Some New Class of Four-Dimensional Absolute Valued Algebras

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Abstract

An absolute valued algebra is a nonzero real algebra that is equipped with a multiplicative norm \( ||ab|| = ||a|| \cdot ||b|| \). We prove that if \( A \) is a four-dimensional absolute valued algebra containing a nonzero central element \( a \) which satisfies the following identities:

1. \( (a^2, a, a) = 0 \) or \( (a, a, a^2) = 0 \),
2. \( (a^2, a^2, a) = 0 \) or \( (a, a^2, a^2) = 0 \),
3. \( (a^2, a^2) = 0 \), with \( (a/a^2) \neq 0 \),
4. \( (a^2, a, a^2) = 0 \).

Then \( A \) contains a commutative sub-algebra of dimension two. Moreover, \( A \) is isomorphic to \( A_1, A_2, A_3, A_4, B_1 \) or \( B_2 \) in the first two cases and isomorphic to \( A_1, A_2, A_3, A_4, B_1, B_2, B_3 \) or \( B_4 \), in the last two cases.

Keywords: Absolute valued algebra, Pre-Hilbert algebra, Commutative algebra, Central element.

1. Introduction

Let \( A \) be a non-necessarily associative real algebra which is normed as real vector space. We say that a real algebra is pre-Hilbert algebra, if its norm \( ||.|| \) comes from an inner product \( (./. ,) \), and it’s said to be absolute valued algebras if its norm satisfies the equality \( ||ab|| = ||a|| \cdot ||b|| \), for all \( a, b \in A \). Note that, the norm of any absolute valued algebras containing a nonzero central idempotent (or finite dimensional) comes from an inner product [3] and [4]. In 1947 Albert proved that the finite dimensional unital absolute valued algebras are classified by \( \mathbb{R}, \mathbb{C}, H \) and \( O \), and that every finite dimensional absolute valued algebra has dimension 1, 2, 4 or 8 [1]. Urbanik and Wright proved in 1960 that all unital absolute valued algebras are classified by \( \mathbb{R}, \mathbb{C}, H \) and \( O \) [9]. It is easily seen that the one-dimensional absolute valued algebras are classified by \( \mathbb{R} \), and it is well-known that the two-dimensional absolute valued algebras are classified by \( \mathbb{C} \), and it is well-known that the two-dimensional absolute valued algebras are classified by \( \mathbb{C} \), \( \mathbb{C}_* \), \( \mathbb{C} \) or \( \mathbb{C} \) (the real algebras obtained by endowing the space \( \mathbb{C} \) with the product \( x \star y = \bar{x}y \), \( x \star y = x\bar{y} \), and \( x \star y = \bar{x}\bar{y} \) respectively) [7]. The four-dimensional absolute valued algebras have been described by M.I. Ramirez Alvarez in 1997 [5]. The problem of classifying all four (eight)-dimensional absolute valued algebras seems still to be open.

In 2016 [5], we classified all four-dimensional absolute valued algebras containing a nonzero central idempotent and we also proved such an algebra contains a commutative sub-algebra of dimension two. Here (theorems 3.1, 3.3 and 3.4) we extend this result to more general situation. Indeed, Let \( A \) be a four-dimensional absolute valued algebra containing a nonzero central element \( a \). If \( A \) has a commutative sub-algebra of dimension two, then \( A \) is isomorphic to a new absolute valued algebras of
dimension four. We also show, in propositions 2.10, 2.11 and 2.13, that A contains a sub-algebra of dimension two if and only if a satisfies one of the following identities:

1) \((a^2, a, a) = 0\) or \((a, a, a^2) = 0\),
2) \((a^2, a^2, a) = 0\) or \((a, a^2, a^2) = 0\),
3) \((a^2, a^2, a^2) = 0\), with \((a/a^2) \neq 0\),
4) \((a^2, a, a^2) = 0\).

(Where \(\ldots\) means associator, that is \((x, y, z) = (xy)z - x(yz)\)). Also we prove that A is isomorphic to \(A_1, A_2, A_3, A_4, B_1\) or \(B_2\) in the first two cases and isomorphic to \(A_1, A_2, A_3, A_4, B_1, B_2, B_3\) or \(B_4\), in the last two cases. We denote that a central idempotent is central element, the reciprocal does not hold in general, and the counter example is given (remark 3.2).

