Some New Class of Four-Dimensional Absolute Valued Algebras

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Abstract

An absolute valued algebra is a nonzero real algebra that is equipped with a multiplicative norm $||ab|| = ||a||||b||$. We prove that if A is a four-dimensional absolute valued algebras containing a nonzero central element a which satisfies one of the following identities:
1) $(a^2, a, a) = 0$ or $(a, a, a^2) = 0$,
2) $(a^2, a^2, a) = 0$ or $(a, a^2, a^2) = 0$,
3) $(a^2, a^2, a^2) = 0$, with $(a/a^2) \neq 0$,
4) $(a^2, a, a^2) = 0$.
Then A contains a commutative sub-algebra of dimension two. Moreover, A is isomorphic to A_1, A_2, A_3, A_4, B_1 or B_2 in the first two cases and isomorphic to $A_1, A_2, A_3, A_4, B_1, B_2$, B_3 or B_4, in the last two cases.

Keywords: Absolute valued algebra, Pre-Hilbert algebra, Commutative algebra, Central element.

1. Introduction

Let A be a non-necessarily associative real algebra which is normed as real vector space. We say that a real algebra is pre-Hilbert algebra, if it's norm $\|.\|$ come from an inner product $(./.)$, and it's said to be absolute valued algebras, if it's norm satisfies the equality $\|ab\| = \|a\|\|b\|$, for all $a, b \in A$. Note that, the norm of any absolute valued algebras containing a nonzero central idempotent (or finite dimensional) comes from an inner product [3] and [4]. In 1947 Albert proved that the finite dimensional unital absolute valued algebras are classified by $\mathbb{R}, \mathbb{C}, \mathrm{H}$ and O, and that every finite dimensional absolute valued algebra has dimension 1, 2, 4 or 8 [1]. Urbanik and Wright proved in 1960 that all unital absolute valued algebras are classified by $\mathbb{R}, \mathbb{C}, \mathrm{H}$ and O [9]. It is easily seen that the one-dimensional absolute valued algebras are classified by $\mathbb{R}$, and it is well-known that the two-dimensional absolute valued algebras are classified by $\mathbb{C}, \mathbb{C}^*$, $^*\mathbb{C}$ or $\overset{*}{\mathbb{C}}$ (the real algebras obtained by endowing the space $\mathbb{C}$ with the product $x * y = \bar{x}y$, $x * y = x\bar{y}$, and $x * y = \bar{x}\,\bar{y}$ respectively) [7]. The four-dimensional absolute valued algebras have been described by M.I. Ramirez Alvarez in 1997 [5]. The problem of classifying all four (eight)-dimensional absolute valued algebras seems still to be open.

In 2016 [5], we classified all four-dimensional absolute valued algebras containing a nonzero central idempotent and we also proved such an algebra contains a commutative sub-algebra of dimension two. Here (**theorems 3.1, 3.3 and 3.4**) we extend this result to more general situation. Indeed, Let A be a four-dimensional absolute valued algebra containing a nonzero central element a. If A has a commutative sub-algebra of dimension two, then A is isomorphic to a new absolute valued algebras of

dimension four. We also show, in **propositions 2.10, 2.11 and 2.13**, that A contains a sub-algebra of dimension two if and only if a satisfies one of the following identities:
1) $(a^2, a, a) = 0$ or $(a, a, a^2) = 0$,
2) $(a^2, a^2, a) = 0$ or $(a, a^2, a^2) = 0$,
3) $(a^2, a^2, a^2) = 0$, with $(a/a^2) \neq 0$,
4) $(a^2, a, a^2) = 0$,
(Where $(.,.,.)$ means associator, that is $(x, y, z) = (xy)z - x(yz)$). Also we prove that A is isomorphic to A_1, A_2, A_3, A_4, B_1 or B_2 in the first two cases and isomorphic to $A_1, A_2, A_3, A_4, B_1, B_2, B_3$ or B_4, in the last two cases. We denote that a central idempotent is central element, the reciprocal does not hold in general, and the counter example is given (**remark 3.2**).

