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Convexities And Concavities of Means Using Vander Monde's Determinant

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Abstract

Greek terms for concavity and convexity are explored in this study, with the outcomes inferred using Vander monde's determinant.

Keywords: Convexity, Concavity, Vander monde's determinant, Classical means.

1. Introduction

The primary influence of the Greeks is found in Pappus of Alxendria's works, which were written in the fourth century A.D. and described the well-known techniques. On the core of proportion, ten Greek meanings are outlined in Pythagorean School, six of which are entitled and four of which are unnamed. Arithmetic mean, Geometric mean, Harmonic mean, and Contra harmonic mean are some of the more well-known named means. The unnamed Greek phrase signifies are $F_7(R,S)$, $F_8(R,S)$, $F_9(R,S)$ and $F_{10}(R,S)$ are given in [1]; Here, we analyse the resources required to write this paper.

$$A(R,S) = \frac{R-m}{m-S} = \frac{R}{R} = \frac{R+S}{2} \qquad (1.1)$$

$$F_7(R,S) = \frac{R-m}{m-S} = \frac{S}{R} = \frac{R-RS+S^2}{R} \qquad (1.2)$$

$$F_8(R,S) = \frac{R-m}{m-S} = \frac{m}{R} = \frac{R^2}{2R-S} \qquad (1.3)$$

$$F_9(R,S) = \frac{R-S}{m-S} = \frac{R}{S} = \frac{S(2R-S)}{R} \qquad (1.4)$$

Results regarding one function's convexity with respect to another function were also thoroughly discussed in [1]. Results for convexity for various key means and their relevance to mean inequalities were discovered in [2-8].

2. Preliminaries (concepts and methods)

DEFINITIONS AND LEMMAS

Recall a few of the lemmas and definitions that were develop to create this essay.

Vander Monde's determinant: Zhang et al. in their research work used the following VanderMonde's determinants extensively and established some remarkable results and limitations.

In linear algebra, Vandermonde's matrix, titled after Alexandre-Thophile Vander monde, is a matrix where each row is a geometric progression.

Also provides an definite form of the upper triangular matrix of the LU decomposition method. Using Gaussian elimination, determinant of Vander monde matrices may also be estimated. It evaluates a polynomial at a set of points.Vander monde determinant is used in the representation theory of the symmetric group and applicable for different types of means like AM, HM, GM, CHM etc.

Definition 2.1. A mean defined as M: $R_+^2 \rightarrow R_+$, which has the property $R \wedge S \leq M\,(R,S) \leq R \vee S$, $\forall\, R, S\ >\ 0$, *where* $R \wedge S = min\,(R,S)$ *and* $R \vee S = max\,(R,S)$.

Definition 2.2. Vander Monde's determinant: Suppose that $\emptyset$ is a continuous function on an interval, $I \subseteq R$, $R = (R_0, R_1, R_2 \ldots \ldots \ldots R_n)$ and $R_i \in I, R_i \neq R_j$ for $i \neq j$ see [8]. Setting for $i \neq j$ see [8]. Setting (2.1)

$$V(R,\emptyset) = \begin{vmatrix} 1 & R_0 & R_0^2 & \ldots & R_0^{n-1} & \emptyset(R_0) \\ 1 & R_1 & R_1^2 & \ldots & R_1^{n-1} & \emptyset(R_1) \\ \ldots & & \ldots & \ldots & & \\ 1 & R_n & R_n^2 & \ldots & R_n^{n-1} & \emptyset(R_n) \end{vmatrix} \quad (2.1)$$

Let $\emptyset(R) = R^{n+h} ln^k R$ in 2.1, we have

$$V(R,\emptyset) = \begin{vmatrix} 1 & R_0 & R_0^2 & \ldots & R_0^{n-1} & R_0^{n+h} ln^k R_0 \\ 1 & R_1 & R_1^2 & \ldots & R_1^{n-1} & R_1^{n+h} ln^k R_1 \\ \ldots & & \ldots & & & \\ 1 & R_n & R_n^2 & \ldots & R_n^{n-1} & R_n^{n+h} ln^k R_n \end{vmatrix} \quad (2.2)$$