We recall that, the classification of four-dimensional absolute valued algebras containing a unique two-dimensional sub-algebra is still an open problem. Also, in [2] we showed that if A is a four-dimensional absolute valued algebra with left unit and containing a nonzero central element, then A contains a sub-algebra of dimension two. It may be conjectured that every four-dimensional absolute valued algebra containing a nonzero central element, has sub-algebra of dimension two.

In section 2 we introduce the basic tools for the study of four-dimensional absolute valued algebras. We also give some properties related to central element satisfying some restrictions on commutativity (lemmas 2.7, 2.8, 2.9, 2.14, 2.15 and propositions 2.10, 2.11, 2.13). Moreover, the section 3 is devoted to construct, by algebraic method, some new class of the four-dimensional absolute valued algebras having commutative sub-algebras of dimension two, namely \(B_1, B_2, B_3\) and \(B_4\). The paper ends with the following main results:

**Theorem 3.1** Let A be a four-dimensional absolute valued algebra containing a nonzero central element a which satisfies one of the following identities:

1) \((a^2, a, a) = 0\) or \((a, a, a^2) = 0\),
2) \((a^2, a^2, a) = 0\) or \((a, a^2, a^2) = 0\).

Then A contains a commutative sub-algebra of dimension two. Moreover, A is isomorphic to \(A_1, A_2, A_3, A_4, B_1\) or \(B_2\).

**Theorem 3.2** Let A be a four-dimensional absolute valued algebra containing a nonzero central element a which satisfies one of the following identities:

1) \((a^2, a^2, a^2) = 0\), with \((a/a^2) \neq 0\),
2) \((a^2, a, a^2) = 0\).

Then A contains a commutative sub-algebra of dimension two. Moreover, A is isomorphic to \(A_1, A_2, A_3, A_4, B_1, B_2, B_3\) or \(B_4\).

2. **Notation And Preliminaries Results**

In this paper all the algebras are considered over the real numbers field \(\mathbb{R}\).
Definition 2.1 Let \( B \) be an arbitrary algebra.

i) \( B \) is called a normed algebra (respectively, absolute valued algebra) if it is endowed with a space norm: \( \| . \| \) such that \( \| xy \| \leq \| x \| \| y \| \) (respectively, \( \| xy \| = \| x \| \| y \| \), for all \( x, y \in B \)).

ii) \( B \) is called a division algebra if the operators \( L_x \) and \( R_x \) of left and right multiplication by \( x \) are bijective for all \( x \in B \setminus \{0\} \).

iii) \( B \) is called a pre-Hilbert algebra if it is endowed with a space norm comes from an inner product \( \langle ., . \rangle \) such that

\[
\begin{align*}
\langle x + y, x + y \rangle &= \frac{1}{4} (\| x + y \|^2 - \| x - y \|^2)
\end{align*}
\]

The most natural examples of absolute valued algebras are \( \mathbb{R}, \mathbb{C}, H \) (the algebra of Hamilton quaternion) and \( O \) (the algebra of Cayley numbers) with norms equal to their usual absolute values \([2] \) and \([8] \). The algebras by \( \mathbb{C}^*, \mathbb{C}^* \) and \( \mathbb{C}^* \) (obtained by endowing the space by \( \mathbb{C}^* \) with the products defined by: \( x \cdot y = \bar{x}y, \ x \cdot y = xy, \text{and} \ x \cdot y = \bar{x} \bar{y} \) respectively) where \( x \rightarrow \bar{x} \) is the standard conjugation of \( \mathbb{C} \).

Note that the algebras \( \mathbb{C}^* \) and \( \mathbb{C}^* \) are the only two-dimensional commutative absolute valued algebras.

We need the following relevant results:

Theorem 2.2 \([3] \) The norm of any finite dimensional absolute valued algebra comes from an inner product.

Theorem 2.3 \([4] \) The norm of any absolute valued algebra containing a nonzero central idempotent comes from an inner product.

Theorem 2.4 \([11] \) Let \( A \) be a two-dimensional absolute valued algebra, then \( A \) is isomorphic to \( \mathbb{C}, \mathbb{C}^*, \mathbb{C}^* \) or \( \mathbb{C}^* \).