We recall that, the classification of four-dimensional absolute valued algebras containing a unique two-dimensional sub-algebra is still an open problem. Also, in [2] we showed that if A is a four-dimensional absolute valued algebra with left unit and containing a nonzero central element, then A contains a sub-algebra of dimension two. It may be conjectured that every four-dimensional absolute valued algebra containing a nonzero central element, has sub-algebra of dimension two.

In section 2 we introduce the basic tools for the study of four-dimensional absolute valued algebras. We also give some properties related to central element satisfying some restrictions on commutativity (**lemmas 2.7, 2.8, 2.9, 2.14, 2.15 and propositions 2.10, 2.11, 2.13**). Moreover, the section 3 is devoted to construct, by algebraic method, some new class of the four-dimensional absolute valued algebras having commutative sub-algebras of dimension two, namely B_1, B_2, B_3 and B_4. The paper ends with the following main results:

Theorem 3.1 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a which satisfies one of the following identities:
1) $(a^2, a, a) = 0$ or $(a, a, a^2) = 0$,
2) $(a^2, a^2, a) = 0$ or $(a, a^2, a^2) = 0$.
Then A contains a commutative sub-algebra of dimension two. Moreover, A is isomorphic to A_1, A_2, A_3, A_4, B_1 or B_2.

Theorem 3.2 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a which satisfies one of the following identities:
1) $(a^2, a^2, a^2) = 0$, with $(a/a^2) \neq 0$,
2) $(a^2, a, a^2) = 0$.
Then A contains a commutative sub-algebra of dimension two. Moreover, A is isomorphic to $A_1, A_2, A_3, A_4, B_1, B_2, B_3$ or B_4.

2. Notation And Preliminaries Results

In this paper all the algebras are considered over the real numbers field $\mathbb{R}$.

Definition 2.1 Let B be an arbitrary algebra.
i) B is called a normed algebra (respectively, absolute valued algebra) if it is endowed with a space norm: $\|\,.\,\|$ such that $\|xy\| \leq \|x\|\|y\|$ (respectively, $\|xy\| = \|x\|\|y\|$, for all $x, y \in B$).
ii) B is called a division algebra if the operators L_x and R_x of left and right multiplication by x are bijective for all $x \in B \setminus \{0\}$.
iii) B is called a pre-Hilbert algebra if it is endowed with a space norm comes from an inner product $(./.)$ such that

$$(./.) : B \times B \longrightarrow \mathbb{R}$$
$$(x, y) \longmapsto \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

The most natural examples of absolute valued algebras are $\mathbb{R}, \mathbb{C}$, H (the algebra of Hamilton quaternion) and O (the algebra of Cayley numbers) with norms equal to their usual absolute values [2] and [8]. The algebras by $\mathbb{C}^*$, $^*\mathbb{C}$ and $\overset{*}{\mathbb{C}}$ (obtained by endowing the space by $\mathbb{C}$ with the products defined by: $x * y = \bar{x}y$, $x * y = x\bar{y}$, and $x * y = \bar{x}\,\bar{y}$ respectively) where $x \rightarrow \bar{x}$ is the standard conjugation of $\mathbb{C}$. Note that the algebras $\mathbb{C}$ and $\overset{*}{\mathbb{C}}$ are the only two-dimensional commutative absolute valued algebras.

We need the following relevant results:

Theorem 2.2 [3] The norm of any finite dimensional absolute valued algebra comes from an inner product.

Theorem 2.3 [4] The norm of any absolute valued algebra containing a nonzero central idempotent comes from an inner product.

Theorem 2.4 [11] Let A be a two-dimensional absolute valued algebra, then A is isomorphic to $\mathbb{C}$, $^*\mathbb{C}$, $\mathbb{C}^*$ or $\overset{*}{\mathbb{C}}$.