Note that $h = 0$ and $k = 0$ is only the determinant of the matrix of Vander Monde. of $(n+1)^{th}$ term. the subsequent lemma (2.1) and (2.2) are distinct types of determinant (2.1) or (2.2)

Lemma 2.1. If $\emptyset(\eta) = \eta^2$ and R= (R_0, R_1, R_2) is the determinant of Vander Monde's matrix of the 3rd order is formed in:

$$V(R; h = 0, k = 0) = \begin{vmatrix} 1 & R_0 & R_0^2 \\ 1 & R_1 & R_1^2 \\ 1 & R_2 & R_2^2 \end{vmatrix} \qquad (2.3)$$

This is analogous

$$V(\mathrm{R}; h = 0, k = 0) = (R_1 - R_0)(R_2 - R_0)(R_2 - R_1) \qquad (2.4)$$

Lemma 2.2. If $\emptyset(\eta) = \eta^{1/2} = \sqrt{\eta}$ and $R = (R_0, R_1, R_2)$ is only the determinant of the third-order Vander Monde matrix and has the following form:

$$V(R; h = 0, k = 0) = \begin{vmatrix} 1 & R_0 & \sqrt{R_0} \\ 1 & R_1 & \sqrt{R_1} \\ 1 & R_2 & \sqrt{R_2} \end{vmatrix} \qquad (2.5)$$

This is analogous

$$V(R; h = -3/2, k = 0) = (\sqrt{R_1} - \sqrt{R_0})(\sqrt{R_2} - \sqrt{R_0})(\sqrt{R_2} - \sqrt{R_1}) \qquad (2.6)$$

Lemma 2.3. [1] If $F(\eta)$ and $G(\eta)$ are two functions, then $G(\eta)$ is said to be convex with regard to $F(\eta)$ for $R \leq S \leq T$ iff

$$\begin{vmatrix} 1 & F(R) & G(R) \\ 1 & F(S) & G(S) \\ 1 & F(T) & G(T) \end{vmatrix} \geq 0 \quad \simeq \quad \begin{vmatrix} 1 & F(R) & G(R) \\ 0 & F(S) - F(R) & G(S) - G(R) \\ 0 & F(T) - F(R) & G(T) - G(R) \end{vmatrix} \geq 0 \qquad (2.7)$$

This is analogous

$$[F(S) - F(R)][G(T) - G(R)] - [F(T) - F(R)][G(S) - G(R)] \geq 0. \qquad (2.8)$$

Placing R= η and S= 1 in equations (1.1) to (1.4), The standard termed means AM, un-termed means $F_7(R, S)$, $F_8(R, S)$ and $F_9(R, S)$ takes the subsequent form:

$$A(\eta, 1) = \frac{\eta+1}{2} \qquad (2.9)$$

$$F_7(\eta, 1) = \frac{\eta^2-\eta+1}{\eta} \qquad (2.10)$$

$$F_8(\eta, 1) = \frac{\eta^2}{2\eta-1} \qquad (2.11)$$

$$F_9(\eta, 1) = \frac{(2\eta-1)}{\eta} \qquad (2.12)$$

3. RESULTS

In this section, the prerequisites and requirements for the convexities and concavities of Arithmetic Mean, $F_7(R, S)$, $F_8(R, S)$ and $F_9(R, S)$ are discussed.

Theorem 3.1 The AM is concave (convex) with regard to $F_7(R, S)$ iff $V(R; h = 0, k = 0) \leq (\geq) 0$.

Proof: The AM and $F_7(R, S)$ in the form;

$A(\eta, 1) = \frac{\eta+1}{2}$ and $F_7(\eta, 1) = \frac{\eta^2-\eta+1}{\eta}$

Let $F(\eta) = \frac{\eta+1}{2}$ and $G(\eta) = \frac{\eta^2-\eta+1}{\eta}$,

by lemma (2.3) we have

$$\begin{vmatrix} 1 & F(R) & G(R) \\ 0 & F(S)-F(R) & G(S)-G(R) \\ 0 & F(T)-F(R) & G(T)-G(R) \end{vmatrix} = \begin{vmatrix} 1 & \frac{R+1}{2} & \frac{R^2-R+1}{R} \\ 0 & \frac{S-R}{2} & \frac{S^2-S+1}{S} - \frac{R^2-R+1}{R} \\ 0 & \frac{T-R}{2} & \frac{T^2-T+1}{T} - \frac{R^2-R+1}{R} \end{vmatrix}$$