Theorem 2.5 \([5] \) Let \( A \) be a four-dimensional absolute valued algebra containing a nonzero central idempotent \( e \), then \( A \) is isomorphic to \( A_1, A_2, A_3 \) or \( A_4 \) defined by: \( (\alpha^2 + \beta^2 = 1) \)

\[
\begin{array}{cccccc}
A_1 & e & i & j & k \\
e & e & i & \alpha j + \beta k & -\beta j + \alpha k \\
i & i & -e & -\beta j + \alpha k & -\alpha j - \beta k \\
j & \alpha j + \beta k & \beta j - \alpha k & -e & i \\
k & -\beta j + \alpha k & \alpha j + \beta k & -i & -e \\
\end{array}
\]

\[
\begin{array}{cccccc}
A_2 & e & i & j & k \\
e & e & i & \alpha j + \beta k & -\beta j + \alpha k \\
i & i & -e & -\beta j + \alpha k & -\alpha j - \beta k \\
j & \alpha j + \beta k & \beta j - \alpha k & -e & i \\
\end{array}
\]
Lemma 2.6 [5] Let $A$ be a finite dimensional absolute valued algebra containing a nonzero central idempotent $e$, then $A$ contains a commutative sub-algebra of dimension two.

Lemma 2.7 Let $A$ be a finite dimensional absolute valued algebra containing a nonzero central element $a$, then
1) $x^2 = -\|x\|^2 a^2$, for all $x \in \{a\}^\perp := \{x \in A, (x/a) = 0\}$
2) $xy + yx = -2(x/y)a^2$ for all $x, y \in \{a\}^\perp$
3) $(xy/yx) = -(x^2/y^2)$ for all $x, y \in \{a\}^\perp$ such that $(x/y) = 0$.

Proof. 1) By theorem 2.2, the norm of $A$ comes from an inner product and we assume that $\|x\| = 1$ (where $x \in \{a\}^\perp$), we have:
$$\|x^2 - a^2\|^2 = \|x - a\|^2 \|x + a\|^2 = 2$$
That is $(x^2/a^2) = -1$, which imply that $\|x^2 + a^2\|^2 = 0$ therefore $x^2 = -a^2$
2) It’s clear.
3) We get this identity by simple linearization of the identity $\|x^2\| = \|x\|^2$.

Lemma 2.8 Let $A$ be a four-dimensional absolute valued algebra containing a nonzero central element $a$ and commutative sub-algebra $B$, then $a \in B$.
Proof. By theorem 2.2, the norm of $A$ comes from an inner product. Applying theorem 2.4, $B$ is isomorphic to $\mathbb{C}$ or $\mathbb{C}$, that is, there exist an idempotent $e$ and an element $i$ such that $B = A(e, i)$, where $i^2 = -e$ and $ie = ei = \pm i$. We distinguish the following cases:
i. If $(a/e) = 0$, by lemma 2.7(1), we have $a^2 = -e$, so $a = \pm i \in B$ $(ai = ia$ and $i^2 = -e)$.
ii. If $(a/e) \neq 0$, we put $c = a - (a/e)e$, this imply that $(c/e) = 0$. Since $ce = ec$, then $c^2 = -\|c\|^2 e = 0$ which means that $c = \pm i$, thus $a = c + (a/e)e \in B$. We can put $a = \lambda e + \mu i$, with $\lambda, \mu \in \mathbb{R}$ $(\lambda^2 + \mu^2 = 1)$ and let $b = \mu e - \lambda i \in B$ and $j \in A$ two elements orthogonal to $a$. As $aj = ja$, we get $\lambda ej + \mu ij = \lambda je + \mu ji$ (1)
Using lemma 2.7(2), we have $bj + jb = 0$. This imply $\mu ej + \lambda ij = -\mu je - \lambda ji$ (2)
From the equalities (1) and (2), we obtain 
\[ 2\lambda \mu e j + ij = (\mu^2 - \lambda^2) ji \] 
\[ (\mu^2 + \lambda^2 = 1) \]

Therefore 
\[ (2\lambda \mu e j + ij/i) = ((\mu^2 - \lambda^2) ji/i) \]

Applying lemma 2.7.(3), we get 
\[ 1 = (\mu^2 - \lambda^2)(ji/i) = - (\mu^2 - \lambda^2)(j^2/i^2) \]  \hspace{1cm} (3)

Since \( a^2 = (\lambda^2 - \mu^2)e \pm 2\lambda \mu i, i^2 = -e \) and \( j^2 = -a^2 \). Then the equality (3) gives \( (\lambda^2 - \mu^2)^2 = 1 \).

