-chromatic set S must contain $\left\lceil \frac{p}{2} \right\rceil$ vertices. Since G is disconnected graph with isolates, anyone component $'g'$ of G must be vertex color critical with $|V(S)| \neq t \leq \left\lceil \frac{p}{2} \right\rceil$ and other remaining vertices are isolates. Hence the graph G takes the structure $G = K_t \cup (p-t)K_1$ where K_t is a complete graph which is vertex color critical and $(p-t)$ isolates.

Subcase: (iii) Let $diam(G) \geq 3$. Then the disconnected graph becomes $G_1 = P_r \cup (p-r)K_1$ or $G_2 = D_{t_1,t_2} \cup (p-(t_1+t_2))K_1$, where P_r is a path on r vertices and D_{t_1,t_2} is a double star with (t_1+t_2) vertices. The majority dom-chromatic number of these graphs G_1 and G_2 is $\gamma_{M\chi}(G) < \left\lceil \frac{p}{2} \right\rceil$. Since $\gamma_{M\chi}(G) = \left\lceil \frac{p}{2} \right\rceil$, G must have a vertex color critical component $'g'$ and isolates. Hence $|V(S)| = t \leq \left\lceil \frac{p}{2} \right\rceil$ and $(p-t)$ isolates. Hence the only graph structure $G = K_t \cup (p-t)K_1$, where K_t is the complete graph of t vertices and $|t| \leq \left\lceil \frac{p}{2} \right\rceil$.

Conversely, let $G = K_t \cup (p-t)K_1$, where $|t| \leq \left\lceil \frac{p}{2} \right\rceil$. Since K_t is the complete graph, it is a vertex color critical. Then by result (1.2) (iv), $\gamma_{M\chi}(G) = p$. If $|t| = \left\lceil \frac{p}{2} \right\rceil$ then the graph $G = K_{\left\lceil \frac{p}{2} \right\rceil} \cup \left(\left\lceil \frac{p}{2} \right\rceil K_1 \right)$ and $\gamma_{M\chi}(G) = \left\lceil \frac{p}{2} \right\rceil$. If $|t| < \left\lceil \frac{p}{2} \right\rceil$ then $|t| = \left\lceil \frac{p}{4} \right\rceil$. The graph G becomes $G = K_{\left\lceil \frac{p}{4} \right\rceil} \cup \left(p - \left\lceil \frac{p}{4} \right\rceil \right) K_1$. The majority dom-chromatic number $\gamma_{M\chi}(G) = \left\lceil \frac{p}{4} \right\rceil + \left(\left\lceil \frac{p}{2} \right\rceil - \left\lceil \frac{p}{4} \right\rceil \right) = \left\lceil \frac{p}{2} \right\rceil$. Suppose $|t| > \left\lceil \frac{p}{2} \right\rceil$ then $G = K_{t'} \cup (p-t')K_1$, where $|t'| > |t|$. Since $K_{t'}$ is a complete graph with t' vertices , $\gamma_{M\chi}(G) = t' > t = \left\lceil \frac{p}{2} \right\rceil$. Hence for a disconnected graph with isolates and $|t| \leq \left\lceil \frac{p}{2} \right\rceil$, $\gamma_{M\chi}(G) = \left\lceil \frac{p}{2} \right\rceil$.

3. $\gamma_{M\chi}$ for complement of a graph G

Proposition: 3.1 Let the bipartite graph G with $diam(G) = 3$. Then $\gamma_{M\chi}(G) = \gamma_{M\chi}(\bar{G})$ if and only if $G = P_4$, where $\bar{G}$ is the complement of G.

Proof: Let the equality holds and uv be the dominating edge of G. Let $|N[u]| = m$, $|N[v]| = n$ and $p = m + n$. In the graph $\bar{G}$, both $N(u)$ and $N(v)$ are of cardinality 2. The set $\{N(u) \cup N(v)\}$ is a K_{m+n-2} graph, $\chi(\bar{G}) = m + n - 2$ *and* $\{N(u) \cup N(v)\}$ be the majority dom-chromatic set for $\bar{G} \Rightarrow \gamma_{M\chi}(\bar{G}) = m + n - 2$. Since $\gamma_{M\chi}(G) = \gamma_{M\chi}(\bar{G})$, $\frac{m+n}{2} = m + n - 2$. It implies that $m + n = 4$. Hence the graph must be P_4 and C_4. The converse is obvious.

