Charecterization Of Graphs Attaining the Upper Bound for Strong Independence Number

Surekha R Bhat

Associate Professor, Department of Mathematics, Milagres College Kallianpur, Udupi, Karnataka

Abstract

R.S.Bhat and S.S.Kamath defined the strong independence number $s\alpha(G)$ and proved that $s\alpha(G) \leq \left\lfloor \frac{p}{2} \right\rfloor$. In this paper we characterize the class of graphs attaining the upper bound. We also characterize the graphs which attain the upper bound in Vizing type results for strong (weak) independence numbers.

Keywords: Independence Number, strong independence number, weak independence number, covering number, strong covering number, weak covering number.

1. Introduction

Let $G = (V, X)$ be any graph. For any $v \in V$ the set $N(v) = \{u \in V \mid uv \in X\}$ is called the *open neighbourhood* of the vertex v. R.S. Bhat and S.S.Kamath [1] and [10] defined the strong (weak) coverings as follows. For an edge $x = uv$, v *strongly covers* the edge x if $\deg(v) \geq \deg(u)$ in G. Then u *weakly covers* x. A set $S \subseteq V$ is a *strong vertex cover* [SVC] (*Weak Vertex Cover* [WVC]) if every edge in G is strongly (weakly) covered by some vertex in S. The *strong* (*weak*) *vertex covering number* $s\beta = s\beta(G)$ ($w\beta = w\beta(G)$) is the minimum cardinality of a SVC (WVC). A minimum vertex covering is written as β -set and a minimum *SVC* (*WVC*) is written as $s\beta$-set ($w\beta$-set).A vertex $v \in V$ is *strong* (*weak*) if $d(v) \geq d(u)$ ($d(v) \leq d(u)$) for every $u \in N(v)$ in G. A strong (weak) set which is independent is called a *strong independent set* [SIS] (*weak independent set* [WIS]). The *strong* (*weak*) *independence number* $s\alpha = s\alpha(G)$ ($w\alpha = w\alpha(G)$) is the maximum cardinality of an SIS (WIS). Later, This concept was extensively studied in [2], [3], [4], [5]. The domination number is the minimum number of vertices needed to dominate all the vertices of G. The concept of domination is well atudied in [6], [7], [8], [9] [11], [12].

S.S. Kamth and R.S. Bhat [1] obtained the relationship between these four new parameters which is similar to Gallais Theorem.

Theorem 1[3]. *For any isolate free graph G with p vertices ,*

$$s\alpha + w\beta = p$$
$$w\alpha + s\beta = p$$

The following bound for strong independence number is obtained in [10].

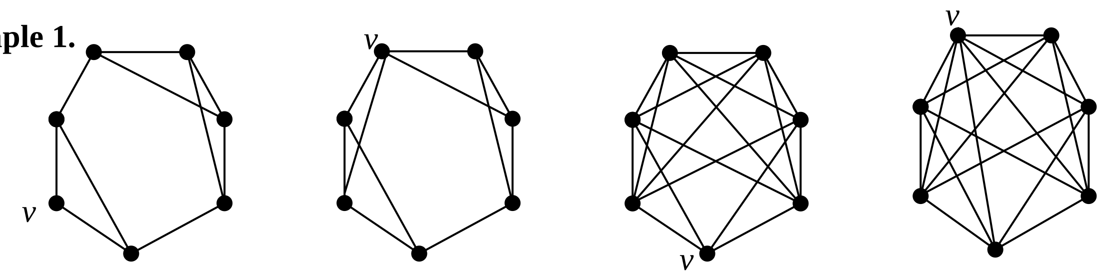
Theorem 2 [10]. *For any connected graph G with p vertices* $s\alpha \leq \left\lfloor \frac{p}{2} \right\rfloor$

We now characterize the graphs for which $s\alpha = \left\lfloor \frac{p}{2} \right\rfloor$. In this direction we define the following new types of graphs.