Hence \( \lambda^2 - \mu^2 = 1 \) or \( \lambda^2 - \mu^2 = -1 \), as \( \lambda^2 + \mu^2 = 1 \), then \( \lambda^2 = 1 \) (because, \( \lambda = (a/e) \neq 0 \)). That is, \( a = \pm e \in B \).

**Lemma 2.9** Let \( B \) be a four-dimensional absolute valued algebra containing a nonzero central element \( a \) and commutative sub-algebra \( B \) of dimension two. If \( x, y \in B^\perp \), then \( xy \in B \).

**Proof.** By theorem 2.2, the norm of \( A \) comes from an inner product. According to theorem 2.4, \( B \) is isomorphic to \( \mathbb{C} \) or \( \mathbb{C}^* \). Then there exist an idempotent \( e \) and an element \( i \) such that \( B = A(e, i) \), where \( i^2 = -e \) and \( ei = ie = \pm i \). Let \( F = \{e, i, j, k\} \) be an orthonormal basis of \( A \), as \( A \) is a division algebra then \( L_j \) is bijective, so there exist \( j' \) such that \( i = jj' \). We have

\[ (j'/e) = (jj'/je) = (i/je) = \pm (ie/je) = \pm (i/j) = 0 \]

and

\[ (j'/i) = (jj'/ji) = (i/ji) = \pm (ei/ji) = \pm (e/j) = 0 \]

so \( j' = \alpha j + \beta k \), with \( \alpha, \beta \in \mathbb{R} \). Consequently we have \( i = jj' = \alpha j^2 + \beta jk \). Since \( a \in B \) (lemma 2.8), then, by lemma 2.7.(1), \( \beta jk = \alpha a^2 + i \in B \). Finally, we pose \( x = pj + qk \) and \( y = p'j + q'k \) with \( p, q, p', q' \in \mathbb{R} \). We have

\[ xy = (pp' + qq')e + (pq' - qp')jk \in B \quad (jk = -kj). \]

We give some conditions implying the existence of two-dimensional sub-algebras.

**Proposition 2.10** Let \( A \) be a four-dimensional absolute valued algebra containing a nonzero central element \( a \), then the following assertions are equivalent:

1) \((a^2, a, a) = 0\)

2) \(A\) contains a commutative sub-algebra of dimension two.

**Proof.** By theorem 2.2, the norm of \( A \) comes from an inner product. We discuss the following two cases:

1) If \( a \) and \( a^2 \) linearly dependent, then \( a \) is a nonzero central idempotent. Therefore \( A \) contains a commutative sub-algebra of dimension two (lemma 2.6) and the three assertions are equivalent.

2) If \( a \) and \( a^2 \) linearly independent

1) \(\Rightarrow\) 2) i) We suppose \((a/a^2) = 0\), by lemma 2.7.(1), we have \((a^2)^2 = -a^2\), so \((a^2)a = -a^2\) which means that \(a^2a = -a\). Hence \(B = A(a^2, a)\) is two dimensional commutative sub-algebra of \( A \).

ii) We assume that \((a/a^2) = m \neq 0\). We pose \(d = a^2 - ma\), this imply that \((d/a) = 0\), using lemma 2.6, we have

\[ d^2 = -d \parallel a^2 = -(1 - m^2)a^2 \]

which means that

\[ -(1 - m^2)a^2 = (a^2 - ma)a^2 = (a^2)^2 - 2ma^2a + m^2a^2 \]

Since \((a^2)^2 = (a^2)a\), thus

\[ -(1 - m^2)a^2 = (a^2)a - 2ma^2a + m^2a^2 \]

Hence

\[ -(1 - m^2)a = a^2a - 2ma^2 + m^2a = da - md \]

Therefore \(ad = da = -(1 - m^2)a + md\) this means that \(A(a, d)\) is a two-dimensional commutative sub-algebra of \( A \).

2) \(\Rightarrow\) 1) Let \( B \) be a two-dimensional commutative sub-algebra of \( A \), according to theorem 2.4, \( B \) is
isomorphic to \( \mathbb{C} \) or \( \mathbb{C}^* \). That is, \( B = A(e, i) \) such that \( ie = ei = \pm i \) and \( i^2 = -e \), where \( e \) is an idempotent of \( A \). On the other hand, the lemma 2.8 proves that \( a = \pm e \) or \( a = \pm i \) and we have the two following cases:

*) \( B \) is isomorphic to \( \mathbb{C} \), hence \( a \) verifies the equality \( (a^2, a^2, a) = 0 \).

