On Hyperbolic Hsu-Structure Manifold, Recurrent and Symmetry

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Abstract- In this paper, we have defined recurrence and symmetry of different kinds in H- Hsu-structure manifold. Some theorem establishing relationship between different kinds of recurrent H- HSU-Structure manifold involving equivalent conditions with respect to projective, conformal, conharmonic and concircular curvature tensors has been discussed recurrence parameter have also been studied. Index Terms- Recurrence parameter,Curvature Tensors,$C^\infty$ -function,Hsu-structure manifold.

1. Introduction

If a differentiable manifold $V_n$, of differentiability class $C^\infty$, there be in $V_n$, a vector valued linear function $F$ of class $C^\infty$, satisfying the algebraic equation

$$\bar{X} = a'X,$$  \hspace{1cm} \text{for arbitrary vector field } X. \hspace{1cm} (1.1)$$

Where $\bar{X}=FX$, $0 \leq r \leq n$ and ‘a’ is real or imaginary number, then $\{F\}$ is said to give to $V_n$ a Hyperbolic HSU-structure defined by the equations$(1.1)$ and the manifold $V_n$ is called a Hyperbolic HSU-structure manifold. Hyperbolic HSU-structure manifold or briefly H-Hsu-structure manifold.

Remark(1.1): The equation $(1.1)$ gives different structures for different values of ‘a’ and $r$.

If $a = \pm 1 \text{ and } r = 2$, it is an almost complex structure.
If $a = \pm i \text{ and } r = 2$, it is an almost product structure or a hyperbolic almost complex structure.
If $a = 0$, it is an almost tangent structure or almost hyperbolic tangent structure.
If $a \neq 0$, It is the hyperbolic $\pi$-structure.

Let the Hsu – structure $V_n$, be endowed with a Hermitian metric tensor $g$, such that

$$g(\bar{X},\bar{Y}) - a' g(X,Y) = 0,$$

Then $\{F,g\}$ is said to give $V_n$ a hyperbolic Hsu-structure metric manifold.

Agreement (1.1): In what follows and the above, the equations containing $X,Y,Z$ ..............,etc. hold for these arbitrary vector in $V_n$.

The curvature tensor $K$, a vector –valued tri-linear function w.r.t the connexion $D$ is given by

$$K(X,Y,Z) = D_X D_Y Z - D_Y D_X Z - D_{\{X,Y\}} Z,$$  \hspace{1cm} (1.2)\hspace{0.5cm} a
Where

\[ [X, Y] = D_X Y - D_Y X \]  \hspace{1cm} (1.2)b

The Ricci tensor in \( V_n \) is given by

\[ \text{Ric}(Y, Z) = (C_1^1 K)(Y, Z). \]  \hspace{1cm} (1.3)

Where by \((C_1^1 K)(Y, Z)\), we mean the contraction of \( K(X, Y, Z) \) with respect the first slot. For Ricci tensor, we also have

\[ \text{Ric}(Y, Z) = \text{Ric}(Z, Y), \]  \hspace{1cm} (1.4)a

\[ \text{Ric}(Y, Z) = g(r(Y), Z) = g(Y, r(Z)), \]  \hspace{1cm} (1.4)b

\[ (C_1^1 r) = R \]  \hspace{1cm} (1.4)c

Let \( W, C, L \) and \( V \) be the Projective, Conformal, conharmonic and concircular curvature tensors respectively given by

\[ W(X, Y, Z) = K(X, Y, Z) - \frac{1}{n-1} [\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y], \]  \hspace{1cm} (1.5)

\[ C(X, Y, Z) = \frac{1}{(n-2)} \{ \text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y - g(X, Z)r(Y) + g(Y, Z)r(X) \} + \frac{R}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y], \]  \hspace{1cm} (1.6)

\[ L(X, Y, Z) = K(X, Y, Z) - \frac{1}{n-2} [\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y - g(X, Z)r(Y) + g(Y, Z)r(X)]. \]  \hspace{1cm} (1.7)

\[ V(X, Y, Z) = K(X, Y, Z) - \frac{R}{n(n-1)} [g(Y, Z)X - g(X, Z)Y]. \]  \hspace{1cm} (1.8)

A manifold is said to be recurrent, if

\[ (\mathcal{F}K)(X, Y, Z, T) = A(T_1)K(X, Y, Z). \]

The recurrent manifold is said to be symmetric, if

\[ A(T_1) = 0, \]  \hspace{1cm} in the equation (1.9).

\[ \text{II RECURRENT AND SYMMETRY OF DIFFERENT KINDS} \]

Let \( Q \), a vector – valued trilinear function be any one of the curvature tensors \( K, W, C, L \) or \( V \). Then we will define recurrence of different kinds as follows:

\[ \text{Definition(2.1).} \] A \( \text{HSU}- \text{structure manifold is said to be (1)} \)-recurrent in \( Q \), if

\[ a^r(\mathcal{F}Q)(X, Y, Z, T) - Q((\mathcal{F}F)(\bar{X}, T), Y, Z) = a^r P_1(T)Q(X, Y, Z) \]  \hspace{1cm} (2.1)
Where $P_1(T)$ is non–vanishing $C^\infty$ function.

Definition (2.2). A H-HSU-structure manifold is said to be (2)-recurrent in $Q$, if
\[
a^r(\nabla Q)(X,Y,Z,T) - Q(X,(\nabla F)(\vec{Y},T),Z) = a^r P_1(T) Q(X,Y,Z),
\]
(2.2)

Definition (2.3). A H-HSU-structure manifold is said to be (3)-recurrent in $Q$, if
\[
a^r(\nabla Q)(X,Y,Z,T) - Q(X,Y(\nabla F)(\vec{Z},T),T) = a^r P_1(T) Q(X,Y,Z),
\]
(2.3)

Definition (2.4). A H-HSU-structure manifold is said to be Ricci-(1)-recurrent, if
\[
a^r(\nabla \text{Ric})(Y,Z,T) - Ric((\nabla F)(\vec{Y},T),Z) = a^r P_1(T) \text{Ric}(Y,Z),
\]
(2.4)

Definition (2.5). A H-HSU-structure manifold is said to be Ricci-(2)-recurrent, if
\[
a^r(\nabla \text{Ric})(Y,Z,T) - Ric(Y,\nabla F(\vec{Z},T)) = a^r P_1(T) \text{Ric}(Y,Z),
\]
(2.5)

Definition (2.6). A HSU-structure manifold is said to be (1,2)-recurrent in $Q$, if
\[
a^r(\nabla Q)(X,\vec{Y},Z,T) - Q(\nabla F)(\vec{X},T),Y,Z) + a^r Q(X,(\nabla F)(Y,T),Z)
= a^r P_1(T) Q(X,\vec{Y},Z),
\]
(2.6a)

or equivalently
\[
a^r(\nabla Q)(X,\vec{Y},Z,T) - Q(\nabla F)(\vec{Y},T),Z) + a^r Q((\nabla F)(X,T),Y,Z)
= a^r P_1(T) Q(X,\vec{Y},Z),
\]
(2.6b)

or equivalently
\[
a^{2r}(\nabla Q)(X,Y,Z,T) - a^r Q((\nabla F)(\vec{X},T),Y,Z) = a^r Q(X,(\nabla F)(\vec{Y},T),Z)
= a^{2r} P_1(T) Q(X,Y,Z),
\]
(2.6c)