Theorem 2.5 [5] Let A be a four-dimensional absolute valued algebra containing a nonzero central idempotent e, then A is isomorphic to A_1, A_2, A_3 or A_4 defined by: $(\alpha^2 + \beta^2 = 1)$

A_1	e	i	j	k
e	e	i	$\alpha j + \beta k$	$-\beta j + \alpha k$
i	i	$-e$	$-\beta j + \alpha k$	$-\alpha j - \beta k$
j	$\alpha j + \beta k$	$\beta j - \alpha k$	$-e$	i
k	$-\beta j + \alpha k$	$\alpha j + \beta k$	$-i$	$-e$

A_2	e	i	j	k
e	e	i	$\alpha j + \beta k$	$-\beta j + \alpha k$
i	i	$-e$	$-\beta j + \alpha k$	$-\alpha j - \beta k$
j	$\alpha j + \beta k$	$\beta j - \alpha k$	$-e$	i

k	$-\beta j+\alpha k$	$\alpha j+\beta k$	$-i$	$-e$

A_3	e	i	j	k
e	e	i	$\alpha j+\beta k$	$-\beta j+\alpha k$
i	i	$-e$	$-\beta j+\alpha k$	$-\alpha j-\beta k$
j	$\alpha j+\beta k$	$\beta j-\alpha k$	$-e$	i
k	$-\beta j+\alpha k$	$\alpha j+\beta k$	$-i$	$-e$

A_4	e	i	j	k
e	e	i	$\alpha j+\beta k$	$-\beta j+\alpha k$
i	i	$-e$	$-\beta j+\alpha k$	$-\alpha j-\beta k$
j	$\alpha j+\beta k$	$\beta j-\alpha k$	$-e$	i
k	$-\beta j+\alpha k$	$\alpha j+\beta k$	$-i$	$-e$

Lemma 2.6 [5] Let A be a finite dimensional absolute valued algebra containing a nonzero central idempotent e, then A contains a commutative sub-algebra of dimension two.

Lemma 2.7 Let A be a finite dimensional absolute valued algebra containing a nonzero central element a, then
1) $x^2 = -\|x\|^2 a^2$, for all $x \in \{a\}^{\perp} := \{x \in A, (x/a) = 0\}$
2) $xy + yx = -2(x/y)a^2$ for all $x, y \in \{a\}^{\perp}$
3) $(xy/yx) = -(x^2/y^2)$ for all $x, y \in \{a\}^{\perp}$ such that $(x/y) = 0$.
Proof. 1) By theorem 2.2, the norm of A comes from an inner product and we assume that $\|x\| = 1$ (where $x \in \{a\}^{\perp}$), we have :

$$\|x^2 - a^2\|^2 = \|x-a\|^2\|x+a\|^2 = 2$$

That is $(x^2/a^2) = -1$, which imply that $\|x^2 + a^2\|^2 = 0$ therefore $x^2 = -a^2$
2) It's clear.
3) We get this identity by simple linearization of the identity $||x^2|| = ||x||^2$.

Lemma 2.8 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a and commutative sub-algebra B, then $a \in B$.
Proof. By theorem 2.2, the norm of A comes from an inner product. Applying theorem 2.4, B is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$, that is, there exist an idempotent e and an element i such that $B = A(e, i)$, where $i^2 = -e$ and $ie = ei = \pm i$. We distinguish the two following cases:
i) If $(a/e) = 0$, by lemma 2.7.(1), we have $a^2 = -e$, so $a = \pm\, i \in B$ $(ai = ia$ and $i^2 = -e)$.
ii) If $(a/e) \neq 0$, we put $c = a - (a/e)e$, this imply that $(c/e) = 0$. Since $ce = ec$, then $c^2 = -\|c\|^2 e = \|c\|^2 i^2$ which means that $c = \pm\|c\|\, i$, thus $a = c + (a/e)e \in B$. We can put $a = \lambda\, e + \mu\, i$, with $\lambda, \mu \in \mathbb{R}$ $(\lambda^2 + \mu^2 = 1)$ and let $b = \mu\, e - \lambda\, i \in B$ and $j \in A$ two elements orthogonal to a. As $aj = ja$, we get $\lambda\, ej + \mu\, ij = \lambda\, je + \mu\, ji$ (1)
Using lemma 2.7.(2), we have $bj + jb = 0$. This imply

$$\mu\, ej + \lambda\, ij = -\mu\, je - \lambda\, ji \quad (2)$$

From the equalities (1) and (2), we obtain $2\lambda\,\mu\,ej + ij = (\mu^2 - \lambda^2)\,ji \quad (\mu^2 + \lambda^2 = 1)$
Therefore $(2\lambda\,\mu\,ej + ij/ij) = ((\mu^2 - \lambda^2)\,ji\,/ij)$
Applying lemma 2.7.(3), we get

$$1 = (\mu^2 - \lambda^2)(ji/\,ij) = -\,(\mu^2 - \lambda^2)(j^2/i^2) \qquad (3)$$