**) \( B \) is isomorphic to \( \mathbb{C}^* \), then \( a = \pm e \) satisfies the identity \( (a^2, a, a) = 0 \). But since \( ie = ei = -i \) and \( i^2 = -e \), thus \( (a^2, a, a) \neq 0 \).

In the same way, we get

**Proposition 2.11** Let \( A \) be a four-dimensional absolute valued algebra containing a nonzero central element \( a \), then the following assertions are equivalent:

1) \( (a, a, a^2) = 0 \)
2) \( A \) contains a commutative sub-algebra of dimension two.

**Remark 2.12** Let \( A \) be a four-dimensional absolute valued algebra containing a nonzero central element \( a \) and commutative sub-algebra \( B \) isomorphic to \( \mathbb{C}^* \). If \( (a^2, a, a) = 0 \) or \( (a, a, a^2) = 0 \), then \( a \) is a central idempotent of \( A \).

**Proposition 2.13** Let \( A \) be a pre-Hilbert absolute valued algebra containing a nonzero central element \( a \) such that \( (a/a^2) \neq 0 \). Then the following assertions are equivalent:

1) \( (a^2, a^2, a^2) = 0 \),
2) \( (a^2, a, a^2) = 0 \),
3) \( A \) contains a commutative sub-algebra of dimension two.

Proof. By theorem 2.2, the norm of \( A \) comes from an inner product.

1) \( \Rightarrow \) 2) Let \( d = a^2 - (a/a^2)a \), \( (d \neq 0) \), we have \( (d/a) = 0 \), by lemma 2.7.(1)

\[
d^2 = -||d||^2 a^2 = -(1 - (a/a^2)^2) a^2
\]

That is

\[
-(1 - (a/a^2)^2) a^2 = (a^2 - (a/a^2)a)^2
\]

\[
-a^2 + (a/a^2)^2 a^2 = (a^2)^2 - 2(a/a^2)a a^2 + (a/a^2)^2 a^2
\]

This gives

\[
(a^2)^2 = 2(a/a^2)a a^2 - a^2
\]

Since \( (a^2, a^2, a^2) = 0 \), then \( (a^2)^2 a^2 = a^2(a^2)^2 \). So \( (a^2 a) a^2 = a^2(aa^2) \), consequently \( (a^2, a, a^2) = 0 \).

2) \( \Rightarrow \) 3) Let \( c = aa^2 - (a/aa^2)a \), we have \( (c/a) = 0 \), by lemma 2.7.(1),

\[
c^2 = -||c||^2 a = -(1 - (a/aa^2)^2) a^2.
\]

*) If \( ||c|| = 0 \), then \( aa^2 = \pm a \). That is

\[
(a^2)^2 = 2(a/a^2)a a^2 - a^2 = \pm 2(a/a^2)a - a^2
\]

This implies that \( A(a, a^2) \) is a two dimensional commutative sub-algebra of \( A \).

**) Assuming that \( ||c|| \neq 0 \), since \( (a^2, a, a^2) = 0 \) thus \( (a^2 a) a^2 = a^2(aa^2) \).

So

\[
(a^2 a) a^2 = a^2(aa^2),
\]

Moreover

\[
d c = (a^2 - (a/a^2) a)(aa^2 - (a/aa^2) a)
\]

\[
= a^2(aa^2) - (a/a^2) a aa^2 - (a/aa^2) a a^2 + (a/a^2)(a/aa^2) a^2
\]

\[
= (aa^2) a^2 - (a/a^2) a aa^2 + (a/aa^2) a a^2 + (a/a^2)(a/aa^2) a^2
\]

\[
= cd
\]

And since \( ||c||^2 d^2 = ||d||^2 c^2 \), then \( ||c|| d = ||d|| c \) or \( ||c|| d = -||d|| c \). We conclude that
\[ ||d||aa^2 = ||c||a^2 + ((a/aa^2)||d|| - (a/a^2)||c||)a \]

Or

\[ ||d||aa^2 = ||c||a^2 + ((a/aa^2)||d|| - (a/a^2)||c||)a \]

Therefore \( A(a, a^2) \) is a two-dimensional commutative sub-algebra of \( A \), thus \( A(a, a^2) \) is isomorphic to \( \mathbb{C} \) or \( \mathbb{C} \) (theorem 2.4).