Proposition: 3.2 If the graph $G = K_p$ is the vertex color critical graph then $1 \leq \gamma_{M\chi}(\bar{G}) \leq \left\lceil \frac{p}{2} \right\rceil$.

Proof: Since the complete graph $G = K_p$ is the vertex color critical graph, $1 \leq \gamma_{M\chi}(G) \leq p$. The complement of K_p is $\bar{G} = \overline{K_p}$. By the result (1.2)(ii), the majority dom-chromatic number is $\gamma_{M\chi}(\bar{G}) = \left\lceil \frac{p}{2} \right\rceil$. And the lower bound attains for $\bar{G} = \overline{K_2}$. Hence the result.

Proposition: 3.3 Let $G = K_{m,n}$, $m \leq n$ *and* $m, n \geq 3$ be a complete bipartite graph. Then majority dom-chromatic number of a complement $\bar{G}$ is $\gamma_{M\chi}(\bar{G}) \geq \left\lceil \frac{p}{2} \right\rceil$ and $\gamma_{M\chi}(G) < \gamma_{M\chi}(\bar{G})$.

Proof: Let $\bar{G} = K_m \cup K_n$ be the complement of G where K_m and K_n both are complete graphs with m and n vertices.

Case: (i) Suppose $m = n , n+1, n+2$. Since K_m and K_n are vertex color critical and $p = m + n$, $\gamma_{M\chi}(\bar{G}) = n$ *or* $n + 1$ and $\gamma_{M\chi}(\bar{G}) = n + 2$. Hence $\gamma_{M\chi}(\bar{G}) = max\{m, n\}$.

Case: (ii) Let $m < n$ and $n \geq m + 3$. Since K_m and K_n are vertex color critical and $p = m + n$, $m < \left\lceil \frac{p}{2} \right\rceil$ *and* $n > \left\lceil \frac{p}{2} \right\rceil$. Hence $\gamma_{M\chi}(\bar{G}) = max\{m, n\}$. If $G = K_{m,n}$, $m \leq n$, then by the result(1.2) (v), $\gamma_{M\chi}(G) = 2$. By case (i), $\gamma_{M\chi}(\bar{G}) = n \text{ or } n + 1 = \left\lceil \frac{p}{2} \right\rceil$ and $\gamma_{M\chi}(\bar{G}) = n + 2 > \left\lceil \frac{p}{2} \right\rceil$. By case (ii), $\gamma_{M\chi}(\bar{G}) = n$, if $m < n$. It implies that $\gamma_{M\chi}(\bar{G}) > \left\lceil \frac{p}{2} \right\rceil$. Hence, $\gamma_{M\chi}(G) < \gamma_{M\chi}(\bar{G})$, if $m, n \geq 3$.

Proposition: 3.4 Let G be a bipartite graph with $diam(G) \geq 6$. Then $\gamma_{M\chi}(\bar{G}) \geq \gamma_M(\bar{G}) + 1$, if $\bar{G}$ is the complement of G and $\gamma_M(\bar{G})$ is the majority dominating number of $\bar{G}$.

Proof : If $diam(G) \geq 6$, then $G = P_p$, $p \geq 7$. The complement $\bar{G}$ contains two vertices with degree $\bar{d}(u_i) = p - 2$, $i = 1, p$ and $\bar{d}(v_i) = p - 3$, $i = 2, \ldots, p - 1$. It gives that there are atleast two vertices with degree $\bar{d}(u_i) \geq \left\lceil \frac{p}{2} \right\rceil - 1$ and the majority dominating number of $\bar{G}$ is $\gamma_M(\bar{G}) = 1$. Since $\bar{G}$ contains a triangle, $\chi(\bar{G}) = 3$ and $\gamma_{M\chi}(\bar{G}) \geq 3$. Hence, $\gamma_{M\chi}(\bar{G}) \geq \gamma_M(\bar{G}) + 1$.

4. Bounds of $\gamma_{M\chi}(G)$

Proposition : 4.1 If G is a vertex color critical and a non-trivial connected graph with $p \geq 2$ then $2 \leq \gamma_{M\chi}(G) \leq p$. These bounds are sharp.

Proof : Since G is connected and non-trivial graph with $p \geq 2$, $\chi(G) \geq 2$ and $\gamma_{M\chi}(G) \geq 2$. Also since G is a vertex color critical graph , by known result (1.2)(iv), $\gamma_{M\chi}(G) = p$. Hence $2 \leq \gamma_{M\chi}(G) \leq p, p \geq 2$. When $G = K_2$ and $G = K_p$, the lower and upper bounds are sharp.