Definition (2.7). A HSU-structure manifold is said to be (1,2)-recurrent in $Q$, if
\[
a^r(\nabla Q)(X,\vec{Z},T) - Q((\nabla F)(\vec{X},T),Y,\vec{Z}) + a^r Q(X,Y,(\nabla F)(Z,T))
= a^r P_1(T) Q(X,Y,\vec{Z}),
\]
(2.7a)

or equivalently
\[
a^r(\nabla Q)(\vec{X},Y,Z,T) = Q(\vec{X},Y(\nabla F)(\vec{Z},T)) + a^r Q((\nabla F)(X,T),Y,Z)
= a^r P_1(T) Q(\vec{X},Y,Z),
\]
(2.7b)

or equivalently
\[
a^{2r}(\nabla Q)(X,Y,Z,T) - a^r Q((\nabla F)(\vec{X},T),Y,Z) = a^r Q(X,Y(\nabla F)(\vec{Z},T))
= a^{2r} P_1(T) Q(X,Y,Z),
\]
(2.7c)

Definition (2.8). A HSU-structure manifold is said to be (2,3)-recurrent in $Q$, if
\[
a^r(\nabla Q)(X,Y,\vec{Z},T) - Q(X,(\nabla F)(\vec{Y},T),\vec{Z}) + a^r Q(X,Y,(\nabla F)(Z,T))
= a^r P_1(T) Q(X,Y,\vec{Z}),
\]
(2.8a)
or equivalently
\[ a^r(\nabla Q)(X, \bar{Y}, Z, T) - Q(X, \bar{Y}, (\nabla F)(\bar{Z}, T) + a^r Q(X, (\nabla F)(Y, T), Z) = a^r P_1(T)Q(X, \bar{Y}, Z), \]
(2.8b)
or equivalently
\[ a^{2r}(\nabla Q)(X, Y, Z, T) - a^r Q(X, (\nabla F)(\bar{Y}, T), Z) - a^r Q(X, (\nabla F)(Y, T), Z) = a^{2r} P_1(T)Q(X, Y, Z). \]
(2.8c)

Definition (2.9). A H-HSU-structure manifold is said to be Ricci- (1,2)-recurrent, if
\[ a^r(\nabla \text{Ric})(Y, \bar{Z}, T) - \text{Ric}\left((\nabla F)(T), Z\right) + a^r \text{Ric}(Y, (\nabla F), Z, T)) = a^r P_1(T)\text{Ric}(Y, \bar{Z}), \]
(2.9a)
or equivalently
\[ a^r(\nabla \text{Ric})(Y, \bar{Z}, Z, T) - \text{Ric}\left((\nabla F)(\bar{Z}, T)\right) + a^r \text{Ric}(\nabla F)(Y, Z, T) = a^r P_1(T)\text{Ric}(\bar{Y}, Z), \]
(2.9b)

Or equivalently
\[ a^{2r}(\nabla \text{Ric})(Y, Z, T) - a^r \text{Ric}\left((\nabla F)(\bar{Y}, T)\right) - a^r \text{Ric}(\nabla F)(Y, Z, T) = a^{2r} P_1(T)\text{Ric}(Y, Z). \]
(2.9c)

Definition (2.10). A –HSU-structure manifold is said to be (1,2,3)-recurrent if
\[ a^r(\nabla Q)(X, \bar{Y}, \bar{Z}, T) - Q((\nabla F)(X, T), \bar{Y}, Z) + a^r Q(X, (\nabla F)(Z, T)) + a^r Q(X, (\nabla F)(Y, T), Z) = a^r P_1(T)Q(X, \bar{Y}, \bar{Z}), \]
(2.10a)
or equivalently
\[ a^r Q(X, (\nabla F)(Y, T), Z) = a^r P_1(T)Q(X, \bar{Y}, \bar{Z}), \]
(2.10b)
or equivalently
\[ a^r(\nabla Q)(X, \bar{Y}, \bar{Z}, T) + a^r Q((\nabla F)(X, T), \bar{Y}, Z) + a^r Q(X, (\nabla F)(Y, T), Z) - Q(X, (\nabla F)(\bar{Y}, T), Z) = a^r P_1(T)Q(X, \bar{Y}, \bar{Z}). \]
(2.10c)

Definition (2.11). A (1,2,3,1,2,1,3,2) and (1,2,3)-recurrent H-HSU-structure manifold is said to be Q-symmetric or Ricc-symmetric, if
\[ P_1(T) = 0 \]

(2.11)

In the above equations,
Theorem (2.1) A Q-(1)-recurrent H-HSU-manifold is Q-(2)-recurrent for same recurrent parameter, if
\[ Q((\nabla F)(\bar{X}, T), Y, Z) = Q(X, (\nabla F)(\bar{Y}, T), Z). \]
(2.12)

Proof: if a Q-(1)-recurrent H-HSU-manifold is Q-(2)-recurrent, then we have
\[ a^r(\mathcal{V}Q)(X,Y,Z,T) - Q((\mathcal{V}F)(\overline{X},T),Y,Z) - a^r P_1(T)Q(X,Y,Z) = a^r(\mathcal{V}Q)(X,Y,Z,T) - Q((\mathcal{V}F)(\overline{Y},T),Z) - a^r P_1(T)Q(X,Y,Z) \]  
(2.13)

From equation (2.13), we have the equation (2.12).
Conversely, let the equation (2.12) is satisfied and the manifold is \( Q(1) \)-recurrent, then using equation in (2.1), we get

\[ a^r(\mathcal{V}Q)(X,Y,Z,T) - Q(X,(\mathcal{V}F)(\overline{Y},T),Z) = a^r P_1(T)Q(X,Y,Z) \]

Which shows that the manifold is \( Q(2) \)-recurrent.

Note (2.1) - Similarly, it can be shown that the \( Q(3) \)-recurrent H-HSU-structure manifold is \( Q(1) \)-recurrent, if
\[ Q((\mathcal{V}F)(\overline{X},T),Y,Z) = Q(X,Y(\mathcal{V}F)(\overline{Z},T)), \]
Or \( Q(2) \)-recurrent, if
\[ Q(X,(\mathcal{V}F)(\overline{Y},T),Z) = Q(X,Y(\mathcal{V}F)(\overline{Z},T)), \]
And Ricci-\( (1) \)-recurrent H-HSU-manifold is Ricci-\( (2) \)-recurrence, if

\[ Ric((\mathcal{V}F)(\overline{Y},T),Z) = Ric(Y,(\mathcal{V}F)(\overline{Z},T)) \] for the same recurrence parameter.

**Theorem (2.2)** A \( Q(1,2) \)-recurrent H-HSU-structure manifold is \( Q(1,3) \)-recurrent for the same recurrence parameter, iff
\[ Q(\overline{X},(\mathcal{V}F)(\overline{Y},T),Z) = Q(\overline{X},Y(\mathcal{V}F)(\overline{Z},T)). \]  
(2.14)

Proof. Assuming that the \( Q(1,2) \)-recurrent H-HSU-structure manifold is \( Q(1,3) \)-recurrent then using equation (2.14) in equation (2.6)b, we get
\[ a^r(\mathcal{V}Q)(\overline{X},Y,Z,T) - Q(\overline{X},Y(\mathcal{V}F)(\overline{Z},T)) + a^r Q((\mathcal{V}F)(X,T),Y,Z) \]
\[ = a^r P_1(T)Q(\overline{X},Y,Z), \]
Which shows that the manifold is \( Q(1,3) \)-recurrent.