3) \( \Rightarrow \) 1) Let \( B \) be a two-dimensional commutative sub-algebra of \( A \), according to theorem 2.4, \( B \) is isomorphic to \( \mathbb{C} \) or \( \mathbb{C} \). That is \( B = A(e, i) \) such that \( ie = ei = \pm i \) and \( i^2 = -e \), where \( e \) is an idempotent of \( A \). On the other hand, the lemma 2.8 imply that \( a = \pm e \) or \( a = \pm i \), which means \( a \) verifies the equality \((a^2, a^2, a^2) = 0\).

**Lemma 2.14** Let \( A \) be a pre-Hilbert absolute valued algebra containing a central element \( a \) such that \((a^2, a^2, a) = 0\). If \( a \) and \( a^2 \) are linearly independent, then \( A(a, a^2) \) is commutative and isomorphic to \( \mathbb{C} \).

Proof. By theorem 2.2, the norm of \( A \) comes from an inner product. Let \( d = a^2 - (a/a^2)a \), \( (d \neq 0) \), we have \((d/a) = 0\), by lemma 2.7.(1)

\[
\begin{align*}
d^2 &= -||d||^2a^2 = -(1 - (a/a^2)^2)a^2 \\
&= a^2 - (a/a^2)^2a^2 = (a^2 - (a/a^2)^2a^2 - 2(a/a^2)a^2a^2 + (a/a^2)^2a^2 \\
&= (a^2)^2 - 2(a/a^2)a^2 - a^2
\end{align*}
\]

\( \) If \((a/a^2) = 0\), then \((a^2)^2 = -a^2 \) and \((a^2)^2a = -a^2a\). That is \( a^2(a^2a) = -a^2a, \) hence \( aa^2 = -a, \) which means that \( A(a, a^2) \) is a two dimensional commutative sub-algebra of \( A \).

\( **\) Assuming that \((a/a^2) \neq 0\), since \((d^2, d^2, d) = 0\), then \((a^2, a^2, a^2) = 0\) thus \((a^2)^2a^2 = a^2(a^2)^2 \)

So \( A(a, a^2) \) is a two dimensional commutative sub-algebra of \( A \) (proposition 2.13). Thus \( A(a, a^2) \) is isomorphic to \( \mathbb{C} \) or \( \mathbb{C} \) (theorem 2.4). We assume that \( A(a, a^2) \) is isomorphic to \( \mathbb{C} \), that is, there exist a basis \{\( f, j \)\} of \( A(a, a^2) \) such that \( f^2 = f, \ j^2 = -f \) and \( jf = fj = -j \).

So \( (j^2, j^2, j) = (f, f, j) = fj - f(jf) = -j - j = -2j \neq 0 \)

Which absurd, therefore \( A(a, a^2) \) is isomorphic to \( \mathbb{C} \).

Similarly, we obtain

| **Lemma 2.15** Let \( A \) be a pre-Hilbert absolute valued algebra containing a central element \( a \) such that \((a^2, a^2, a) = 0\). If \( a \) and \( a^2 \) are linearly independent, then \( A(a, a^2) \) is commutative and isomorphic to \( \mathbb{C} \).

| **Remark 2.16** Let \( A \) be a four-dimensional absolute valued algebra containing a nonzero central element \( a \) and commutative sub-algebra \( B \) isomorphic to \( \mathbb{C} \). If \((a^2, a^2, a) = 0\) or \((a, a^2, a^2) = 0\), then \( a \) is a central idempotent of \( A \).

3. **Main Results**

**Theorem 3.1** Let \( A \) be a four-dimensional absolute valued algebra containing a nonzero central element \( a \) and commutative sub-algebra \( B = A(e, i) \), where \( i^2 = -e, ie = ei = \pm i \), then \( A \) is isomorphic to \( A_1, A_2, A_3, A_4, B_1, B_3 \) or \( B_4 \).

Proof. By theorem 2.2, the norm of \( A \) comes from an inner product and by lemma 2.8, we have the following cases:
1) If \( a = \pm e \) is a central idempotent of \( A \), then by theorem 2.5, \( A \) is isomorphic to \( A_1, A_2, A_3 \) or \( A_4 \).