Proposition: 4.2 Let G be a connected bipartite graph with p vertices. Then $\gamma_{M\chi}(G) = p$ if and only if $G = K_p$, $p = 2$.

Proof: Let G be a connected bipartite graph with p vertices. Since $\gamma_{M\chi}(G) = p$, then the graph must be a vertex color critical. The only connected bipartite vertex color critical graph is K_2. It implies that $G = K_2$. The converse is obvious.

Proposition: 4.3 If G be a graph of $diam(G) = 3$ then $\gamma_{M\chi}(G) = 2$ and $\gamma_{M\chi}(G) = \gamma_M(G) + 1$.

Proof: Let G be a connected graph and $diam(G) = 3$. Then the graph G has the structure with two central vertices u *and* v which are adjacent with some pendants. Then $G = P_4$ and $G = D_{r,s}$, $r \leq s$ where r and s number of pendants at u and v respectively. Then by result ((i) 1.2), $\gamma_M(G) = |\{v\}| = 1$.

Case: (i) If $s = r, r+1, r+2$ then both u and v are adjacent to some number of pendant vertices. Since $\chi(G) = 2$, $S = \{u, v\}$ be the majority dom-chromatic set of G and $\gamma_{M\chi}(G) = |S| = 2$. Hence $\gamma_{M\chi}(G) = \gamma_M(G) + 1$.

Case: (ii) If $r < s$ and $s \geq r+3$. Choose $S = \{u, v\}$, where u and v are central vertices of G . Then $|N[S]| = d(u) + d(v) = r + s + 2 = p > \left\lceil \frac{p}{2} \right\rceil$.

Therefore, S is majority dominating set of G. Also $\chi(G) = 2 = \chi(< S >)$.

Hence S will be the majority dom-chromatic set of G and $\gamma_{M\chi}(G) = |S| = 2$. Since $\gamma_M(G) = 1, \gamma_{M\chi}(G) = \gamma_M(G) + 1$. This result is true for $G = P_4$.

Proposition: 4.4 Let G be a bipartite graph of $diam(G) \leq 5$. Then $\gamma_{M\chi}(G) = 2$ and $\gamma_{M\chi}(G) = \gamma_M(G) + 1$.

Proof: Since the graph G is bipartite, the graph structures are P_p , $p \leq 6$, $K_{1,n}$, C_4 and K_2.

Case : (i) Suppose $diam(G) = 1$, then the bipartite graph G becomes only K_2 . By result [5], $\gamma_M(G) = 1$ and $\chi(G) = 2$ and by result (1.2)(iv), $\gamma_{M\chi}(G) = 2$. Hence $\gamma_{M\chi}(G) = \gamma_M(G) + 1$.

case: (ii) If $diam(G) = 2$, then the graph structures be $G = P_3$ or $K_{1,n}$. By the result (1.2)(i) , $\gamma_M(G) = 1$. Also by result (1.2)(vii), $\gamma_{M\chi}(G) = 2$. In both graphs, $\gamma_{M\chi}(G) = \gamma_M(G) + 1$.

case : (iii) Let $diam(G) = 3$. Then the graph becomes $G = P_4$ or and $D_{r,s}$. By Proposition (4.3), the result is true.

case : (iv) when $diam(G) = 4 \ and \ 5$, the bipartite graph is P_p , $p \leq 6$. By the result (1.2)(i) , $\gamma_M(G) = 1$. Since $\chi(G) = 2$, the set $\{v_2, v_3\}$ be the majority dom-chromatic set of G, where $v_2, v_3 \in V(P_5)$. Hence $\gamma_{M\chi}(G) = 2 = \gamma_M(G) + 1$.

Hence, for all cases, $\gamma_{M\chi}(G) = \gamma_M(G) + 1$.

Proposition: 4.5 Let G be a bipartite graph with $diam(G) \geq 6$. Then

(i) $\gamma_{M\chi}(G) = \gamma_M(G)$, if $p = 1,2 \ (mod \ 6)$

(ii) $\gamma_{M\chi}(G) = \gamma_M(G) + 1$, if $p = 0,3,4,5 \ (mod \ 6)$.