Since $a^2 = (\lambda^2 - \mu^2)e \pm 2\lambda\mu\, i, i^2 = -e$ and $j^2 = -a^2$. Then the equality (3) gives $(\lambda^2 - \mu^2)^2 = 1$. Hence $\lambda^2 - \mu^2 = 1$ or $\lambda^2 - \mu^2 = -1$, as $\lambda^2 + \mu^2 = 1$, then $\lambda^2 = 1$ (because, $\lambda = (a/e) \neq 0$). That is, $a = \pm\, e \in B$.

Lemma 2.9 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a and commutative sub-algebra B of dimension two. If $x, y \in B^{\perp}$, then $xy \in B$.
Proof. By theorem 2.2, the norm of A comes from an inner product. According to theorem 2.4, B is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$. Then there exist an idempotent e and an element i such that $B = A(e, i)$, where $i^2 = -\,e$ and $ei = ie = \pm\, i$. Let $F = \{e, i, j, k\}$ be an orthonormal basis of A, as A is a division algebra then L_j is bijective, so there exist j' such that $i = jj'$. We have

$$(j'/e) = (jj'/je) = (i/je) = \pm\,(ie/je) = \pm\,(i/j) = 0$$

and

$$(j'/i) = (jj'/ji) = (i/ji) = \pm\,(ei/ji) = \pm\,(e/j) = 0$$

so $j' = \alpha j + \beta k$, with $\alpha, \beta \in \mathbb{R}$. Consequently we have $i = jj' = \alpha j^2 + \beta jk$. Since $a \in B$ (lemma 2.8), then, by lemma 2.7.(1), $\beta jk = \alpha a^2 + i \in B$. Finally, we pose $x = p\,j + q\,k$ and $y = p'\,j + q'\,k$ with $p, q, p', q' \in \mathbb{R}$. We have

$$x\,y = (pp' + qq')e + (pq' - qp')\,jk \in B \qquad (jk = -kj).$$

We give some conditions implying the existence of two-dimensional sub-algebras.
Proposition 2.10 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a, then the following assertions are equivalent:
1) $(a^2, a, a) = 0$
2) A contains a commutative sub-algebra of dimension two.

Proof. By theorem 2.2, the norm of A comes from an inner product. We discuss the following two cases:
1) If a and a^2 linearly dependent, then a is a nonzero central idempotent. Therefore A contains a commutative sub-algebra of dimension two (lemma 2.6) and the three assertions are equivalent.
2) If a and a^2 linearly independent
1) ⇒ 2) i) We suppose $(a/a^2) = 0$, by lemma 2.7.(1), we have $(a^2)^2 = -a^2$, so $(a^2a)a = -a^2$ which means that $a^2a = -a$. Hence $B = A(a^2, a)$ is two dimensional commutative sub-algebra of A.
ii) We assume that $(a/a^2) = m \neq 0$. We pose $d = a^2 - ma$, this imply that $(d/a) = 0$, using lemma 2.6, we have $d^2 = -\|\,d\,\|^2\, a^2 = -(1 - m^2)a^2$
which means that $-(1 - m^2)a^2 = (a^2 - ma)^2 = (a^2)^2 - 2ma^2a + m^2a^2$
Since $(a^2)^2 = (a^2a)a$, thus $-(1 - m^2)a^2 = (a^2a)a - 2ma^2a + m^2a^2$
Hence $-(1 - m^2)a = a^2a - 2ma^2 + m^2a = da - md$
Therefore $ad = da = -(1 - m^2)a + md$ this means that $A(a, d)$ is a two-dimensional commutative sub-algebra of A.
2) ⇒ 1) Let B be a two-dimensional commutative sub-algebra of A, according to theorem 2.4, B is

isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$. That is $B := A(e,i)$ such that $ie = ei = \pm i$ and $i^2 = -e$, where e is an idempotent of A. On the other hand, the lemma 2.8 proves that $a = \pm e$ or $a = \pm i$ and we have the two following cases:

*) B is isomorphic to $\mathbb{C}$, hence a verifies the equality $(a^2, a^2, a) = 0$.

