Effectiveness Of Preconditioned M-Order Gauss-Seidel Method for Linear System

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Abstract
Focusing on the current and the proposed preconditioner, this work examines the efficacy of the preconditioned m-order Gauss-Seidel method. Type I + S and I+N preconditioning are used for the current and proposed preconditioner respectively. Preconditioning algorithms for a linear system are constructed using iterative approaches. MATLAB are used to get the findings. The effectiveness of iterative method is compared concerning convergence, condition number, determinant, spectral radius, and the number of iterations for the current and proposed preconditioner. The numerical results show that for a linear system, the preconditioned m-order Gauss-Seidel method converges at a faster rate and the proposed preconditioner succeeds where the current preconditioner fails.

Keywords: Condition Number, Preconditioner, Spectral Radius

1. Introduction

Numerical Linear algebra is one that is employed throughout the vast majority of the several subfields that make up modern mathematics. The theory of linear systems is the most basic and important component of Numerical Linear algebra. The computational techniques that are utilized to find the solutions also play a significant role in Numerical Linear algebra. In general, one should prefer the direct method to solve linear systems but it will be better to apply iterative techniques in the case of matrices that have a significant number of zero elements. Iterative techniques need less time and space on hard drives than other types of approaches do. Research studies [2, 4, 5, 6, 8, 9, 10, 11, 13, 15] indicated that a lot of work has been conducted on preconditioned Gauss-Seidel method, [3] worked on preconditioned Symmetric Gauss-Seidel methods but very few are available on m-order Gauss-Seidel method [1]. So, the researchers plan to work on preconditioned m-order Gauss-Seidel method to increase convergence and robustness. This study indicates that when certain iterative approaches are used to certain preconditioned systems, the results are faster than the original system under specific assumptions.

Consider the system of linear equations represented by

$$Ax = b \qquad (1)$$

where $A \in R^{n\times n}, b \in R^{n\times 1}$ are given and $x \in R^{n\times 1}$ is unknown. We are interested in solving the system (1), with iterative techniques that can be written in the form

$$x^{(k+1)} = Tx^{(k)} + d \qquad k = 0,1,2,\ldots \qquad (2)$$

where $x^{(k+1)}$ represents the solution approximation of iteration matrix. It is well-known that the iterative process represented by equation (2) converges if $\rho(T) < 1$, where $\rho(T)$ represents the spectral radius of the iteration matrix T. Consider the splitting of the matrix as:

$$A = D + L + U \tag{3}$$

where D, L and U are diagonal, strictly lower triangular and upper triangular component of the coefficient matrix A, respectively.

$$(D + L + U)x = b$$
$$(D + L)x = -Ux + b$$
$$x = -(D + L)^{-1}Ux + (D + L)^{-1}b$$

After k^{th} iterations we have,

$$x^{(k+1)} = -(D + L)^{-1}Ux^{(k)} + (D + L)^{-1}b \qquad k = 0,1,2,\dots \tag{4}$$

By [1], the iterative approach represented by equation (4) is known as the **Gauss-Seidel** method and the ***m*-order Gauss-Seidel** method is

$$x^{(k+1)} = [-(D + L)^{-1}U]^m x^{(k)} + [I + \textstyle\sum_{l=1}^{m-1}(-1)^{m-l}[-(D + L)^{-1}U]^{m-l}](D + L)^{-1}b \tag{5}$$

2. Preconditioned m-Order Gauss-Seidel Method

By [3] the preconditioned linear system for current preconditioner S is

$$\hat{A}x = \hat{b} \tag{6}$$

where

$$\hat{A} = (I + S)A \tag{7}$$

$$\hat{b} = (I + S)b \tag{8}$$

and

$$S = \begin{bmatrix} 0 & 0 & \cdots & -a_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{9}$$