Note (2.2). Similarly, it can be shown that the \( Q(2,3) \)-recurrent H-HSU-structure manifold is \( Q(1,2) \)-recurrent, iff
\[ Q((\mathcal{V}F)(\overline{X},T),\overline{Y},Z) = Q(X,\overline{Y},(\mathcal{V}F)(\overline{Z},T)) \]
Or \( Q(1,3) \)-recurrent, iff
\[ Q(X,(\mathcal{V}F)(\overline{Y},T),Z) = Q((\mathcal{V}F)(\overline{X},T),Y,\overline{Z}) \]
For the same recurrence parameter.

**Theorem (2.3)** A \( Q(1,2) \)-recurrent H-HSU-structure manifold is \( Q(1) \)-recurrent for the same recurrence parameter provided
\[ a^r Q(X,(\mathcal{V}F)(Y,T),Z) = 0 \]  
(2.15)

Proof. Let the manifold is \( (1) \)-recurrent in \( Q \) then barring \( Y \) in equation (2.1), we get.
\[ \alpha^r (\nabla Q)(X, Y, Z, T) - Q((\nabla F)(\bar{X}, T), \bar{Y}, Z) = \alpha^r P_1(T)Q(X, Y, Z) \]  
(2.16)

Now, assuming that a Q-(1,2)-recurrent H-HSU-structure-manifold is Q-(1)-recurrent then comparing equation (2.6)a and (2.16), we get the equation (2.15).

Note(1.3). Similarly, it can be shown that a Q-(1,2)-recurrent H-HSU-structure manifold is Q(2)-recurrent for the same recurrence parameter, provided.

\[ \alpha^r Q((\nabla F)(X, T), Y, Z) = 0. \]  
(2.17)

Remark(1.1). Theorems of the type (2.3) can also be proved taking Q-(1,3) or Q(2,3)-recurrent H-HSU-structure manifold instead of Q-(1,2)-recurrent and and Q-(1) or Q-(3) or Q-(2)-recurrent manifold in place of Q-(1)-recurrent manifold.

**Theorem (2.4)** A Q(1,2,3)-recurrent H-HSU-structure manifold is Q-(1)-recurrent for the same recurrent parameter provided.

\[ \alpha^r Q(X, (\nabla F)(Y, T), \bar{Z}) + \alpha^r Q(X, Y, (\nabla F)(Z, T)) = 0. \]  
(2.18)

Proof. Let the manifold is Q-(1)-recurrent, then barring Y and Z in equation (2.1), we get

\[ \alpha^r (\nabla Q)(X, Y, Z, T) - Q((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z}) = \alpha^r P_1(T)Q(X, Y, Z) \]  
(2.19)

Now assuming that a Q-(1,2,3)-recurrent manifold is Q-(1)-recurrent, then comparing equations (2.10)a and (2.19), we get the equation (2.18).

Note(2.4). Theorems of the type (1.4) can also be proved taking Q-(2) or Q(3)-recurrent H-HSU-structure manifold instead of Q-(1)-recurrent H-HSU-structure manifold.

**Theorem (1.5)** A Q-(1,2,3)-recurrent H-HSU-structure manifold is Q-(1,2)-recurrent H-HSU-manifold for the same recurrence parameter, provided.

\[ \alpha^r Q(X, Y, (\nabla F)(Z, T)) = 0 \]  
(2.20)

Proof. Let the manifold is Q-(1,2)-recurrent, then barring Z in equation (2.6)a, we get

\[ \alpha^r (\nabla Q)(X, Y, \bar{Z}, T) - Q((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z}) + \alpha^r Q(X, (\nabla F)(Y, T), \bar{Z}) \]

\[ = \alpha^r P_1(T)Q(X, Y, \bar{Z}), \]  
(2.21)

Now, assuming that a Q-(1,2,3)-recurrent H-HSU-structure manifold is Q-(1,2)-recurrent and then comparing equations (2.10)a and (2.21), we get the equation (2.20).

Note(1.5). Theorems of the type (2.5) can also be proved taking Q-(1,3) or Q-(2,3)-recurrent H-HSU-structure manifold instead of Q-(1,2)-recurrent H-HSU-structure manifold.
Theorem (2.6). In a recurrent H-HSU-structure manifold, if any two of following conditions hold for the same recurrence parameter, then the third also hold:

(a) It is conformal (1)-recurrent,
(b) It is conharmonic (1)-recurrent,
(c) It is concircular (1)-recurrent.

Proof. From the equation (1.6), (1.7), (1.8) we have

\[ C(X, Y, Z) = L(X, Y, Z) + \frac{n}{(n-2)} [K(X, Y, Z) - V(X, Y, Z)] \] (2.22)

Barring X in equations (2.22), we get

\[ C(\bar{X}, Y, Z) = L(\bar{X}, Y, Z) + \frac{n}{(n-2)} [K(\bar{X}, Y, Z) - V(\bar{X}, Y, Z)]. \] (2.23)

Now, from equation from from (1.1) and (2.23), we have

\[ a^r P_1(T)C(X, Y, Z) = a^r P_1(T)L(X, Y, Z) + \frac{na^r}{(n-2)} P_1(T)\{K(X, Y, Z) - V(X, Y, Z)\} \] (2.24)

Differentiating equation (2.23) with respect to T, using equation (2.23) and then barring X in the resulting equation, we get

\[ -a^r(\nabla C)(X, Y, Z, T) + C(\nabla F)(\bar{X}, T), Y, Z) = -a^r((\nabla L)(X, Y, Z, T)
+ L(\nabla F)(\bar{X}, T), Y, Z) + \frac{n}{(n-2)} (-a^r(\nabla K)(X, Y, Z, T)
+ K(\nabla F)(\bar{X}, T), Y, Z) + a^r(\nabla V)(X, Y, Z, T)
- V(\nabla F)(X, T), Y, Z). \] (2.25)

Adding equation (2.24) and (2.25), we get

\[ -a^r(\nabla C)(X, Y, Z, T) + C(\nabla F)(\bar{X}, T), Y, Z) + a^r P_1(T)C(X, Y, Z)
= -a^r((\nabla L)(X, Y, Z, T) + L(\nabla F)(\bar{X}, T), Y, Z) + a^r P_1(T)L(X, Y, Z)
+ \frac{n}{(n-2)} (-a^2(\nabla K)(X, Y, Z, T) + K(\nabla F)(\bar{X}, T), Y, Z) + a^r P_1(T)K(X, Y, Z)
+ a^r(\nabla V)(X, Y, Z, T) - V(\nabla F)(X, T), Y, Z) - a^r P_1(T)K(X, Y, Z) \] (2.26)

If a (1)-recurrent H-HSU-Structure manifold is conformal-(1) recurrent and conharmonic-(1)-recurrent for the same recurrence parameter then from equation (2.16), we get

\[ a^r(\nabla V)(X, Y, Z, T) - V(\nabla F)(\bar{X}, T), Y, Z) = a^r P_1(T)V(X, Y, Z), \]

Which shows that the manifolds is concircular –(1)-recurrent.
Similarly, it can be shown that if the recurrent manifold is either conformal-(1)-recurrent and concircular-(1)-recurrent or conharmonic-(1)-recurrent and concircular-(1)-recurrent then it is either conharmonic-(1)-recurrent or conformal-(1)-recurrent for same recurrence parameter.