On Simplifying the determinant leads to

$$\frac{(S-R)(T-R)(T-S)}{2RST} = \frac{V(R;h=0,k=0)}{2RST} \geq 0. \qquad (3.1)$$

Similarly by taking $F(y) = \frac{\eta^2-\eta+1}{\eta}$ and $G(y) = \frac{\eta+1}{2}$, by lemma (2.3) we have

$$\frac{(S-R)(T-R)(S-T)}{2RST} = \frac{V(R;h=0,k=0)}{2RST} \leq 0. \qquad (3.2)$$

The theorem (3.1) is proved by combining equations (3.1) and (3.2).

Theorem 3.2. The AM is concave (convex) with regard to$F_8(R,S)$ iff $V(R;h=0,k=0) \leq (\geq) 0$.

Proof: **:** The AM and $F_8(\mathrm{R},S)$ in the form;

$$A(\eta,1) = \frac{\eta+1}{2} \text{ and } F_8(y,1) = \frac{\eta^2}{2\eta-1}$$

Let $F(\eta) = \frac{\eta+1}{2}$ and $G(\eta) = \frac{\eta^2}{2\eta-1}$, by lemma (2.3) we have

$$\begin{vmatrix} 1 & F(R) & G(R) \\ 0 & F(S)-F(R) & G(S)-G(R) \\ 0 & F(T)-F(R) & G(T)-G(R) \end{vmatrix} = \begin{vmatrix} 1 & \frac{R+1}{2} & \frac{R^2}{2R-1} \\ 0 & \frac{S-R}{2} & \frac{S^2}{2S-1} - \frac{R^2}{2R-1} \\ 0 & \frac{T-R}{2} & \frac{T^2}{2T-1} - \frac{R^2}{2R-1} \end{vmatrix}$$

On Simplifying the determinant leads to

$$\frac{(S-R)(T-R)(T-S)}{2(2R-1)(2S-1)(2T-1)} = \frac{V(R;h=0,k=0)}{2(2R-1)(2S-1)(2T-1)} \geq 0 \qquad (3.3)$$

By taking into account $F(y) = \frac{\eta^2}{2\eta-1}$ & $G(y) = \frac{\eta+1}{2}$, by lemma (2.3) we have

$$\frac{(S-R)(T-R)(T-S)}{2(2R-1)(2S-1)(2T-1)} = \frac{V(R;h=0,k=0)}{2(2R-1)(2S-1)(2T-1)} \leq 0 \qquad (3.4)$$

The theorem (3.2) is proved by combining equations (3.3) and (3.4).

Theorem.3.3. The AM is concave (convex) with regard to$F_9(\mathrm{R},\mathrm{S})$ iff $V(R;h=0,k=0) \leq (\geq) 0$.

Proof:
Consider the AM and $F_9(R,S)$ in the form;

$A(\eta,1)=\frac{\eta+1}{2}$ and $F_9(\eta,1)=\frac{2\eta-1}{\eta}$

Let $\mathrm{F}(\eta)=\frac{\eta+1}{2}$ & $G(\eta)=\frac{2\eta-1}{\eta}$, by lemma (2.3) we have

z

$$\begin{vmatrix} 1 & F(R) & G(R) \\ 0 & F(S)-F(R) & G(S)-G(R) \\ 0 & F(T)-F(R) & G(T)-G(R) \end{vmatrix} = \begin{vmatrix} 1 & \frac{R+1}{2} & \frac{2R-1}{R} \\ 0 & \frac{S-R}{2} & \frac{2S-1}{S}-\frac{2R-1}{R} \\ 0 & \frac{T-R}{2} & \frac{2T-1}{T}-\frac{2R-1}{R} \end{vmatrix}$$

As a result of the determinant's simplification,

$$\frac{(S-R)(T-R)(S-T)}{2RST}=\frac{V(R;h=0,k=0)}{2RSR}\leq 0 \qquad (3.5)$$

By taking into account $F(\eta)=\frac{2\eta-1}{\eta}$ & $G(\eta)=\frac{\eta+1}{2}$, by lemma (2.3) we get

$$\frac{(S-R)(T-R)(S-T)}{2RST}=\frac{V(R;h=0,k=0)}{2RSR}\geq 0 \qquad (3.6)$$

The theorem (3.3) is proved by combining equations (3.5) and (3.6).