Without loss of generality, let the matrix $\hat{A}$ be

$$\hat{A} = \hat{D} + \hat{L} + \hat{U}$$

In this case, the diagonal matrix is designated by $\hat{D}$, the strictly lower and the strictly upper triangular matrices produced from $\hat{A}$ are denoted by $\hat{L}$ and $\hat{U}$, respectively. Then preconditioned **Gauss-Seidel** method is

$$x^{(k+1)} = -\left(\hat{D} + \hat{L}\right)^{-1} \hat{U}x^{(k)} + \left(\hat{D} + \hat{L}\right)^{-1} \hat{b} \tag{10}$$

And the **preconditioned *m*-order Gauss-Seidel** method is:

$$x^{(k+1)} = \left[-\left(\hat{D} + \hat{L}\right)^{-1} \hat{U}\right]^m x^{(k)} + \left[I + \textstyle\sum_{l=1}^{m-1}(-1)^{m-l}\left[-\left(\hat{D} + \hat{L}\right)^{-1} \hat{U}\right]^{m-l}\right]\left(\hat{D} + \hat{L}\right)^{-1} \hat{b} \tag{11}$$

The general form of preconditioned m-order Gauss-Seidel method is defined as follows:

$$x^{(k+1)} = [T_1]^m x^{(k)} + \left[I + \sum_{l=1}^{m-1}(-1)^{m-l}\,[T_1]^{m-l}\right] d_1$$

where T_1 is the iteration matrix and d_1 is column vector of preconditioned m-order Gauss-Seidel method. The development of subsequences that are based on their predecessors can also be understood as an alternate interpretation of the employment of m-order techniques. It is conceivable to demonstrate that m-order operations will converge more quickly if their antecedents do so.

Let the preconditioned linear system for **proposed preconditioner *N*** is

$$\tilde{A}x = \tilde{b} \tag{12}$$

where $$\tilde{A} = (I + N)A \tag{13}$$

$$\tilde{b} = (I + N)b \tag{14}$$

and $$N = \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 0 & \cdots & -a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 0 \end{bmatrix} \tag{15}$$

Considering the splitting of the matrix as:

$$\tilde{A} = \tilde{D} + \tilde{L} + \tilde{U}$$

where $\tilde{D}, \tilde{L}$ *and* $\tilde{U}$ represents the strictly diagonal, strictly lower triangular and strictly upper triangular parts of the matrix $\tilde{A}$, respectively. The **proposed preconditioned Gauss-Seidel** method for preconditioned linear system represented by equation (12) is:

$$x^{(k+1)} = -(\tilde{D} + \tilde{L})^{-1}\,\tilde{U}x^{(k)} + (\tilde{D} + \tilde{L})^{-1}\,\tilde{b} \tag{16}$$

After m-intermediate steps of **proposed preconditioned Gauss-Seidel** method is:

$$x^{(k+1)} = -(\tilde{D} + \tilde{L})^{-1}\,\tilde{U}x^{(k+(m-1)/m)} + (\tilde{D} + \tilde{L})^{-1}\,\tilde{b}$$

$$x^{(k+(m-1)/m)} = -(\tilde{D} + \tilde{L})^{-1}\,\tilde{U}\,x^{(k+(m-2)/m)} + (\tilde{D} + \tilde{L})^{-1}\,\tilde{b}$$

$$\vdots = \vdots$$

$$x^{(k+1/m)} = -(\tilde{D} + \tilde{L})^{-1}\,\tilde{U}x^{(k)} + (\tilde{D} + \tilde{L})^{-1}\,\tilde{b}$$

After eliminating all m-intermediate steps, the **proposed preconditioned m-order Gauss-Seidel** method is:

$$x^{(k+1)} = \left[-(\tilde{D} + \tilde{L})^{-1}\tilde{U}\right]^{m} x^{(k)} + \left[I + \sum_{l=1}^{m-1}(-1)^{m-l}\left[-(\tilde{D} + \tilde{L})^{-1}\,\tilde{U}\right]^{m-l}\right](\tilde{D} + \tilde{L})^{-1}\,\tilde{b} \tag{17}$$

Theorem 1([1]): Let an iterative method represented by (4), its corresponding m-order method represented by (5) and $x^{(0)} = 0$ be the same initial guess for both iterative methods. If the precursor method represented by (4) is convergent with rate of convergence R_K and R_V, then the m-order method represented by (5) is convergent with rate of convergence $\hat{R}_K \geq (m-1)R_V + R_K$.