Theorem (2.7) In a (1, 2) recurrent H-HSU-structure manifold, if any two of following conditions hold for the same recurrence parameter, then the third also hold:

(a) It is conformal (1,2)-recurrent ,
(b) It is conharmonic (1,2)-recurrent,
(c) It is concircular (1,2)-recurrent.

Barring X and Y in equations (2.22), we get

\[ C(\bar{X}, \bar{Y}, Z) = L(\bar{X}, \bar{Y}, Z) + \frac{n}{(n-2)}[K(\bar{X}, \bar{Y}, Z) - V(\bar{X}, \bar{Y}, Z)]. \]  \hspace{1cm} (2.27)

Now, from equation (1.1) and (2.27), we have

\[ a^rP_1(T)C(X, \bar{Y}, Z) = a^rP_1(T)L(X, \bar{Y}, Z) + \frac{n a^r}{(n-2)}P_1(T)[K(X, \bar{Y}, Z) - V(X, \bar{Y}, Z)] \]  \hspace{1cm} (2.28)

Differentiating equation (2.27) with respect to T, using equation (2.27) and then barring X in the resulting equation, we get

\[ -a^r(\nabla C)(X, \bar{Y}, Z, T) + C((\nabla F)(\bar{X}, T), \bar{Y}, Z) - a^rC(X, (\nabla F)(Y, T), Z) \]
\[ = -a^r((\nabla L)(X, \bar{Y}, Z, T) + L((\nabla F)(\bar{X}, T), \bar{Y}, Z) - a^rL(X, (\nabla F)(Y, T), Z) \]
\[ + \frac{n}{(n-1)}[-a^r(\nabla K)(X, \bar{Y}, Z, T) + K((\nabla F)(\bar{X}, T), \bar{Y}, Z) - a^rK(X, (\nabla F)(Y, T), Z) \]
\[ + a^r(\nabla V)(X, \bar{Y}, Z, T) - V((\nabla F)(\bar{X}, T), \bar{Y}, Z)] + a^rV(X, (\nabla F)(Y, T), Z)]. \]  \hspace{1cm} (2.29)

Adding equations (2.28) and (2.29), we get

\[ -a^r(\nabla C)(X, \bar{Y}, Z, T) + C((\nabla F)(\bar{X}, T), \bar{Y}, Z) - a^rC(X, (\nabla F)(Y, T), Z) + a^rP_1(T)C(X, \bar{Y}, Z) \]
\[ = -a^r((\nabla L)(X, \bar{Y}, Z, T) + L((\nabla F)(\bar{X}, T), \bar{Y}, Z) - a^rL(X, (\nabla F)(Y, T), Z) \]
\[ + a^rP_1(T)L(X, \bar{Y}, Z) \]
\[ + \frac{n}{(n-2)}[-a^r(\nabla K)(X, \bar{Y}, Z, T) + K((\nabla F)(\bar{X}, T), \bar{Y}, Z) \]
\[ - a^rK(X, (\nabla F)(Y, T), Z) + a^rP_1(T)K(X, \bar{Y}, Z) \]
\[ + a^r(\nabla V)(X, \bar{Y}, Z, T) - V((\nabla F)(\bar{X}, T), \bar{Y}, Z)] + a^rV(X, (\nabla F)(Y, T), Z) \]  \hspace{1cm} (2.30)

If a (1,2)-recurrent H-HSU-Structure manifold is conformal-(1,2) recurrent and conharmonic-(1,2)-recurrent for the same recurrence parameter then from equation (2.30), we get

\[ a^r(\nabla V)(X, \bar{Y}, Z, T) - V((\nabla F)(\bar{X}, T), \bar{Y}, Z) + a^rV(X, (\nabla F)(Y, T), Z) = a^rP_1(T)V(X, \bar{Y}, Z). \]

Which shows that the manifolds is concircular –(1,2)-recurrent.
Similarly, it can be shown that if the recurrent manifold is either conformal-(1,2)-recurrent and concircular-(1,2)-recurrent or conharmonic-(1,2)-recurrent and concircular-(1,2)-recurrent then it is either conharmonic-(1,2)-recurrent or conformal-(1,2)-recurrent for same recurrence parameter.

Theorem (2.8) ) In a (1,2,3) recurrent H-HSU-structure manifold, if any two of the following conditions hold for the same recurrence parameter, then the third also hold:

(a) It is conformal (1,2,3)-recurrent,
(b) It is conharmonic (1,2,3)-recurrent,
(c) It is concircular (1,2,3)-recurrent.

Barring X,Y and Z in equations (2.22), we get
\[ C(\bar{X}, \bar{Y}, \bar{Z}) = L(\bar{X}, \bar{Y}, \bar{Z}) + \frac{n}{n-2}[K(\bar{X}, \bar{Y}, \bar{Z}) - V(\bar{X}, \bar{Y}, \bar{Z})]. \] (2.31)

Now, from equation (1.1) and (2.31), we have
\[ a^r P_1(T) C(X, \bar{Y}, Z) = a^r P_1(T) L(X, \bar{Y}, Z) + \frac{n a^r}{(n-2)} P_1(T) \{ K(X, \bar{Y}, Z) - V(X, \bar{Y}, Z) \} \] (2.32)

Differentiating equation (2.31) with respect to T, using equation (2.31) and then barring X in the resulting equation, we get
\[ -a^r (\nabla C)(X, \bar{Y}, Z, T) + C(\nabla F(X, Y, T), \bar{Y}, \bar{Z}) - a^r C(X, (\nabla F(Y, T), \bar{Z}) - a^r C(X, \bar{Y}, (\nabla F)(Z, T) \]
\[ = a^r((\nabla L)(X, \bar{Y}, Z, T)) + L((\nabla F(X, Y, T), \bar{Y}, \bar{Z}) - a^r L(X, (\nabla F)(Y, T), \bar{Z}) \]
\[ -a^r K(X, (\nabla F)(Y, T), \bar{Z}) - (X, \bar{Y}(\nabla F)(Z, T) + a^r(\nabla V)(X, \bar{Y}, \bar{Z}, T) - V((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z}) \]
\[ + a^r V(X, (\nabla F)(Y, T), \bar{Z}) + a^r V(X, \bar{Y}(\nabla F)(Z, T))]. \] (2.33)