Proof: If the bipartite graph G with $diam(G) \geq 6$, then $G = P_p$, a path with $p > 6$. By the result (1.2)(i), $\gamma_M(G) = \left\lceil \frac{p}{6} \right\rceil$, for all $p \geq 7$ and by the result(1.2)(vii),

$$\gamma_{M\chi}(G)=\begin{cases}\left\lceil\frac{p}{6}\right\rceil = \gamma_M(G) \quad , if\ p\equiv 1,2\ (mod\ 6)\\ \left\lceil\frac{p}{6}\right\rceil+1=\gamma_M(G)+1\ , if\ \ p\equiv 0,3,4,5\ (mod\ 6).\end{cases}$$

Hence the result.

Proposition: 4.6 Let G be a 3-regular bipartite graph with p vertices. Then

$$\gamma_{M\chi}(G)=\begin{cases}\left\lceil\frac{p}{8}\right\rceil \quad\quad , if\ p\equiv 2,4\ (mod\ 8)\\ \left\lceil\frac{p}{8}\right\rceil+1\ , if\ p\equiv 0,6\ (mod\ 8).\end{cases}$$

Proof : Let $V_1(G)=\left\{v_1,v_2,\dots,v_{\frac{p}{2}}\right\}$ and $V_2(G)=\left\{u_1,u_2,\dots,u_{\frac{p}{2}}\right\}$ with $p=2m$.

Case: (i) Let $p\equiv 2,4\ (mod\ 8)$. Let $S=\{v_1,u_1,v_j,v_{j+1},\dots,v_t\}$ be the subset of G with $|S|=t=\gamma_{M\chi}(G)$ such that $d(v_1,u_1)=1$ and $d(v_i,u_j)\geq 4$. Then

$$|N[S]|=|N[v_1]+N[u_1]|+\sum_{j=1}^{t-2}d(u_j)-(t-2)=6+4(t-2)=4t-2$$

$\geq\left\lceil\frac{p}{2}\right\rceil$. Let $p=8r+2$. Then $|N[S]|=4t-2=4\left\lceil\frac{p}{8}\right\rceil-2=4\left(\frac{8r+2}{8}\right)-2=\frac{p}{2}-2+2>\left\lceil\frac{p}{2}\right\rceil$. Let $p=8r+4$. Then $|N[S]|=4t-2=4\left\lceil\frac{p}{8}\right\rceil-2=4\left(\frac{8r+4}{8}\right)-2=\frac{p}{2}-2+2>\left\lceil\frac{p}{2}\right\rceil$. Since $d(v_1,u_1)=1$, the induced subgraph $<S>$ contains K_2 and $\chi(<S>)=2=\chi(G)$. Thus S is a majority dom-chromatic set of G and $\gamma_{M\chi}(G)\leq|S|=\left\lceil\frac{p}{8}\right\rceil$.

(1)

Suppose that $S=\{v_1,u_1,v_j,\dots,v_t\}$ with $|S|=t=\gamma_{M\chi}(G)$ such that $d(v_1,u_1)=1$, $d(v_i,v_j)\geq 4$ and $|N[S]|\geq\left\lceil\frac{p}{2}\right\rceil$. Since S contains the induced subgraph K_2 and $\chi(<S>)=2=\chi(G)$. Therefore $|N[S]|\leq 4t=4\gamma_{M\chi}(G)$. Since $|N[S]|\geq\left\lceil\frac{p}{2}\right\rceil$, $\left\lceil\frac{p}{2}\right\rceil\leq 4\gamma_{M\chi}(G)$. It implies that $\gamma_{M\chi}(G)\geq\frac{1}{4}\left\lceil\frac{p}{2}\right\rceil$.

Hence $\gamma_{M\chi}(G)\geq\left\lceil\frac{p}{8}\right\rceil$. (2)

Combining (1) and (2) , $\gamma_{M\chi}(G)=\left\lceil\frac{p}{8}\right\rceil$, if $p\equiv 2,4\ (mod\ 8)$.

Case : (ii) Let $p\equiv 0,6\ (mod\ 8)$. Let $S_1=\{v_1,u_1,v_j\dots,v_t\}$ be the subset of $V(G)$ with $|S_1|=t_1=\left\lceil\frac{p}{8}\right\rceil+1=\gamma_{M\chi}(G)$ and $\chi(<S_1>)=2$. Let $p=8r$. Then $|N[S_1]|=4t-$