II. *K*-SEMIREGULAR GRAPHS

A graph G is said to be *k-semiregular* if all the vertices in G are of degree k except one vertex of degree $k \pm 1$. We thus have two class of *k-semiregular* graphs. A graph G is said to be *k-semiregular graph of first kind* or Δ*-semiregular graph* if all the vertices are of degree $k = \Delta$, except one vertex of degree $k - 1$. On the other hand, a graph G is said to be *k-semiregular graph of second kind* or δ-*semiregular graph* if all the vertices are of degree $k = \delta$, except one vertex which is of degree $k + 1$. The k-semiregular graphs of both kinds for $k = 3, 5$ are shown in the Fig.2.1. The vertex v shown in each figure represents the one vertex of degree $k \pm 1$. Immediately we observe that the number of edges in any k-semiregular graph of first kind is $\frac{pk-1}{2}$ and that of second kind is $\frac{pk+1}{2}$. Hence if G is any k-semiregular graph of first kind by joining the minimum degree vertex v with any other nonadjacent vertex we get a k-semiregular graph of second kind.

Example 1.



3-Semiregular graph of first kind | 3-semiregular graph of second kind | 5-Semiregular graph of first kind | 5-Semiregular graph of second kind

Fig. 2.1

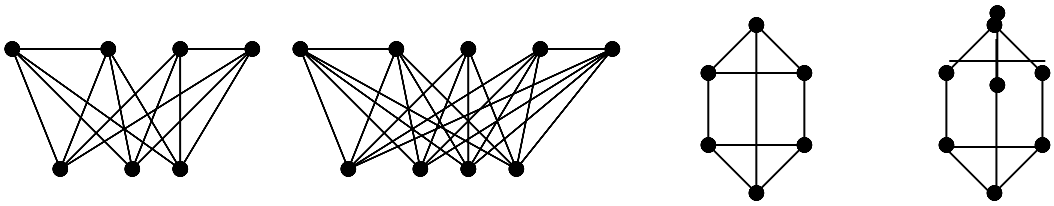
II. B- graph

A $\left\lfloor \frac{p}{2} \right\rfloor$ regular or semiregular graph G is said to be a *B- graph* if the vertex set V of G can be partitioned in to two sets V_1 and V_2 satisfying the following

(i) V_1 is an SIS and (ii) $|V_1| = \left\lfloor \frac{p}{2} \right\rfloor$.

Any B-graph with p vertices is denoted as $B_{\left\lfloor \frac{p}{2} \right\rfloor, \left\lceil \frac{p}{2} \right\rceil}$. Note that the complete bipartite graph $K_{\left\lfloor \frac{p}{2} \right\rfloor, \left\lceil \frac{p}{2} \right\rceil}$ is a subgraph of $B_{\left\lfloor \frac{p}{2} \right\rfloor, \left\lceil \frac{p}{2} \right\rceil}$.

Example 2.



G_1	G_2	G_3	G_4
4-regular Bhat's graph	5-semiregular Bhat's graph	3-regular graph but not a Bhat's graph	3-semiregular graph but not a Bhat's graph

Fig. 3.1

$K_{n,n}$ is a n- regular B-graph on $2n$ vertices. G_1 and G_2 in Fig. 3.1, are the examples of B-graph. Note that any $\left\lfloor \frac{p}{2} \right\rfloor$ regular graph or semiregular graph is not a B-graph. For example, the graph G_3 is a 3-regular graph but not a B-graph as V cannot be partitioned in to two sets V_1 and V_2 such that V_1 is an SIS and $|V_1| = \left\lfloor \frac{p}{2} \right\rfloor$. For the same reason G_4 is a 3-semiregular graph which is not a B-graph.

Proposition 3.1 For any connected graph G with p vertices, $s\alpha = \left\lfloor \frac{p}{2} \right\rfloor$ if and only if G is a spanning subgraph of B- graph such that V_1 is an SIS.