Adding equations (2.32) and (2.33), we get
\[ -a^r (\nabla C)(X, \bar{Y}, Z, T) + C(\nabla F(X, Y, T), \bar{Y}, \bar{Z}) - a^r C(X, (\nabla F)(Y, T), \bar{Z}) \]
\[ -a^r C(X, \bar{Y}(\nabla F)(Z, T)) + a^r P_1(T) C(X, \bar{Y}, Z) \]
\[ = -a^r((\nabla L)(X, \bar{Y}, Z, T)) + L((\nabla F(X, Y, T), \bar{Y}, \bar{Z}) - a^r L(X, (\nabla F)(Y, T), \bar{Z}) \]
\[ -a^r L(X, \bar{Y}(\nabla F)(Z, T)) + a^r P_1(T) L(X, \bar{Y}, Z) \]
\[ + \frac{n}{n-1} \{-a^r(\nabla K)(X, \bar{Y}, Z, T) \]
\[ + K((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z}) - a^r K(X, (\nabla F)(Y, T), \bar{Z}) - a^r K(X, \bar{Y}(\nabla F)(Z, T)) \]
\[ + a^r P_1(T) K(X, \bar{Y}, \bar{Z}) + a^r(\nabla V)(X, \bar{Y}, \bar{Z}, T) - V((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z}) \].
\[ a^r V(X, (\nabla F)(Y, T), \bar{Z}) - a^r P_1(T)V(X, \bar{Y}, \bar{Z}) \]  

(2.34)

If a \((1,2,3)\)-recurrent H-HSU-Structure manifold is conformal-\((1,2,3)\) recurrent and conharmonic-\((1,2,3)\)-recurrent for the same recurrence parameter then from equation (2.34), we get

\[
a^r (\nabla V)(X, \bar{Y}, \bar{Z}, T) - V((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z}) + a^r V(X, (\nabla F)(Y, T), \bar{Z}) + a^r V(X, \bar{Y}(\nabla F)(Z, T))
= a^r P_1(T)V(X, \bar{Y}, \bar{Z}).
\]

Which shows that the manifolds is concircular \(-(1,2,3)\)-recurrent.

Similarly, it can be shown that if the recurrent manifold is either conformal-\((1,2,3)\)-recurrent and concircular-\((1,2,3)\)-recurrent or conharmonic-\((1,2,3)\)-recurrent and concircular-\((1,2,3)\)-recurrent then it is either conharmonic-\((1,2,3)\)-recurrent or conformal-\((1,2,3)\)-recurrent for same recurrence parameter.

References


On Hyperbolic Hsu-Structure Manifold,BIRrecurrant and Symmetry

1.Introduction

If a differentiable manifold \(V^n\), of differentiability class \(C^\infty\).there be in \(V^n\),a vector valued linear function \(F\) of class \(C^\infty\), satisfying the algebraic equation

\[
\bar{x} = a'x,
\]

for arbitrary vector field \(x\).

(1.1)

Where \(\bar{x} = FX\), \(0 \leq r \leq n\) and \('a'\)is real or imaginary number,then \(\{F\}\) is said to give to \(V^n\) a Hyperbolic Hsu-structure defined by the equations(1.1) and the manifold \(V^n\) is called a Hyperbolic Hsu –structure manifold. Hyperbolic Hsu-structure manifold or briefly H-Hsu-structure manifold.
Remark(1.1): The equation (1.1) gives different structures for different values of ‘a’ and r.

If \( a = \pm 1 \) and \( r = 2 \), it is an almost complex structure.
If \( a = \pm i \) and \( r = 2 \), it is an almost product structure or a hyperbolic almost complex structure.
If \( a = 0 \), it is an almost tangent structure or almost hyperbolic tangent structure.
If \( a \neq 0 \), it is the hyperbolic \( \pi \)-structure.

Let the Hsu – structure \( V_n \), be endowed with a Hermitian metric tensor \( g \), such that
\[
g(\bar{X}, \bar{Y}) - a^r g(X, Y) = 0,
\]
Then \( \{F, g\} \) is said to give \( V_n \) a hyperbolic Hsu-structure metric manifold.

Agreement (1.1): In what follows and the above, the equations containing \( X, Y, Z \), etc. hold for these arbitrary vector in \( V_n \).

The curvature tensor \( K \), a vector –valued tri-linear function w.r.t the connexion \( D \) is given by
\[
K(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z,
\]  
(1.2)a

Where
\[
[X, Y] = D_X Y - D_Y X
\]  
(1.2)b

The Ricci tensor in \( V_n \) is given by
\[
Ric(Y, Z) = (C^1_1 K)(Y, Z).
\]  
(1.3)

Where by \( (C^1_1 K)(Y, Z) \), we mean the contraction of \( K(X, Y, Z) \) with respect the first slot.

For Ricci tensor, we also have
\[
Ric(Y, Z) = Ric(Z, Y),
\]  
(1.4)a

\[
Ric(Y, Z) = g(r(Y), Z) = g(Y, r(Z)),
\]  
(1.4)b

\[
(C^1_1 r) = R
\]  
(1.4)c

Let \( W, C, L \) and \( V \) be the Projective, Conformal, conharmonic and concircular curvature tensors respectively given by
\[
W(X, Y, Z) = K(X, Y, Z) - \frac{1}{(n-1)} [Ric(Y, Z)X - Ric(X, Z)Y]
\]  
(1.5)
\[ C(X,Y,Z) = -\frac{1}{(n-2)}\{Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)r(Y) + g(Y,Z)r(X)\} + \]
\[ \frac{R}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \]  
(1.6)

\[ L(X,Y,Z) = K(X,Y,Z) - \frac{1}{(n-2)}\{Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)r(Y) + g(Y,Z)r(X)\}. \] 
(1.7)

\[ V(X,Y,Z) = K(X,Y,Z) - \frac{R}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \]  
(1.8)

A manifold is said to be recurrent, if

\[ (\forall K)(X,Y,Z,T) = A(T_1)K(X,Y,Z). \]

The recurrent manifold is said to be symmetric, if

\[ A(T_1) = 0, \] in the equation (1.9).

II  BIRECURRENCE AND SYMMETRY OF DIFFERENT KINDS

Let \( Q \), a vector – valued trilinear function be any one of the curvature tensors \( K,W,C,L \) or \( V \). Then we will define recurrence of different kinds as follows:

**Definition(2.1).** A –HSU-structure manifold is said to be (1)-birecurrent in \( Q \), if

\[ a^* (\nabla Q)(X,Y,Z,T,S) - (\nabla Q)((\nabla F)(\bar{X},T),Y,Z,S) - (\nabla Q)((\nabla F)(\bar{X},S),Y,Z,T) - Q((\nabla F)(\bar{X},T,S),Y,Z) = a^* P_2(T,S)Q(X,Y,Z) \]

(2.1)

Where \( P_2(T,S) \) is non – vanishing \( C^\infty \)-called birecurrent parameter.