$2 = 4\left(\left\lceil\frac{p}{8}\right\rceil + 1\right) - 2 = 4\left(\frac{8r}{8} + 1\right) - 2 = 4\left\lceil\frac{p}{8}\right\rceil + 2 > \frac{p}{2} + 2 > \left\lceil\frac{p}{2}\right\rceil$. Let $p = 8r + 6$. Then $|N[S_1]| = 4t_1 - 2 = 4\left(\left\lceil\frac{p}{8}\right\rceil + 1\right) - 2 = 4\left(\frac{8r+6}{8} + 1\right) - 2 = 4\left\lceil\frac{p}{8}\right\rceil + 2 > \frac{p}{2} + 2 > \left\lceil\frac{p}{2}\right\rceil$. Hence $|N[S_1]| \geq \left\lceil\frac{p}{2}\right\rceil$. Therefore $S_1$is a majority dom-chromatic set of G and $\gamma_{M\chi}(G) \leq |S_1| = t_1 = \left\lceil\frac{p}{8}\right\rceil + 1$. Applying the same argument as in case (i), $\gamma_{M\chi}(G) \geq \left\lceil\frac{p}{8}\right\rceil + 1$.

Hence $\gamma_{M\chi}(G) = \left\lceil\frac{p}{8}\right\rceil + 1$, if $p \equiv 0, 6 \ (mod\ 8)$.

Proposition: 4.6 If the graph G is a bipartite with $diam(G) \leq 2$ then $\gamma_{M\chi}(G) \leq p - \Delta(G) + 1$ and $\gamma_{M\chi}(G) = p - \Delta(G) + 1$ if and only if $G = K_2$, P_3 and $K_{1,p-1}$, $p \geq 2$.

Proof: Let G be a bipartite graph with $diam(G) \leq 2$. If $\Delta(G) = 1$, the graph G becomes K_2. By the result (), $\gamma_{M\chi}(G) = 2 = p - \Delta(G) + 1$, if $G = K_2$. If $\Delta(G) = 2$, the graph structures becomes P_p , a path and $K_{2,2}$. Since $diam(G) \leq 2$, if $G = P_3$, by the result (1.2)(vii), $\gamma_{M\chi}(G) = 2 = p - \Delta(G) + 1$ and $\gamma_{M\chi}(K_{2,2}) = 2 < p - \Delta(G) + 1$. Suppose $\Delta(G) = 3$. Then $G = K_{3,3}$. By the result (1.2)(v), $\gamma_{M\chi}(K_{3,3}) = 2 < p - \Delta(G) + 1$. If $\Delta(G) \geq 4$ then the graph G becomes $K_{m,n}$, $m = n \geq 4$. By the result (1.2)(v), $\gamma_{M\chi}(G) = 2 < p - \Delta(G) + 1$. This is true for $\Delta(G) = 1, 2, 3, \ldots, (p-2)$. Suppose $\Delta(G) = p - 1$. Then the only bipartite graph $G = K_{1,p-1}$. By the result (1.2)(ix), $\gamma_{M\chi}(G) = 2 = p - \Delta(G) + 1$. Hence from the above cases, $\gamma_{M\chi}(G) \leq p - \Delta(G) + 1$. Also from the above cases, $\gamma_{M\chi}(G) = p - \Delta(G) + 1$ is true if and only if $G = K_2$, P_3 and $K_{1,p-1}$, $p \geq 2$.

Proposition: 4.7 Let G be a bipartite graph with $diam(G) = 3$. Then $\gamma_{M\chi}(G) \leq p - \Delta(G)$. Also $\gamma_{M\chi}(G) = p - \Delta(G)$ if and only if $G = P_4$ and $D_{r,s}$, $r = 1$ and $s = p - 3$.

Proof: Let G be a bipartite graph with $diam(G) = 3$. By the result (1.2)(x), $\gamma_{\chi}(G) \leq p - \Delta(G)$. Since $\gamma_{M\chi}(G) \leq \gamma_{\chi}(G)$, $\gamma_{M\chi}(G) \leq \gamma_{\chi}(G) \leq p - \Delta(G)$. Hence $\gamma_{M\chi}(G) \leq p - \Delta(G)$.

Let $\gamma_{M\chi}(G) = p - \Delta(G)$. (1)

Case : (i) Since $diam(G) = 3$, the graph G has a dominating edge uv with some pendants at u and v. Let $V(G) = \{u, v, u_1, \ldots, u_r, v_1, v_2, \ldots, v_s\}$ where $u_i, i = 1, \ldots, r\ and\ v_j, j == 1, \ldots, s$ are pendants with $r \leq p - 3$ and $s \geq 1$. Clearly , since G is bipartite, $\chi(G) = 2$. By the assumption (1), $S = \{u, v, v_1, \ldots, u_t\}$ is a majority dom-chromatic set with $|S| = p - \Delta(G) = t$.