**) B is isomorphic to $\overset{*}{\mathbb{C}}$, then a = ±e satisfies the identity $(a^2, a, a) = 0$. But since ie = ei = −i and $i^2 = -e$, thus $(a^2, a, a) \neq 0$.

In the same way, we get

Proposition 2.11 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a, then the following assertions are equivalent:

1) $(a, a, a^2) = 0$

2) A contains a commutative sub-algebra of dimension two.

Remark 2.12 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a and commutative sub-algebra B isomorphic to $\overset{*}{\mathbb{C}}$. If $(a^2, a, a) = 0$ or $(a, a, a^2) = 0$, then a is a central idempotent of A.

Proposition 2.13 Let A be a pre-Hilbert absolute valued algebra containing a nonzero central element a such that $(a/a^2) \neq 0$. Then the following assertions are equivalent:

1) $(a^2, a^2, a^2) = 0$,

2) $(a^2, a, a^2) = 0$,

3) A contains a commutative sub-algebra of dimension two.

Proof. By theorem 2.2, the norm of A comes from an inner product.

1) ⇒ 2) Let $d = a^2 - (a/a^2)a$, $(d \neq 0)$, we have $(d/a) = 0$, by lemma 2.7.(1)

$$d^2 = -||d||^2 a^2 = -(1 - (a/a^2)^2)a^2$$

That is
$$-(1 - (a/a^2)^2)a^2 = (a^2 - (a/a^2)a)^2$$
$$-a^2 + (a/a^2)^2 a^2 = (a^2)^2 - 2(a/a^2)aa^2 + (a/a^2)^2 a^2$$

This gives
$$(a^2)^2 = 2(a/a^2)aa^2 - a^2$$

Since $(a^2, a^2, a^2) = 0$, then $(a^2)^2 a^2 = a^2(a^2)^2$. So $(a^2 a)a^2 = a^2(aa^2)$, consequently $(a^2, a, a^2) = 0$.

2) ⇒ 3) Let $c = aa^2 - (a/aa^2)a$, we have $(c/a) = 0$, by lemma 2.7.(1),

$$c^2 = -||c||^2 a^2 = -(1 - (a/aa^2)^2)a^2.$$

*) If $||c|| = 0$, then $aa^2 = \pm a$. That is

$$(a^2)^2 = 2(a/a^2)aa^2 - a^2 = \pm 2(a/a^2)a - a^2$$

This implies that $A(a, a^2)$ is a two dimensional commutative sub-algebra of A.

**) Assuming that $||c|| \neq 0$, since $(a^2, a, a^2) = 0$ thus $(a^2 a)a^2 = a^2(aa^2)$.

So
$$(aa^2)a^2 = a^2(aa^2),$$

Moreover
$$\begin{aligned} dc &= (a^2 - (a/a^2)a)(aa^2 - (a/aa^2)a) \\ &= a^2(aa^2) - (a/a^2)a(aa^2) - (a/aa^2)aa^2 + (a/a^2)(a/aa^2)a^2 \\ &= (aa^2)a^2 - (a/a^2)a(aa^2) - (a/aa^2)aa^2 + (a/a^2)(a/aa^2)a^2 \\ &= cd \end{aligned}$$

And since $||c||^2 d^2 = ||d||^2 c^2$, then $||c||d = ||d||c$ or $||c||d = -||d||c$. We conclude that

$$||d||aa^2 = ||c||a^2 + ((a/aa^2)||d|| - (a/a^2)||c||)a$$

Or $$||d||aa^2 = ||c||a^2 + ((a/aa^2)||d|| - (a/a^2)||c||)a$$

Therefore $A(a,a^2)$ is a two-dimensional commutative sub-algebra of A, thus $A(a,a^2)$ is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$ (theorem 2.4).