2) We assume \( a = \pm i \) is a central element of \( A \) and let \( F = \{ e, i, j, k \} \) be an orthonormal basis of \( A \), since \( j^2 = k^2 = -i^2 = e \) and \( jk = -kj \in B \) (lemma 2.7.(3)), then \( (jk/e) = (jk/j^2) = (k/j) = 0 \). So \( jk = \pm i \), the set \( \{ e, i, j, ik \} \) is an orthonormal basis of \( A \), then
\[
(ej/e) = (ej/e^2) = (ej/j) = 0, (ej/i) = \pm (ej/ei) = \pm (j/i) = 0 \text{ and } (ej/ij) = (e/j) = 0.
\]
Which imply that \( e = \pm jk \), similarly \( (ek/e) = (ek/e^2) = (e/k) = 0, (ek/i) = \pm (ek/ei) = \pm (k/i) = 0 \text{ and } (ek/ik) = (e/k) = 0 \).

Then \( e(k)j = e(j + k) = -ik - jk = -ik - i = -ki - e - i = -(e + k)j \).

Which gives \( i = -j \) (\( A \) has no zero divisors), a contradiction. Moreover, if \( e(k)j = i \) then
\[
(e + j)k = ek + jk = i + i = e + j \cdot i
\]
The last gives \( k = i \), which is absurd. We pose \( ej = \alpha j + \beta k \) (where, \( \alpha^2 + \beta^2 = 1 \)), then \( ik = \alpha i + \beta j \). Likewise, \( ek = \lambda j + \mu k \), where \( \lambda^2 + \mu^2 = 1 \).

Since \( (ek/ej) = 0 \), we get \( \alpha \lambda + \beta \mu = 0 \). So
\[
(\alpha \mu - \beta \lambda)^2 = \alpha^2 \mu^2 - 2\alpha \mu \beta \lambda + \beta^2 \lambda^2
= \alpha^2 \mu^2 + 2\alpha^2 \beta \lambda + \beta^2 \lambda^2
= \alpha^2 \mu^2 + \lambda^2(\alpha^2 + \beta^2)\lambda^2
= \alpha^2 + \lambda^2
\]

On the other hand we have.
\[
(\alpha \mu - \beta \lambda)^2 = \alpha^2 \mu^2 - 2\alpha \mu \beta \lambda + \beta^2 \lambda^2
= \alpha^2 \mu^2 + 2\alpha^2 \beta \lambda + \beta^2 \lambda^2
= \mu^2(\alpha^2 + \beta^2) + \beta^2(\mu^2 + \lambda^2)
= \mu^2 + \beta^2
\]

So \( \alpha^2 + \lambda^2 = \mu^2 + \beta^2 = 2 - (\alpha^2 + \lambda^2) \), which means that \( \alpha^2 + \lambda^2 = 1 \) and consequently \( \alpha \mu - \beta \lambda = \pm 1 \).

*) If \( \alpha \mu - \beta \lambda = 1 \), then \( \mu = \mu(\alpha \mu - \beta \lambda) = \alpha \mu^2 - \beta \lambda \mu = \alpha \mu^2 + \alpha \lambda = \alpha \)
And
\( \lambda = \lambda(\alpha \mu - \beta \lambda) = \alpha \mu \lambda - \beta \lambda^2 = -\beta \mu^2 - \beta \lambda^2 = -\beta \)
Therefore the multiplication table of \( A \) is given by:

<table>
<thead>
<tr>
<th>( B_1 )</th>
<th>( e )</th>
<th>( i )</th>
<th>( j )</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>( e )</td>
<td>( i )</td>
<td>( \alpha j + \beta k )</td>
<td>( -\beta j + \alpha k )</td>
</tr>
<tr>
<td>( i )</td>
<td>( i )</td>
<td>( -e )</td>
<td>( -\beta j + \alpha k )</td>
<td>( \alpha j + \beta k )</td>
</tr>
<tr>
<td>( j )</td>
<td>( -\alpha j - \beta k )</td>
<td>( -\beta j + \alpha k )</td>
<td>( e )</td>
<td>( i )</td>
</tr>
<tr>
<td>( k )</td>
<td>( \beta j - \alpha k )</td>
<td>( \alpha j + \beta k )</td>
<td>( -i )</td>
<td>( e )</td>
</tr>
</tbody>
</table>