Proof. If G is any spanning subgarph of a B-graph, then V_1 is a maximum SIS and hence $s\alpha = \left\lfloor \frac{p}{2} \right\rfloor$.

Conversely, suppose $s\alpha(G) = \left\lfloor \frac{p}{2} \right\rfloor$. If G is a B-graph the theorem is proved. So let us assume that G is not a B-graph. Since $s\alpha(G) = \left\lfloor \frac{p}{2} \right\rfloor$, there exists an SIS D such that $|D| = s\alpha$. Then $V_1 = D$ and and $V_2 = V\text{-}D$ is a partition of G such that V_1 is an SIS. To show that G is a spanning subgraph of a B-graph, we construct a B-graph G_1 starting from G as follows. Since G is connected and D is an SIS, we have any vertex in G is at most of degree $\left\lceil \frac{p}{2} \right\rceil$. If u and v are not adjacent in G, then join uv by an edge for all $u \in D$ and $v \in V\text{-}D$. Then every vertex in D is of degree $\left\lceil \frac{p}{2} \right\rceil$ and any vertex in $V\text{-}D$ is of degree $\left\lceil \frac{p}{2} \right\rceil$ or $\left\lfloor \frac{p}{2} \right\rfloor$. Let $S_1 = \{v \in V\text{-}D \mid d(v) = \left\lceil \frac{p}{2} \right\rceil\}$and $S_2 = \{v \in V\text{-}D \mid d(v) = \left\lfloor \frac{p}{2} \right\rfloor\}$.(S_1 may be empty and then $S_2 = V\text{-}D$).

Claim: S_2 *is independent*. Suppose x and y are adjacent in S_2. Since any vertex in S_2 is adjacent to every vertex in D we have x is adjacent to $\left\lfloor \frac{p}{2} \right\rfloor+1$ vertices and therefore $d(x) = \left\lfloor \frac{p}{2} \right\rfloor+1= \left\lceil \frac{p}{2} \right\rceil$, a contradiction. Hence our claim. Let $|S_2| = k$.

Case 1. If k is even then $k = 2n$ and we can pair the vertices of S_2 as (x_i, y_i) $1 \le i \le n$ and join each x_i with y_i. Then every vertex in S_2 is of degree $\left\lceil \frac{p}{2} \right\rceil$ and thus the new graph G_1 is a $\left\lceil \frac{p}{2} \right\rceil$-regular graph and hence a Bhat's graph.

Case 2. If k is odd then $k = 2n+1$ for some n. Then we can pair the vertices of S_2 as (x_i, y_i) say $1 \le i \le n$ and exactly one vertex v is left with no pair. Now join each x_i with y_i. Then every vertex in S_2 is of degree $\left\lceil \frac{p}{2} \right\rceil$ and the one vertex v is of degree $\left\lfloor \frac{p}{2} \right\rfloor$. Thus the new graph G_1 is a $\left\lceil \frac{p}{2} \right\rceil$- semiregular graph and hence a B- graph. ■

The following bound for the number of edges when the strong independence number is given was obtained in [3].

Theorem 3.2 [3]**.** *Let G be a simple connected graph with p vertices, q edges and strong independence number* $s\alpha = k$. *Then* $q \le \left\lfloor \frac{p(p-k)}{2} \right\rfloor$.

Here we characterize the class of graphs for which $q = \left\lfloor \frac{p(p-k)}{2} \right\rfloor$.