3) ⇒ 1) Let B be a two-dimensional commutative sub-algebra of A, according to theorem 2.4, B is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$. That is $B := A(e,i)$ such that $ie = ei = \pm i$ and $i^2 = -e$, where e is an idempotent of A. On the other hand, the lemma 2.8 imply that $a = \pm e$ or $a = \pm i$, which means a verifies the equality $(a^2,a^2,a^2) = 0$.

Lemma 2.14 Let A be a pre-Hilbert absolute valued algebra containing a central element a such that $(\mathrm{a}^2,\mathrm{a}^2,\mathrm{a}) = 0$. If a and a^2 are linearly independent, then $A(a,a^2)$ is commutative and isomorphic to $\mathbb{C}$.

Proof. By theorem 2.2, the norm of A comes from an inner product. Let $d = a^2 - (a/a^2)a$, $(d \neq 0)$, we have $(d/a) = 0$, by lemma 2.7.(1)

$$d^2 = -||d||^2a^2 = -(1-(a/a^2)^2)a^2$$

That is $$-(1-(a/a^2)^2)a^2 = (a^2-(a/a^2)a)^2$$

$$-a^2 + (a/a^2)^2a^2 = (a^2)^2 - 2(a/a^2)aa^2 + (a/a^2)^2a^2$$

This gives $$(a^2)^2 = 2(a/a^2)aa^2 - a^2$$

*) If $(a/a^2) = 0$, then $(a^2)^2 = -a^2$ and $(a^2)^2a = -a^2a$. That is $a^2(a^2a) = -a^2a$, hence $aa^2 = -a$, which means that $A(a,a^2)$ is a two dimensional commutative sub-algebra of A.

**) Assuming that $(a/a^2) \neq 0$, since $(d^2,d^2,d) = 0$, then $(a^2,a^2,a^2) = 0$ thus $(a^2)^2a^2 = a^2(a^2)^2$

So $A(a,a^2)$ is a two dimensional commutative sub-algebra of A (proposition 2.13). Thus $A(a,a^2)$ is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$ (theorem 2.4). We assume that $A(a,a^2)$ is isomorphic to $\overset{*}{\mathbb{C}}$, that is, there exist a basis $\{f,j\}$ of $A(a,a^2)$ such that $f^2 = f,\ j^2 = -f\ \ and\ jf = fj = -j.$

So $$(j^2,j^2,j) = (f,f,j) = fj - f(fj) = -j - j = -2j \neq 0$$

Which absurd, therefore $A(a,a^2)$ is isomorphic to $\mathbb{C}$.

Similarly, we obtain

Lemma 2.15 Let A be a pre-Hilbert absolute valued algebra containing a central element a such that $(\mathrm{a},\mathrm{a}^2,\mathrm{a}^2) = 0$. If a and a^2 are linearly independent, then $A(a,a^2)$ is commutative and isomorphic to $\mathbb{C}$.

Remark 2.16 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a and commutative sub-algebra B isomorphic to $\overset{*}{\mathbb{C}}$. If $(a^2,a^2,a) = 0$ or $(a,a^2,a^2) = 0$, then a is a central idempotent of A.

3. Main Results

Theorem 3.1 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a and commutative sub-algebra $B = A(e,i)$, where $i^2 = -e, ie = ei = \pm i$, then A is isomorphic to $A_1, A_2, A_3, A_4, B_1, B_3$ or B_4.