**Definition(2.2).** A H-HSU- structure manifold is said to be Ricci (1)-birecurrent ,if

\[ a^* (\nabla Ric)(Y,Z,T,S) - (\nabla Ric)(((\nabla F)(\bar{Y},T),Z,S) - (\nabla Ric) \]

\[ ((\nabla F)(\bar{Y},S),Z,T) - Q((\nabla Ric)(\bar{Y},T,S),Z) = a^* P_2(T,S)Ric(Y,Z). \]

(2.2)

**Definition(2.3).** A H-HSU- structure manifold is said to be- (1,2)-birecurrent in\( Q \), if

\[ a^* (\nabla Q)(X,\bar{Y},Z,T,S) - (\nabla Q)(((\nabla F)(\bar{X},T),\bar{Y},Z,S) - (\nabla Q)(((\nabla F)(\bar{X},S),\bar{Y},Z,T) + \]

\[ a^* (\nabla Q)(X,\nabla F)(Y,T),Z,S) + a^* (\nabla Q)(X,\nabla F)(Y,S),Z,T) - Q((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z) - \]

\[ Q((\nabla F)(\bar{X},S),(\nabla F)Y,T),Z) - Q((\nabla F)(\bar{X},T,S),\bar{Y},Z) + a^* Q(X,(\nabla F)(Y,T),S,Z) = \]

\[ a^* P_2(T,S)Q(X,\bar{Y},Z). \]

(2.3)

**Definition(2.4).** A H-HSU- structure manifold is said to be Ricci- (1)- recurrent ,if

\[ a^* (\nabla Ric)(Y,Z,T) - Ric((\nabla F)(\bar{Y},T),Z) = a^* P_1(T)Ric(Y,Z). \]

(2.4)

**Definition(2.5).** A H-HSU- structure manifold is said to be Ricci- (2)- recurrent ,if

\[ a^* (\nabla Ric)(Y,Z,T) - Ric(Y,(\nabla F)(\bar{Z},T) = a^* P_1(T)Ric(Y,Z). \]

(2.5)

**Definition(2.6).** A –HSU-structure manifold is said to be (1,2)-recurrent in \( Q \), if

\[ a^* (\nabla Q)(X,\bar{Y},Z,T) - Q((\nabla F)(\bar{X},T),\bar{Y},Z) + a^* Q(X,(\nabla F)(Y,T),Z) \]
\begin{align}
\mathcal{R}(\tilde{Y}) &= a^r P_1(T) Q(X, \tilde{Y}, Z), \quad \text{(2.6a)} \\
\text{or equivalently} \\
a^r (\nabla Q)(X, Y, Z, T) - Q(X(\nabla F)(\tilde{Y}, T), Z) + a^r Q((\nabla F)X, T), Y, Z) \\
&= a^r P_1(T) Q(X, Y, Z), \quad \text{(2.6b)} \\
\text{or equivalently} \\
a^2 r (\nabla Q)(X, Y, Z, T) - a^r Q((\nabla F)(X, T), Y, Z) - a^r Q(X, (\nabla F)(\tilde{Y}, Z), T) \\
&= a^2 r P_1(T) Q(X, Y, Z), \quad \text{(2.6c)} \\
\end{align}

**Definition (2.7).** A HSU-structure manifold is said to be (1,2)-recurrent in Q, if
\begin{align}
a^r (\nabla Q)(X, Y, \tilde{Z}, T) - Q((\nabla F)(\tilde{X}, T), Y, \tilde{Z}) + a^r Q(X, Y, (\nabla F)(Z, T)) \\
&= a^r P_1(T) Q(X, Y, \tilde{Z}), \quad \text{(2.7a)} \\
\text{or equivalently} \\
a^r (\nabla Q)(\tilde{X}, Y, \tilde{Z}, T) - Q(\tilde{X}, (\nabla F)(\tilde{Y}, T), \tilde{Z}) + a^r Q(X, (\nabla F)(Z, T)) \\
&= a^r P_1(T) Q(X, Y, \tilde{Z}), \quad \text{(2.7b)} \\
\text{or equivalently} \\
a^2 r (\nabla Q)(X, Y, Z, T) - a^r Q((\nabla F)(\tilde{X}, T), Y, Z) - a^r Q(X, Y, (\nabla F)(\tilde{Z}, T)) \\
&= a^2 r P_1(T) Q(X, Y, \tilde{Z}), \quad \text{(2.7c)} \\
\end{align}

**Definition (2.8).** A HSU-structure manifold is said to be (2,3)-recurrent in Q, if
\begin{align}
a^r (\nabla Q)(X, Y, \tilde{Z}, T) - Q(X, (\nabla F)(\tilde{Y}, T), Z) + a^r Q(X, Y, (\nabla F)(Z, T)) \\
&= a^r P_1(T) Q(X, Y, \tilde{Z}), \quad \text{(2.8a)} \\
\text{or equivalently} \\
a^r (\nabla Q)(X, Y, \tilde{Z}, T) - Q(X, (\nabla F)(\tilde{Y}, T), \tilde{Z}) + a^r Q(X, (\nabla F)(Z, T), \tilde{Z}) \\
&= a^r P_1(T) Q(X, Y, \tilde{Z}), \quad \text{(2.8b)} \\
\text{or equivalently} \\
a^2 r (\nabla Q)(X, Y, Z, T) - a^r Q((\nabla F)(\tilde{X}, T), Y, Z) - a^r Q(X, Y, (\nabla F)(\tilde{Z}, T), \tilde{Z}) \\
&= a^2 r P_1(T) Q(X, Y, \tilde{Z}), \quad \text{(2.8c)} \\
\end{align}

**Definition (2.9).** A H-HSU-structure manifold is said to be Ricci- (1,2)-recurrent, if
\begin{align}
a^r (\nabla R)(Y, \tilde{Z}, T) - Ric((\nabla F)(\tilde{Y}, T), Z) + a^r Ric(Y, (\nabla F), Z, T)) \\
&= a^r P_1(T) Ric(Y, \tilde{Z}), \quad \text{(2.9a)} \\
\text{or equivalently} \\
a^r (\nabla R)(\tilde{Y}, Z, T) - Ric((\nabla F)(\tilde{Z}, T)) + a^r Ric((\nabla F)(Y, T), Z) \\
&= a^r P_1(T) Ric(\tilde{Y}, Z), \quad \text{(2.9b)} \\
\end{align}

Or equivalently
\[ a^{2r}(\nabla \text{Ric})(Y, Z, T) - a^r(\text{Ric}((\nabla F)(\bar{Y}, T), Z) - a^r \text{Ric}(\bar{Y}, (\nabla F)(Z, T)) \]
\[ = a^{2r} P_1(T) \text{Ric}(Y, Z). \quad (2.9)c \]

Definition (2.10). A -HSU-structure manifold is said to be (1,2,3)-recurrent if

\[ a^r((\nabla Q)(X, \bar{Y}, \bar{Z}, T) - Q((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z}) + a^r Q(X, \bar{Y}, (\nabla F)(Z, T)) \]
\[ + a^r Q(X, (\nabla F)(Y, T), \bar{Z}) = a^r P_1(T) Q(X, \bar{Y}, \bar{Z}), \quad (2.10)a \]

or equivalently

\[ a^r((\nabla Q)(\bar{X}, Y, \bar{Z}, T) - Q(\bar{X}, (\nabla F)(\bar{Y}, T), \bar{Z}) + a^r Q((\nabla F)(X, T), Y, \bar{Z}) \]
\[ + a^r Q(\bar{X}, Y, (\nabla F)(Z, T)) = a^r P_1(T) Q(\bar{X}, Y, Z), \quad (2.10)b \]

or equivalently

\[ a^r((\nabla Q)(\bar{X}, \bar{Y}, \bar{Z}, T) + a^r Q((\nabla F)(X, T), \bar{Y}, Z) + a^r Q(\bar{X}, (\nabla F)(Y, T), Z) \]
\[ - Q(\bar{X}, \bar{Y}, (\nabla F)(\bar{Z}, T)) = a^r P_1(T) Q(\bar{X}, \bar{Y}, Z). \quad (2.10)c \]