Theorem 3.3. *Let G be a simple connected graph with p vertices, q edges and strong independence number* $s\alpha = k$. *Then* $q = \left\lfloor \frac{p(p-k)}{2} \right\rfloor$ *if and only if G is a (p-k)-regular or semiregular graph.*

Proof. Let G be a $(p-k)$-regular or semiregular graph with $s\alpha = k$. Since any r-regular or r-semiregular graph has $\frac{pr}{2}$ edges we have G has $q = \left\lfloor \frac{p(p-k)}{2} \right\rfloor$. Conversely, let $q = \left\lfloor \frac{p(p-k)}{2} \right\rfloor$. Then $2q = p(p-k)$ or $p(p-k)$ -1. If $2q = p(p-k)$ then this implies all the vertices in G are of degree $(p-k)$ and hence G is a $(p-k)$- regular graph. On the other hand if $2q = p(p-k) - 1 = p(p-k) - (p-k) + (p-k) - 1 = (p-1)(p-k) + (p - k - 1)$. This implies that G has $(p-1)$ vertices of degree $(p-k)$ and one vertex of degree $(p-k-1)$. Hence G is a $(p-k)$ -semiregular graph. ■

The next result gives a bound for number of edges when the weak independence number is given which is proved in [10].

Theorem 3.4. *Let G be a simple connected graph with p vertices, q edges and weak independence number* $w\alpha = k$. *Then* $q \le \frac{(p+k-1)(p-k)}{2}$.

We now characterize the graphs for which $q = \frac{(p+k-1)(p-k)}{2}$.

IV SPLIT GRAPHS.

A graph $G = (V, E)$ is said to be a *split graph* if V can be partitioned in to two sets V_1 and V_2 such that V_1 is independent and V_2 induces a complete graph and every vertex in V_1 is adjacent to every vertex in V_2. In other words G is a split graph if $G \cong K_{p-k} + \overline{K}_k$ with $V = V_1 \cup V_2$, $|V_1| = k$ and $|V_2| = p-k$.

Theorem 4.1 *Let G be a simple connected graph with p vertices, q edges and weak independence number* $w\alpha = k$. *Then* $q = \dfrac{(p+k-1)(p-k)}{2}$ if and only if G is a Split graph.

Proof. Let G be a Split graph with $w\alpha = k$. Then the vertex set of V can be partitioned in to two sets V_1 and V_2 such that $V = V_1 \cup V_2$, $|V_1| = k$ and $|V_2| = p-k$. The k vertices in V_1 are of degree (p-k) and the (p-k) vertices in V_2 are of degree (p-1). Hence $2q = k(p-k) + (p-k)(p-1) = (p-k)(p+k-1)$. Hence the result follows. Converse is trivial. ■

In the next result we refine the bound given in Theorem 4.1. We recall the following definition defined in [3]. Let $N_w(v)=\{u \in V \mid d(v) \le d(u)\}$. Then the *weak degree* of the vertex v is defined as $d_w(v) = |N_w(v)|$ and the maximum weak degree of G is denoted as $\Delta_w(G) = \max_{v \in V} d_w(v)$. It is proved that $\Delta_w(G) \le \delta(G)$ where $\delta(G)$ is the minimum
degree of the graph G.

Theorem 4.2. *Let G be a simple connected graph with p vertices, q edges and weak independence number* $w\alpha = k$. *Let* $\Delta_w > \delta$ so that $\Delta_w - \delta = r$ *where r is any positive integer. Then* $q \le \dfrac{(p+k-1)(p-k)-2r}{2}$. *Further this bound is sharp.*

This theorem suggests a better upper bound for $w\alpha$ in terms of order and size of the graph.

Corollary 4.2.1: *Let G be a simple connected graph with p vertices and q edges.*

Then $w\alpha \le \dfrac{1}{2} + \sqrt{p(p-1) - 2q - 2r + \dfrac{1}{4}}$.

V SEMISPLIT GRAPH.

Let G be a *Split graph*. Identify any one vertex v in V_1 and remove any r edges incident on v. The new graph G' so obtained is called a *Semisplit graph*. Any semisplit graph has (k-1) vertices of degree (p-k), r vertices of degree (p-2), k-r vertices of degree (p-1) and one vertex of degree δ.

Example 3. The semisplit graph G' obtained from the split graph $G = K_4 + \overline{K}_6$ shown in the Fig.8, satisfies $q = \dfrac{(p+k-1)(p-k)-2r}{2} = \dfrac{(15 \times 4)-4}{2} = 28$.