Proof. By theorem 2.2, the norm of A comes from an inner product and by lemma 2.8, we have the following cases:

1) If $a = \pm e$ is a central idempotent of A, then by theorem 2.5, A is isomorphic to A_1, A_2, A_3 or A_4.
2) We assume $a = \pm i$ is a central element of A and let $F = \{e, i, j, k\}$ be an orthonormal basis of A, since $j^2 = k^2 = -i^2 = e$ and $jk = -kj \in B$ (lemma 2.7.(3)), then $(jk/e) = (jk/j^2) = (k/j) = 0$
So $jk = \pm i$, the set $\{e, i, ij, ik\}$ is an orthonormal basis of A, then

$$(ej/e) = (ej/e^2) = (e/j) = 0, (ej/i) = \pm(ej/ei) = \pm(j/i) = 0 \text{ and } (ej/ij) = (e/j) = 0.$$

Which imply that $ej = \pm jk$, similarly

$$(ek/e) = (ek/e^2) = (e/k) = 0, (ek/i) = \pm(ek/ei) = \pm(k/i) = 0 \text{ and } (ek/ik) = (e/k) = 0.$$

Then $ek = \pm ij$. According to lemma 2.7.(3), $(ej/je) = -(e^2/j^2) = (e/i^2) = -1$ hence $ej = -je$. Also $(ek/ke) = -(e^2/k^2) = (e/i^2) = -1$, thus $ek = -ke$. We assume that $jk = i$ and we distinguish the following cases:
i) B isomorphic to $\mathbb{C}$, we have $ei = ie = i$ and $i^2 = -e$. So $ej = ik$ and $ek = -ij$. Indeed, if $ej = -ik$ then $(e + k)j = ej + kj = -ik - jk = -ik - i = -ki - ei = -(e + k)i$.
Which gives $i = -j$ (A has no zero divisors), a contradiction. Moreover, if $ek = ij$ then

$$(e + j)k = ek + jk = ij + i = ji + ei = (e + j)i$$

The last gives $k = i$, which is absurd. We pose $ej = \alpha j + \beta k$ (where, $\alpha^2 + \beta^2 = 1$), then $ik = ki = \alpha e + \beta j$. Likewise, $ek = \lambda j + \mu k$, where $\lambda^2 + \mu^2 = 1$.
Since $(ek/ej) = 0$, we get $\alpha\lambda + \beta\mu = 0$. So

$$\begin{aligned}(\alpha\mu - \beta\lambda)^2 &= \alpha^2\mu^2 - 2\alpha\mu\beta\lambda + \beta^2\lambda^2 \\ &= \alpha^2\mu^2 + 2\alpha^2\lambda^2 + \beta^2\lambda^2 \\ &= \alpha^2(\mu^2 + \lambda^2) + \lambda^2(\alpha^2 + \beta^2)\lambda^2 \\ &= \alpha^2 + \lambda^2\end{aligned}$$

On the other hand we have.

$$\begin{aligned}(\alpha\mu - \beta\lambda)^2 &= \alpha^2\mu^2 - 2\alpha\mu\beta\lambda + \beta^2\lambda^2 \\ &= \alpha^2\mu^2 + 2\alpha^2\lambda^2 + \beta^2\lambda^2 \\ &= \mu^2(\alpha^2 + \beta^2) + \beta^2(\mu^2 + \lambda^2) \\ &= \mu^2 + \beta^2\end{aligned}$$

So $\alpha^2 + \lambda^2 = \mu^2 + \beta^2 = 2 - (\alpha^2 + \lambda^2)$, which means that $\alpha^2 + \lambda^2 = 1$ and consequently $\alpha\,\mu - \beta\,\lambda = \pm 1$.
*) If $\alpha\mu - \beta\lambda = 1$, then $\mu = \mu(\alpha\mu - \beta\lambda) = \alpha\mu^2 - \beta\lambda\mu = \alpha\mu^2 + \alpha\lambda^2 = \alpha$
And $\lambda = \lambda(\alpha\mu - \beta\lambda) = \alpha\mu\lambda - \beta\lambda^2 = -\beta\mu^2 - \beta\lambda^2 = -\beta$
Therefore the multiplication table of A is given by:

B_1	e	i	j	k
e	e	i	$\alpha j + \beta k$	$-\beta j + \alpha k$
i	i	$-e$	$-\beta j + \alpha k$	$\alpha j + \beta k$
j	$-\alpha j - \beta k$	$-\beta j + \alpha k$	e	i
k	$\beta j - \alpha k$	$\alpha j + \beta k$	$-i$	e