Definition (2.11). A (1),(2),(3),(1,2),(1,3),(2,3) and (1,2,3)- recurrent H-HSU-structure manifold is said to be Q-symmetric or Ricc-symmetric, if

\[ P_1(T) = 0 \quad (2.11) \]

In the above equations,

Theorem (2.1) A Q-(1)-recurrent H-HSU-manifold is Q-(2)- recurrent for same recurrent parameter ,if

\[ Q((\nabla F)(\bar{X}, T), Y, Z) = Q(X, (\nabla F)(\bar{Y}, T), Z). \quad (2.12) \]

Proof: if a Q-(1)-recurrent H-HSU-manifold is Q-(2)-recurrent ,then we have

\[ a^r((\nabla Q)(X, Y, Z, T) - Q((\nabla F)(\bar{X}, T), Y, Z) - a^r P_1(T) Q(X, Y, Z) = a^r((\nabla Q)(X, Y, Z, T) - Q(X, (\nabla F)(\bar{Y}, T), Z) - a^r P_1(T) Q(X, Y, Z) \quad (2.13) \]

From equation (2.13),we have the equation (2.12).

Conversely, let the equation (2.12)is satisfied and the manifold is Q(1)-recurrent ,then using equation in (2.1),we get

\[ a^r((\nabla Q)(X, Y, Z, T) - Q(X, (\nabla F)(\bar{Y}, T), Z) = a^r P_1(T) Q(X, Y, Z) \]

Which shows that the manifold is Q-(2)-recurrent.

Note(2.1)-Similarly, it can be shown that the Q-(3)-recurrent H-HSU-structure manifold is Q-(1)- recurrent, if

\[ Q((\nabla F)(\bar{X}, T), Y, Z) = Q(X, Y(\nabla F)(\bar{Z}, T)), \]

Or Q-(2)-recurrent, if

\[ Q(X, (\nabla F)(\bar{Y}, T), Z) = Q(X, Y(\nabla F)(\bar{Z}, T)), \]

And Ricci-(1)-recurrent H-HSU-manifold is Ricci-(2)-recurrence, if
\[
Ric((\nabla F)(\bar{Y}, T), Z) = Ric(Y, (\nabla F)(\bar{Z}, T)) \text{ for the same recurrence parameter.}
\]

**Theorem (2.2)** A Q-(1,2)-recurrent H-HSU-structure manifold is Q-(1,3)-recurrent for the same recurrence parameter, iff
\[
Q(\bar{X}, (\nabla F)(\bar{Y}, T), Z) = Q(\bar{X}, Y (\nabla F)(\bar{Z}, T)).
\]  

(2.14)

Proof. Assuming that the Q-(1,2)-recurrent H-HSU-structure manifold is Q-(1,3)-recurrent then using equation (2.14) in equation (2.6)b, we get
\[
a^r Q(\bar{X}, Y, Z, T) = Q(\bar{X}, Y (\nabla F)(\bar{Z}, T)) + a^r Q((\nabla F)(X, T), Y, Z)
= a^r P_1(T)Q(\bar{X}, Y, Z),
\]

Which shows that the manifold is Q-(1,3)-recurrent.

Note (2.2). Similarly, it can be shown that the Q-(2,3)-recurrent H-HSU-structure manifold is Q-(1,2)-recurrent, iff
\[
Q((\nabla F)(\bar{X}, T), Y, Z) = Q(X, \bar{Y}, (\nabla F)(\bar{Z}, T))
\]

Or Q-(1,3)-recurrent, iff
\[
Q(X, (\nabla F)(\bar{Y}, T), Z) = Q((\nabla F)(\bar{X}, T), Y, \bar{Z})
\]

For the same recurrence parameter.

**Theorem (2.3)** A Q-(1,2)-recurrent H-HSU-structure manifold is Q-(1)-recurrent for the same recurrence parameter provided
\[
a^r Q(X, (\nabla F)(Y, T), Z) = 0 \tag{2.15}
\]

Proof. Let the manifold is (1)-recurrent in Q then barring Y in equation (2.1), we get
\[
a^r (\nabla Q)(X, Y, Z, T) = Q((\nabla F)(\bar{X}, T), \bar{Y}, Z) = a^r P_1(T)Q(X, \bar{Y}, Z) \tag{2.16}
\]

Now, assuming that a Q-(1,2)-recurrent H-HSU-structure manifold is Q-(1)-recurrent then comparing equation (2.6)\(a\) and (2.16), we get the equation (2.15).

Note (1.3). Similarly, it can be shown that a Q-(1,2)-recurrent H-HSU-structure manifold is Q(2)-recurrent for the same recurrence parameter, provided
\[
a^r Q((\nabla F)(X, T), Y, Z) = 0. \tag{2.17}
\]

Remark (1.1). Theorems of the type (2.3) can also be proved taking Q-(1,3) or Q(2,3)-recurrent H-HSU-structure manifold instead of Q-(1,2)-recurrent and and Q-(1) or Q-(3) or Q-(2)-recurrent manifold in place of Q-(1)-recurrent manifold.

**Theorem (2.4)** A Q(1,2,3)-recurrent H-HSU-structure manifold is Q-(1)-recurrent for the same recurrence parameter provided.
\[ a^r Q(X, (\nabla F)(Y, T), \bar{Z}) + a^r Q(X, \bar{Y}, (\nabla F)(Z, T)) = 0. \]  

(2.18)

Proof. Let the manifold is Q-(1)-recurrent, then barring Y and Z in equation (2.1), we get

\[ a^r (\nabla Q)(X, \bar{Y}, \bar{Z}, T) - Q((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z}) = a^r (\nabla P_1)(T)Q(X, \bar{Y}, \bar{Z}) \]  

(2.19)

Now, assuming that a Q-(1,2,3)-recurrent manifold is Q-(1)-recurrent, then comparing equations (2.10)a and (2.19), we get the equation (2.18).

Note(2.4). Theorems of the type (1.4) can also be proved taking Q-(2) or Q-(3)-recurrent H-HSU-structure manifold instead of Q-(1)-recurrent H-HSU-structure manifold.

Theorem(1.5) A Q-(1,2,3)-recurrent H-HSU-structure manifold is Q-(1,2)-recurrent H-HSU-manifold for the same recurrence parameter, provided.

\[ a^r Q(X, \bar{Y}(\nabla F)(Z, T) = 0 \]  

(2.20)

Proof. Let the manifold is Q-(1,2)-recurrent, then barring Z in equation (2.6)a, we get

\[ a^r (\nabla Q)(X, \bar{Y}, \bar{Z}, T) - Q((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z}) + a^r Q(X, (\nabla F)(Y, T), \bar{Z}) = a^r (\nabla P_1)(T)Q(X, \bar{Y}, \bar{Z}), \]  

(2.21)

Now, assuming that a Q-(1,2,3)-recurrent H-HSU-structure manifold is Q-(1,2)-recurrent and then comparing equations (2.10)a and (2.21), we get the equation (2.20).

Note(1.5). Theorems of the type(2.5) can also be proved taking Q-(1,3)-or Q-(2,3)-recurrent H-HSU-structure manifold instead of Q-(1,2)-recurrent H-HSU-structure manifold.