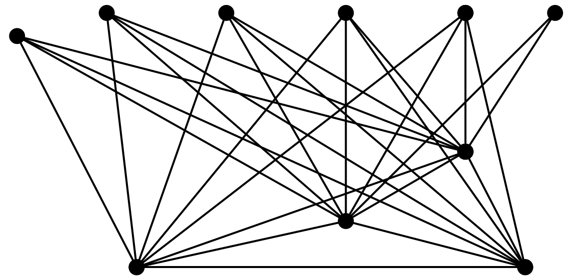


Fig. 5.1. A semisplit graph

Theorem 5.1. *Let G* (*p*, *q*) *be any graph with* $w\alpha = k$. Then $q = \frac{(p+k-1)(p-k)-2r}{2}$ *if and only if G is a Semisplit graph.*

Proof. Suppose $2q = \frac{(p+k-1)(p-k)-2r}{2}$. Then $2q = (p+k-1)(p-k)-2r = (k-1)(p-k)+(p-k-r)+(k-r)(p-1)+r(p-2)$. This implies *G* has (*k*-1) vertices of degree (*p*-*k*), *r* vertices of degree (*p*-2), *k*-*r* vertices of degree (*p*-1) and one vertex of degree δ = *p*-*k*-*r*. This implies that *G* is a semisplit graph. Converse is trivial.

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3. Authors' Biography

Surekha R Bhat is an Associate Professor of Mathematics at Milagres First grade college. She has got a teaching experience of more than 35 years and research experience of more than 10 years. Her research interests are Graph theory in general and domination, independence and coverings in particular. She has published more than 20 research articles in National and international refereed indexed journals.

References

1. R.S.Bhat, A study of strong (weak) domination and related parameters of a graph, Ph.D Thesis, National Institute of Technology Surathkal, 2006.
2. R.S. Bhat, S.S. Kamath And Surekha, Strong / Weak Edge Edge Domination Number Of A Graph, Appl. Math. Sc., Vol. 6, 2012, 5525 – 5531.
3. R.S. Bhat, S.S. Kamath And Surekha, Strong / Weak Edge Vertex Domination Number Of A Graph, Int.J.Math.Sc., Vol.11 (3-4), 2012, 433- 444.
4. R.S. Bhat, S.S. Kamath And Surekha, Strong / Weak Matchings And Edge Coverings Of A Graph, Int. J. Math. And Comp. Appl. Research, Vol.2 (3), 2012, 85-91.
5. R.S.Bhat, S.S.Kamath And Surekha, A Bound On Weak Domination Number Using Strong (Weak) Degree Concepts In Graphs, J. Int. Acad. Phy. Sc., Vol 15, No.11, (2011), 1-15
6. J.F.Fink, M.S.Jacobsons, L.F.Kinch, and J.Roberts, 1985. n graphs having domination number half their order, Period.Math.Hungar.,16: 287-293
7. Harary .F, 1969. *Graph Theory*, Addison Wesley .

8. T.W.Haynes, S.T.Hedetniemi, P.J.Slater, 1999. Fundamentals of domination in graphs Marcel Dekker, Inc., New York..
9. S.M.Hedetnimi, S.T Hedetnimi, Renu C.Laskar, P.J. Slater, 2006 Disjoint Dominating sets, Proceedings of International conference on Discrete Mathematics 88-102.
10. S.S.Kamath and R.S.Bhat, Strong (Weak) Independence and Covering Numbers of a Graph, Disc. Math. , Vol. 307, 2007, 1136 – 1145.
11. Routenbach D, 1998. Domination and Degree, Mathematic, Shaker Verlag, ISBN 3- 8265-3899-4.
12. V.G.Vizing, 1965. A bound on the external stability number of a graph. Dokl.Akad.Nauk SSSR, 164:729-731.