Theorem.3.4. The un-named mean $F_7(\mathrm{R},\mathrm{S})$ is concave (convex) with respect to$F_9(R,S)$ if and only if $V(R;h=0,k=0)\leq(\geq)0$.
Proof: Consider the $F_7(R,S)$ and $F_9(R,S)$ in the following way;

$F_7(\eta,1)=\frac{\eta^2-\eta+1}{\eta}$ and $F_9(\eta,1)=\frac{2\eta-1}{\eta}$

Let $F(\eta)=\frac{\eta^2-\eta+1}{\eta}$ and $G(\eta)=\frac{2\eta-1}{\eta}$, by lemma (2.3) we have

$$\begin{vmatrix} 1 & F(R) & G(R) \\ 0 & F(S)-F(R) & G(S)-G(R) \\ 0 & F(T)-F(R) & G(T)-G(R) \end{vmatrix} = \begin{vmatrix} 1 & \frac{R^2-R+1}{R} & \frac{2R-1}{R} \\ 0 & \frac{S^2-S+1}{S}-\frac{R^2-R+1}{R} & \frac{2S-1}{S}-\frac{2R-1}{R} \\ 0 & \frac{T^2-T+1}{T}-\frac{R^2-R+1}{R} & \frac{2T-1}{T}-\frac{2R-1}{R} \end{vmatrix}$$

As a result of the determinant's simplification,

$$\frac{(S-R)(T-R)(S-T)}{RST}=\frac{V(R;h=0,k=0)}{RST}\leq 0 \qquad (3.7)$$

By taking into account $F(y) = \frac{2\eta-1}{\eta}$ and $G(y) = \frac{\eta^2-\eta+1}{\eta}$, by lemma (2.3) we have

$$\frac{(S-R)(T-R)(S-T)}{RST} = \frac{V(R;h=0,k=0)}{RST} \geq 0 \qquad (3.8)$$

The theorem (3.4) is proved by combining equations (3.7) and (3.8).

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References:

1. Bullen P. S., Handbook of Means and Their Inequalities, Kluwer Academic Publishers, Dordrecht, 2003.
2. Lokesha., K. M. Nagaraja., Relation between series and important means, Advances in Theoretical and Applied Mathematics, 2007, 2(1), 31-36.
3. Nagaraja K. M., Murali K., Lokesha V., Schur convexity and concavity of Gnan mean, Proceedings of the Jangjeon Mathematical Society, 2014, 17(3), 355-367.
4. Nagaraja K. M., Reddy P. S. K., Naveen Kumar B., Refinement of Inequality Involving Ratio of Means for Four Positive Arguments, Bulletin of the International Mathematical Virtual Institute, 2003, 3, 135-

138.
5. Toader G., Toader S., Greek means and Arithmetic mean and Geometric mean, RGMIA Monograph, (2005) Australia.
6. Vimala T., Nagaraja K. M., Convexity and Concavity of Means, International Journal of Research-Grathaalayah, 2017, 5, 92-97.
7. Nagaraja K. M., Vimala T., A Harmonic and Heron Mean Inequalities for Arguments in Different Intervals, International Journal of Mathematical Archive, 2019, 47-52.
8. Nagaraja K. M., Vimala T., Mean inequalities for the arguments in index and conjugate Index set, Turkish Journal of Computer and Mathematics Education, 2021, 12(1), 216-222.
9. Nagaraja K. M., Vimala T., Inequalities for the Arguments lying on Linear and Curved Path, Honam Mathematical Journal, 2020, 42, 747-755.
10. Ambika M. H., Venkataramana B. S., Nagaraja K. M., Sampath Kumar R., The properties of one parameter power exponential mean and its invariants, Advances and Applications in Mathematical Sciences, 2021, 20, 1197-1208.
11. Sampath Kumar R., Harish A., Nagaraja K. M., Venkataramana B. S., The Ky Fan type inequalities for power exponential mean and its invariant, Gis science journal, 2021, 8, 106-115.