Proof: A sequence $\{x^{(k)}\}$ is said to be convergent if and only if its every subsequence $\{x^{l(k)}\}$ is convergent. The convergent precursor technique creates the sequence $\{x^{(k)}\}$ if $x^{(0)}$ is the same initial guess for both iterative approaches, and its m-order method creates a subsequence $\{\hat{x}^{(l)} = x^{l(k)}\}$ such that $\hat{x}^{(l)} = x^{(k)}$ if $k = lm$ with $= 1,2,\ldots$. The subsequence $\hat{x}^{(l)}$ is therefore convergent and exhibits an accelerated rate of convergence.

Keep in mind that the m-order technique has $\hat{\epsilon}^{(k)} = x - \hat{x}^{(k)}$, $\hat{\epsilon}^{(k)} = [T^m]^{(k)}\hat{\epsilon}^{(0)}$. If $\hat{x}^{(0)} = x^0$ it follows that $\hat{\epsilon}^{(0)} = \epsilon^{(0)}$ and $\hat{\epsilon}^{(k)} = [T^k]^{(m-1)}\epsilon^{(k)}$. Therefore,

$$\|\hat{\epsilon}^{(k)}\| \leq \|[T^k]^{(m-1)}\|\|\epsilon^{(k)}\| \leq \|T^k\|^{m-1}\|\epsilon^{(k)}\| \leq \|T\|^{k(m-1)}\|\epsilon^{(k)}\|$$

It is possible to define by using the definition of [14], $\hat{R}_K = -\dfrac{\ln\left(\frac{\|\hat{\epsilon}^{(k)}\|}{\|\hat{\epsilon}^{(0)}\|}\right)}{k}$ for the m-order method. Hence,

$\hat{R}_K \geq -\frac{\ln\left(\left\|[T^k]^{(m-1)}\right\|\right)}{k} + R_K \geq (m-1)R_V + R_K$ Where $R_K = -\frac{\ln\left(\frac{\|\epsilon^{(k)}\|}{\|\epsilon^{(0)}\|}\right)}{k}$ and $R_V = -\frac{\ln(\|T^k\|)}{k}$, and T is the iteration matrix of m-order method which completes the proof. This theorem shows that the m-order method is always iterative faster than its precursor method for k iterations.

3. Numeric Experiment

Considering example 4.2 of [12], the three matrices are selected for the experiment as follows:

$$A_1 = \begin{bmatrix} 3 & 0 & 4 \\ 7 & 4 & 2 \\ -1 & 1 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 7 & 6 & 9 \\ 4 & 5 & -4 \\ -7 & -3 & 8 \end{bmatrix}, A_3 = \begin{bmatrix} -3 & 3 & -6 \\ -4 & 7 & -8 \\ 5 & 7 & -9 \end{bmatrix}$$

The vector b is selected in this case so that it yields the same initial guess of $x^{(0)} = 0$ and the exact answer of $x_i = i, \forall i = 1,2, \ldots, n$ and the stooping criteria is $\max_{1 \le i \le n}\left|x_i^{(k+1)} - x_i^{(k)}\right| < 10^{-14}$.

Algorithms for the m-order Gauss-Seidel and preconditioned m-order Gauss-Seidel methods are prepared and their efficiency is evaluated by the MATLAB software [7]. For each approach, tables show the condition number, determinant, spectral radius and iterations respectively.

Table 1: Matrix A_1 without preconditioner

Method	Determinant	Condition number	Spectral radius	Iterations
Gauss-Seidel	0	∞	1.5833	Divergent
2nd order Gauss-Seidel	0	∞	2.5069	Divergent
10th order Gauss-Seidel	0	∞	99.0201	Divergent

Table 1 shows that matrix $A_1$is divergent without preconditioner

Table 2: Matrix A_1 with the current preconditioner

Method	Determinant	Condition number	Spectral Radius	Iterations
Gauss-Seidel	0	∞	0.3780	38
2nd order Gauss-Seidel	0	∞	0.1429	20
10th order Gauss-Seidel	0	∞	5.9499e-05	5

Table 2 indicates that matrix A_1 is convergent with current preconditioner.

