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Self-Similar Solution of Population Balance Equation for Aggregation with Constant Kernel

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Abstract
Self-similar solution of population balance equation includes the number density function which remains invariant or contains a part that is invariant. This paper describes self-similar solution of aggregation population balance model with constant kernel. Using this constant kernel, moment of the population balance system achieves the form $\mu_i(t) \propto t^{i-1}$ and the aggregation population balance equation with constant kernel reduces to a first order linear ordinary integro-differential equation whose solution is exponential function with negative power.

Keywords: Integro-differential equation, Number density function, Kernel, Moment, Self-similarity.

1. Introduction

Population balance equation mainly introduces aggregation and breakage process. In aggregation process, two or more particles are combined to form a large particle whereas in breakage process a particle breaks into two or more particles. Aggregation was introduced through Smoluchowski coagulation equation by Marian Smoluchowski in 1916 [8]. Direct solution of population balance equation for aggregation consists of finding the population density function for given kernel. Mathematical model of aggregation is

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\int_0^x a(x - x', x')f(x - x', t)f(x', t)dx' - f(x,t)\int_0^\infty a(x,x')\, f(x',t)\, dx' \qquad (1)$$

where $f(x,t)$ is population density function and $a(x,x')$ is aggregation kernel or aggregation frequency for particles of masses x and x′. Self-similarity is very important in case of solving inverse breakage model [6]. In this paper, self-similarity is used to solve aggregation model. Similarity transformation for population balance equation has been used by Lifshitz and Slyozov [7] for agglomerating crystals, Friedlander [4,5] and Friedlander and Wang [2] for coagulating aerosols, Narsimhan et al. [3] and Ramkrishna [1] for break-up of bubbles and drops, Kapur [9] for comminution of powders.

2. Preliminaries

Definition 2.1. Self-similar solution is a solution which is obtained by some transformation in solution such that number of co-ordinates is decreases by at least one. One of the forms of self-similar solution of equation (1) that is the population density function $f(x,t)$ is

$$f(x,t) = h_1(t)\,\varphi(\xi), \qquad \xi = h_2(t)\,x \qquad (2)$$

where ξ is the similarity variable and $h_1(t)$, $h_2(t)$ and $\varphi(\xi)$ are nonnegative, smooth and bounded functions.

Definition 2.2 Moment for the population density function $f(x,t)$ is

$$\mu_i(t) = \int_0^\infty x^i \, f(x,t)\, dx \tag{3}$$

where $\mu_0(t)$ represents the total number of particles at time t and $\mu_1(t)$ represents the total mass of the system at time t.

2.1. Moment in term of similarity variables

Using equations (2) and (3), the moment $\mu_i(t)$ has form,

$$\mu_i(t) = \frac{h_1(t)}{h_2(t)^{i+1}} \int_0^\infty \xi^i \, \varphi(\xi)\, d\xi \tag{4}$$

Moment for similarity variable is defined as [10],

$$K_i = \int_0^\infty \xi^i \, \varphi(\xi)\, d\xi \tag{5}$$

The definitions of $\mu_0(t)$ and $\mu_1(t)$ [10] requires that

$$K_0 = \int_0^\infty \varphi(\xi)\, d\xi = 1 \tag{6}$$

$$K_1 = \int_0^\infty \xi \, \varphi(\xi)\, d\xi = 1 \tag{7}$$

From equations (4), (6) and (7), we get $h_1(t)$ and $h_2(t)$ in terms of moment as

$$h_1(t) = \frac{\mu_0(t)^2}{\mu_1} \tag{8}$$

$$h_2(t) = \frac{\mu_0(t)}{\mu_1} \tag{9}$$

By use of $h_1(t)$, $h_2(t)$ from equations (8) and (9) and using similarity transformation from equation (2), the moment reduces in the form,

$$\mu_i(t) = \frac{{\mu_1}^i}{\mu_0(t)^{i-1}} \, K_i \tag{10}$$

3. Population balance model as moment equation

For constant kernel, $a(x,x') = c$ (constant), equation (1) reduces to

$$\frac{\partial f(x,t)}{\partial t} = \frac{c}{2} \int_0^x f(x-x',t)\, f(x',t)\, dx' - c\, f(x,t) \int_0^\infty f(x',t)\, dx' \tag{11}$$

Multiplying equation (11) by x^i and integrating over semi-infinite interval, it reduces to

$$\frac{d\mu_i(t)}{dt} = \frac{c}{2}\int_0^\infty \int_0^\infty (x' + x)^i\, f(x,t)\, f(x',t)\, dx\, dx' - c\int_0^\infty x^i\, f(x,t)\int_0^\infty f(x',t)\, dx' dx \qquad (12)$$

Using the similarity transformation (2) and the values of $h_1(t)$ and $h_2(t)$ from equations (8) and (9), equation (12) reduces to

$$\frac{d\mu_i(t)}{dt} = c\, \frac{\mu_1{}^i}{\mu_0(t)^{i-2}} \left[\frac{1}{2}\sum_{r=0}^{i} \binom{i}{r} K_r\, K_{i-r} - K_i\right] \qquad (13)$$

3.1. Zeroth order moment equation

For, $i = 0$, equation (13) becomes

$$\frac{d\mu_0(t)}{dt} = -\frac{1}{2} c\, \mu_0(t)^2 \qquad (14)$$

3.2. Solution of moment equation

By solving equation (14), we get

$$\frac{1}{\mu_0(t)} = \frac{1}{2} ct + \frac{1}{\mu_0(0)} \qquad (15)$$

Using equation (15), i^{th} moment (10) becomes

$$\mu_i(t) = K_I\, \mu_1{}^i \left(\frac{1}{2} ct + \frac{1}{\mu_0(0)}\right)^{i-1} \qquad (16)$$

4. Self-similar solution

Using the similarity transformations

$$f(x,t) = h_1(t)\, \varphi(\xi), \qquad \xi = h_2(t)\, x$$

$$f(x',t) = h_1(t)\, \varphi(\xi'), \qquad \xi' = h_2(t)\, x'$$

$$f(x - x',t) = h_1(t)\, \varphi(\xi''), \qquad \xi'' = h_2(t)\, (x - x') = \xi - \xi'$$

equation (11) reduces to

$$\xi \frac{d\varphi(\xi)}{d\xi} = -\int_0^\xi \varphi(\xi - \xi')\, \varphi(\xi')\, d\xi' \qquad (17)$$

This is a linear ordinary first order integro-differential equation. On solving equation (17) by using Laplace transformation, we get

$$\varphi(\xi) = -\frac{1}{b} e^{\xi/b} \tag{18}$$

By using equation (7), we get, $b = -1$. With this value of b, from equation (18), we get

$$\varphi(\xi) = e^{-\xi}$$

which is the self-similar solution that is of exponential nature.

5. Example of self-similarity

Take $c = 2,\ \ \mu_1 = 1, \mu_0(0) = 100$, from equation (15), we get

$$\mu_0(t) = \frac{1}{t+0.01}$$

From equation (9), we get

$$h_2(t) = \frac{1}{t+0.01}$$

From equation (8), we get

$$h_1(t) = \left(\frac{1}{t+0.01}\right)^2$$

From equation (2), we get

$$f(x,t) = \left(\frac{1}{t+0.01}\right)^2 e^{-\left(\frac{x}{t+0.01}\right)}$$

which is the solution in this case and

$$\xi = \frac{x}{t+0.01}$$

is the similarity variable.

6. Conclusion

In this paper, population balance equation for aggregation has been solved for constant kernel by introducing moment equation. Similarity transformation has been used to find self-similar solution.

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