Subcase: (i) Let $d(u) = p - 2$ and $d(v) = 2$. Since G has a dominating edge $e = uv$, $\gamma_{M\chi}(G) = |S| = 2$. By the assumption (1), $\gamma_{M\chi}(G) = p - \Delta(G)$. It implies that $2 = p -$

$d(u) \Longrightarrow 2 = p - (p-2)$. It gives the structure of the graph G with $d(u) = p-2, d(v) = 2$ and the graph is $G = D_{r,s}, r < s$ with $r = 1$ and $s = p-3$.

Subcase: (ii) Let $d(u) \leq p-3$ and $d(v) \geq 3$. The majority dom-chromatic set for the graph G is $S = \{u, v\}$. It implies that $\gamma_{M\chi}(G) = |S| = 2$. By the assumption (1), $\gamma_{M\chi}(G) = p - \Delta(G) = p - d(u) = p - (p-3) = 3$. Hence, $\gamma_{M\chi}(G) < p - \Delta(G)$.

Subcase: (iii) If $d(u) = p-2$ and $d(v) = p-2$ then the majority dom-chromatic set becomes $S = \{u, v\}$. It implies that $\gamma_{M\chi}(G) = |S| = 2$. By the assumption (1), $\gamma_{M\chi}(G) = p - \Delta(G) = p - d(u) \Longrightarrow 2 = p - (p-2)$. Since $d(u) = p-2$ and $d(v) = p-2$, $r = s = 1 \Longrightarrow p = r + s + 2 = 4$.

Hence the graph G with $p = 4$ vertices and $diam(G) = 3$ is P_4.

Case: (ii) Suppose G has no dominating edge $e = uv$. Then the graph G is a wounded spider with $diam(G) = 3$ and the graph contains a vertex u with $d(u) = \frac{p}{2}$ and $d(u_i) \leq 2, u_i \in (V(G) - \{u\})$. Hence $S = \{u, u_1\}$ be the majority dom-chromatic set of G with $d(u_1) = 2$, where $d(u, u_1) = 1$ and $\gamma_{M\chi}(G) = |S| = 2$. By the assumption (1), $\gamma_{M\chi}(G) = p - \Delta(G) = p - \frac{p}{2} = \frac{p}{2}$. Hence $\gamma_{M\chi}(G) < p - \Delta(G)$.

Thus, $\gamma_{M\chi}(G) = p - \Delta(G)$ if and only if $G = P_4$ and $D_{r,s}$, $r = 1$ and $s = p-3$.

5. Conclusion

In this paper, we studied majority dom-chromatic number for a bipartite graph. The characterisation theorems on $\gamma_{M\chi}(G)$ for bipartite graphs are established and its relationship with other domination parameters are discussed. Some results of a disconnected graph and the majority dom-chromatic number for the complement $\bar{G}$ of the graph G are investigated.

REFERENCES

[1] Chaluvaraju, B. and Appajigowda, C. - *The dom-chromatic number graph*, Malaya Journal of Matematik, vol 4(1), 1-7, (2016).

[2] Haynes, T.W., Hedetniemi Peter, S.T. and Slater, T. – *Fundamentals of domination in graphs*, Marcel Dekker Inc, Newyork , (1998).

[3] Janakiraman, T.N. and Poobalaranjani, M. - *Dom-chromatic sets of graphs*, International Journal of Engineering Science, Advanced

Computing and Bio-Technology, Vol. 2, No.2, April-June 2011, 88-103,

[4] Joseline Manora, J. and Mekala, R.- *Majority dom-chromatic set of a graph* , Bulletin of Pure and Applied Sciences , Vol.38E(Math & Stat), No.1, 289-296, (2019).

[5] Joseline Manora, J. and Swaminathan, V. - *Results on majority dominating sets*- Scientia Magna - Dept.of mathematics, Northwest university, X'tian, P.R. China), vol.7, No.3, 53-58, (2011).

[6] Joseline Manora, J. and Swaminathan, V. - *Majority dominating sets in graphs – I*, Jamal Academic Research Journal , Vol.3, No.2, 75-82, (2006).

[7] Paulraj Jayasimman, I. , Joseline Mnora, J. and Swaminathan, V. , Results on majority domatic number of a graph, IEEE conference proceedings Advances in Engineering, Science and Management (ICAESM), 2012.