**) If $\alpha\,\mu - \beta\,\lambda = -1$, then $\mu = -\mu\,(\alpha\,\mu - \beta\,\lambda) = -\alpha\,\mu^2 + \beta\,\lambda\,\mu = -\alpha\,\mu^2 - \alpha\,\lambda^2 = -\alpha$
And $\lambda = -\lambda\,(\alpha\,\mu - \beta\,\lambda) = -\alpha\,\mu\,\lambda + \beta\,\lambda^2 = \beta\,\mu^2 + \beta\,\lambda^2 = \beta$
Therefore the multiplication table of A is given by:

B_2	e	i	j	k
e	e	i	$\alpha j + \beta k$	$\beta j - \alpha k$
i	i	$-e$	$\beta j - \alpha k$	$\alpha j + \beta k$
j	$-\alpha j - \beta k$	$\beta j - \alpha k$	e	i
k	$-\beta j + \alpha k$	$\alpha j + \beta k$	$-i$	e

ii) B isomorphic to $\overset{*}{\mathbb{C}}$, we have $ei = ie = -i, i^2 = -e$ and $jk = i$. If we define a new multiplication on A by $x * y = \bar{x}\,\bar{y}$, we obtain an algebra $\overset{*}{A}$ which contains a sub-algebra isomorphic to $\mathbb{C}$. Therefore $\overset{*}{A}$ has an orthonormal basis which the multiplication tables are given previously. Consequently, the multiplication tables of the elements of the base F of A are given by :

B_3	e	i	j	k
e	e	$-i$	$-\alpha j - \beta k$	$\beta j - \alpha k$
i	$-i$	$-e$	$-\beta j + \alpha k$	$\alpha j + \beta k$
j	$\alpha j + \beta k$	$-\beta j + \alpha k$	e	i
k	$-\beta j + \alpha k$	$\alpha j + \beta k$	$-i$	e

and

B_4	e	i	j	k
e	e	$-i$	$-\alpha j - \beta k$	$-\beta j + \alpha k$
i	$-i$	$-e$	$\beta j - \alpha k$	$\alpha j + \beta k$
j	$\alpha j + \beta k$	$\beta j - \alpha k$	e	i
k	$\beta j - \alpha k$	$\alpha j + \beta k$	$-i$	e

Remark 3.2 Note that, central idempotent is a central element. The reciprocal case does not hold in general, and the counter example is given (B_1 and B_2).

Theorem 3.3 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a which satisfies one of the following identities:
1) $(a^2, a^2, a^2) = 0$, with $(a/a^2) \neq 0$,
2) $(a^2, a, a^2) = 0$.
Then A contains a commutative sub-algebra of dimension two. Moreover, A is isomorphic to A_1, A_2, A_3, A_4, B_1, B_2, B_3 or B_4.
Proof. *) If a and a^2 are linearly dependent, then a is a central idempotent. By lemma 2.6, A contains a commutative sub-algebra isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$ and is isomorphic to A_1, A_2, A_3 or A_4.
**) If a and a^2 are linearly independent, using proposition 2.13, $A(a, a^2)$ is two-dimensional commutative sub-algebra of A, thus $A(a, a^2)$ is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$. Hence the result is consequence of the theorem 3.1.

Theorem 3.4 Let A be a four-dimensional absolute valued algebra containing a nonzero central element a which satisfies one of the following identities:
1) $(a^2, a, a) = 0$ or $(a, a, a^2) = 0$,
2) $(a^2, a^2, a) = 0$ or $(a, a^2, a^2) = 0$.
Then A contains a commutative sub-algebra of dimension two. Moreover, A is isomorphic to A_1, A_2, A_3, A_4, B_1 or B_2.
Proof. *) If a and a^2 are linearly dependent, then a is a central idempotent. By lemma 2.6, A contains a commutative sub-algebra isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$. Then A is isomorphic to A_1, A_2, A_3 or A_4.
**) If a and a^2 are linearly independent, by proposition 2.10 and lemma 2.14, $A(a, a^2)$ is a two-dimensional commutative sub-algebra of A, thus $A(a, a^2)$ is isomorphic to $\mathbb{C}$. Hence the theorem 3.1 completes the proof

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