**) If \( \alpha \mu - \beta \lambda = -1 \), then \( \mu = -\mu(\alpha \mu - \beta \lambda) = -\alpha \mu^2 + \beta \lambda \mu = -\alpha \mu^2 - \alpha \lambda = -\alpha \)
And
\( \lambda = -\lambda(\alpha \mu - \beta \lambda) = -\alpha \mu \lambda + \beta \lambda^2 = \beta \mu^2 + \beta \lambda^2 = \beta \)
Therefore the multiplication table of \( A \) is given by:
ii) $B$ isomorphic to $\mathbb{C}$, we have $e i = i e = -i, i^2 = -e$ and $j k = i$. If we define a new multiplication on $A$ by $x \cdot y = \bar{x} \bar{y}$, we obtain an algebra $\hat{A}$ which contains a sub-algebra isomorphic to $\mathbb{C}$. Therefore $\hat{A}$ has an orthonormal basis which the multiplication tables are given previously. Consequently, the multiplication tables of the elements of the base $F$ of $A$ are given by:

$$
\begin{array}{|c|c|c|c|c|}
\hline
B_2 & e & i & j & k \\
\hline
\hline
e & e & i & \alpha j + \beta k & \beta j - \alpha k \\
i & i & -e & \beta j - \alpha k & \alpha j + \beta k \\
j & -\alpha j - \beta k & \beta j - \alpha k & e & i \\
k & -\beta j + \alpha k & \alpha j + \beta k & -i & e \\
\hline
\end{array}
$$

and

$$
\begin{array}{|c|c|c|c|c|}
\hline
B_3 & e & i & j & k \\
\hline
\hline
e & e & -i & -\alpha j - \beta k & \beta j - \alpha k \\
i & -i & -e & -\beta j + \alpha k & \alpha j + \beta k \\
j & \alpha j + \beta k & -\beta j + \alpha k & e & i \\
k & -\beta j + \alpha k & \alpha j + \beta k & -i & e \\
\hline
\end{array}
$$

and

$$
\begin{array}{|c|c|c|c|c|}
\hline
B_4 & e & i & j & k \\
\hline
\hline
e & e & -i & -\alpha j - \beta k & -\beta j + \alpha k \\
i & -i & -e & \beta j - \alpha k & \alpha j + \beta k \\
j & \alpha j + \beta k & \beta j - \alpha k & e & i \\
k & \beta j - \alpha k & \alpha j + \beta k & -i & e \\
\hline
\end{array}
$$

Remark 3.2 Note that, central idempotent is a central element. The reciprocal case does not hold in general, and the counter example is given ($B_1$ and $B_2$).

Theorem 3.3 Let $A$ be a four-dimensional absolute valued algebra containing a nonzero central element $a$ which satisfies one of the following identities:

1) $(a^2, a^2, a^2) = 0$, with $(a/a^2) \neq 0$,
2) $(a^2, a, a^2) = 0$.

Then $A$ contains a commutative sub-algebra of dimension two. Moreover, $A$ is isomorphic to $A_1, A_2, A_3, A_4, B_1, B_2, B_3$ or $B_4$.

Proof. *) If $a$ and $a^2$ are linearly dependent, then $a$ is a central idempotent. By lemma 2.6, $A$ contains a commutative sub-algebra isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$ and is isomorphic to $A_1, A_2, A_3$ or $A_4$.

**) If $a$ and $a^2$ are linearly independent, using proposition 2.13, $A(a, a^2)$ is two-dimensional commutative sub-algebra of $A$, thus $A(a, a^2)$ is isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$. Hence the result is consequence of the theorem 3.1.
Theorem 3.4 Let $A$ be a four-dimensional absolute valued algebra containing a nonzero central element $a$ which satisfies one of the following identities:
1) $(a^2, a, a) = 0$ or $(a, a, a^2) = 0$,
2) $(a^2, a^2, a) = 0$ or $(a, a^2, a^2) = 0$.
Then $A$ contains a commutative sub-algebra of dimension two. Moreover, $A$ is isomorphic to $A_1, A_2, A_3, A_4, B_1$ or $B_2$.
Proof. *) If $a$ and $a^2$ are linearly dependent, then $a$ is a central idempotent. By lemma 2.6, $A$ contains a commutative sub-algebra isomorphic to $\mathbb{C}$ or $\mathbb{C}$. Then $A$ is isomorphic to $A_1, A_2, A_3$ or $A_4$.
**) If $a$ and $a^2$ are linearly independent, by proposition 2.10 and lemma 2.14, $A(a, a^2)$ is a two-dimensional commutative sub-algebra of $A$, thus $A(a, a^2)$ is isomorphic to $\mathbb{C}$. Hence the theorem 3.1 completes the proof.

4. Acknowledgement
The author expresses their deep gratitude to the referee for the carefully reading of the manuscript and the valuable comments that have improved the final version of the same.

5. References

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