Table 3: Matrix A_2 without preconditioner

Method	Determinant	Condition number	Spectral Radius	Iterations
Gauss-Seidel	0	∞	0.7745	136
2nd order Gauss-Seidel	0	∞	0.6000	70
10th order Gauss-Seidel	0	∞	0.0777	16

It is clear from Table 3 that matrix $A_2$is convergent without preconditioner.

Table 4: Matrix A_2 with the current preconditioner

Method	Determinant	Condition	Spectral	Iterations

		number	Radius	
Gauss-Seidel	0	∞	0.7610	121
2nd order Gauss-Seidel	0	∞	0.5791	62
10th order Gauss-Seidel	0	∞	0.0651	14

Table 4 shows that matrix A_2 is not only convergent but also have better results with the current preconditioner.

Table 5: Matrix A_3 without preconditioner

Method	Determinant	Condition number	Spectral Radius	Iterations
Gauss-Seidel	0	∞	1.1111	Divergent
2nd order Gauss-Seidel	0	∞	1.2345	Divergent
10th order Gauss-Seidel	0	∞	2.8679	Divergent

Table 5 indicates that matrix A_3 is divergent without preconditioner.

Table 6: Matrix A_3 with the current preconditioner

Method	Determinant	Condition number	Spectral Radius	Iterations
Gauss-Seidel	0	∞	1.4062	Divergent
2nd order Gauss-Seidel	0	∞	1.9774	Divergent
10th order Gauss-Seidel	0	∞	30.2245	Divergent

Table 6 indicates the failure of the current preconditioner.

At this stage new preconditioner represented by (15) was proposed.

Table 7: Matrix A_3 with proposed preconditioner

Method	Determinant	Condition number	Spectral Radius	Iterations
Gauss-Seidel	0	∞	0.7759	1180
2nd order Gauss-Seidel	0	∞	0.9505	604
10th order Gauss-Seidel	0	∞	0.9749	128

Table 7 indicates that matrix A_3 is convergent with proposed preconditioner.

4. Results and Discussions

Results show that the $\boldsymbol{m}$-order Gauss-Seidel method is inconveniently divergent for matrix $\boldsymbol{A_1}$ and $\boldsymbol{A_3}$ (Table 1 and Table 5) and convergent for matrix $\boldsymbol{A_2}$ (Table 3). The preconditioned $\boldsymbol{m}$-order Gauss-Seidel method is advantageously convergent for matrix $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ (Table 2 and Table 4) also divergent for matrix $\boldsymbol{A_3}$ (Table 6) with current preconditioner**.** At this stage, new preconditioner works and indicates the convergence of preconditioned m-order Gauss-Seidel method with proposed preconditioner (Table7). The preconditioned $\boldsymbol{m}$-order Gauss-Seidel method for matrix $\boldsymbol{A_1}$and $\boldsymbol{A_2}$ both are indicating reduction in the spectral radius and the number of iterations, respectively (Table 2 and Table 4). The preconditioned $\boldsymbol{m}$-order Gauss-Seidel method with proposed preconditioner is also convergent with spectral radius less than one for matrix $\boldsymbol{A_3}$ (Table7).

5. Conclusion

In this study, the development of the preconditioned $\boldsymbol{m}$-order Gauss-Seidel technique is prepared. According to Theorem 1, the $\boldsymbol{m}$-order technique will almost always be iteratively quicker than its precursor method for a given number of iterations. It has been established that the preconditioned iterations satisfy the conventional convergence criterion using relatively less restrictions that have been carried out on the coefficient matrix of the linear system. According to the numerical findings, the preconditioned $\boldsymbol{m}$-order Gauss-Seidel method converges at a quicker rate for a linear system and the proposed new preconditioner works where the current preconditioner fails.

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