Theorem(2.6). In a recurrent H-HSU-structure manifold, if any two of following conditions hold for the same recurrence parameter, then the third also hold:

(d) It is conformal (1)-recurrent,
(e) It is conharmonic (1)-recurrent,
(f) It is concircular (1)-recurrent.

Proof. From the equation (1.6),(1.7),(1.8) we have

\[ C(X, Y, Z) = L(X, Y, Z) + \frac{n}{(n-2)}[K(X, Y, Z) - V(X, Y, Z)] \]  

(2.22)

Barring X in equations (2.22), we get

\[ C(\bar{X}, Y, Z) = L(\bar{X}, Y, Z) + \frac{n}{(n-2)}[K(\bar{X}, Y, Z) - V(\bar{X}, Y, Z)]. \]  

(2.23)

Now, from equation from from (1.1) and (2.23), we have
\[ a^r P_1(T)C(X, Y, Z) = a^r P_1(T)L(X, Y, Z) + \frac{n a^r}{(n-2)} P_1(T)\{K(X, Y, Z) - V(X, Y, Z)\} \]  

(2.24)

Differentiating equation (2.23) with respect to \( T \), using equation (2.23) and then barring \( X \) in the resulting equation, we get

\[-a^r (\nabla C)(X, Y, Z, T) + C((\nabla F)(\bar{X}, T), Y, Z) = -a^r ((\nabla L)(X, Y, Z, T)
+ L((\nabla F)(\bar{X}, T), Y, Z) + \frac{n}{(n-1)}(a^r (\nabla K)(X, Y, Z, T)
+ K((\nabla F)(\bar{X}, T), Y, Z) + a^r (\nabla V)(X, Y, Z, T)
- V((\nabla F)(X, T), Y, Z)) \].  

(2.25)

Adding equation (2.24) and (2.25), we get

\[-a^r (\nabla C)(X, Y, Z, T) + C((\nabla F)(\bar{X}, T), Y, Z) + a^r P_1(T)C(X, Y, Z)
= -a^r ((\nabla L)(X, Y, Z, T) + L((\nabla F)(\bar{X}, T), Y, Z) + a^r P_1(T)L(X, Y, Z)
+ \frac{n}{(n-2)}(-a^2 (\nabla K)(X, Y, Z, T) + K((\nabla F)(\bar{X}, T), Y, Z) + a^r P_1(T)K(X, Y, Z)
+ a^r (\nabla V)(X, Y, Z, T) - V((\nabla F)(X, T), Y, Z) - a^r P_1(T)K(X, Y, Z)) \]

(2.26)

If a (1)-recurrent H-HSU-Structure manifold is conformal-(1) recurrent and conharmonic-(1)-recurrent for the same recurrence parameter then from equation (2.16), we get

\[ a^r (\nabla V)(X, Y, Z, T) - V((\nabla F)(\bar{X}, T), Y, Z) = a^r P_1(T)V(X, Y, Z), \]

Which shows that the manifolds is concircular –(1)-recurrent.

Similarly, it can be shown that if the recurrent manifold is either conformal-(1)-recurrent and concircular-(1)-recurrent or conharmonic-(1)-recurrent and concircular-(1)-recurrent then it is either conharmonic-(1)-recurrent or conformal-(1)-recurrent for same recurrence parameter.

Theorem(2.7) In a (1,2) recurrent H-HSU-structure manifold, if any two of following conditions hold for the same recurrence parameter, then the third also hold:

(c) It is conformal (1,2)-recurrent,
(d) It is conharmonic (1,2)-recurrent,
(c) It is concircular (1,2)-recurrent.

Barring \( X \) and \( Y \) in equations (2.22), we get

\[ C(\bar{X}, \bar{Y}, Z) = L(\bar{X}, \bar{Y}, Z) + \frac{n}{(n-2)} [K(\bar{X}, \bar{Y}, Z) - V(\bar{X}, \bar{Y}, Z)]. \]

(2.27)

Now, from equation (1.1) and (2.27), we have
\[
a^r P_1(T)C(X, Y, Z) = a^r P_1(T)L(X, Y, Z) + \frac{n a^r}{(n - 2)} P_1(T)\{K(X, Y, Z) - V(X, Y, Z)\}
\]

(2.28)

Differentiating equation (2.27) with respect to T, using equation (2.27) and then barring X in the resulting equation, we get

\[
-a^r(\nabla C)(X, Y, Z, T) + C((\nabla F)(X, Y, Z) - a^r C(X, (\nabla F)(Y, T), Z)
\]

\[
= -a^r((\nabla L)(X, Y, Z, T) + L((\nabla F)(X, Y, Z) - a^r L(X, (\nabla F)(Y, T), Z)
\]

\[
+ \frac{n}{(n - 1)} \{-a^r(\nabla K)(X, Y, Z, T) + K((\nabla F)(X, Y, Z) - a^r K(X, (\nabla F)(Y, T), Z)
\]

\[
+ a^r(\nabla V)(X, Y, Z, T) - V((\nabla F)(X, Y, Z) + a^r V(X, (\nabla F)(Y, T), Z)\}.
\]

(2.29)

Adding equations (2.28) and (2.29), we get

\[
-a^r(\nabla C)(X, Y, Z, T) + C((\nabla F)(X, Y, Z) - a^r C(X, (\nabla F)(Y, T), Z) + a^r P_1(T)C(X, Y, Z)
\]

\[
= -a^r((\nabla L)(X, Y, Z, T) + L((\nabla F)(X, Y, Z) - a^r L(X, (\nabla F)(Y, T), Z)
\]

\[
+ \frac{n}{(n - 1)} \{-a^r(\nabla K)(X, Y, Z, T) + K((\nabla F)(X, Y, Z) - a^r K(X, (\nabla F)(Y, T), Z)
\]

\[
- V((\nabla F)(X, Y, Z)) + a^r V(X, (\nabla F)(Y, T), Z) - a^r P_1(T) V(X, Y, Z)\} \quad (2.30)
\]

If a (1,2)-recurrent H-HSU-Structure manifold is conformal-(1,2) recurrent and conharmonic-(1,2)-recurrent for the same recurrence parameter then from equation (2.30), we get

\[
a^r(\nabla V)(X, Y, Z, T) - V((\nabla F)(X, Y, Z) + a^r V(X, (\nabla F)(Y, T), Z) = a^r P_1(T) V(X, Y, Z).
\]

Which shows that the manifolds is concircular -(1,2)-recurrent.

Similarly, it can be shown that if the recurrent manifold is either conformal-(1,2)-recurrent and concircular-(1,2)-recurrent or conharmonic-(1,2)-recurrent and concircular-(1,2)-recurrent then it is either conharmonic-(1,2)-recurrent or conformal-(1,2)-recurrent for same recurrence parameter.

Theorem (2.8) ) In a (1,2,3) recurrent H-HSU-structure manifold, if any two of following conditions hold for the same recurrence parameter, then the third also hold:

- (b) It is conformal (1,2,3)-recurrent ,
- (b) It is conharmonic (1,2,3)-recurrent,
- (c) It is concircular (1,2,3)-recurrent.

Barring X,Y and Z, in equations (2.22), we get

\[
C(X, Y, Z) = L(X, Y, Z) + \frac{n}{(n - 2)} [K(X, Y, Z) - V(X, Y, Z)].
\]

(2.31)
Now, from equation (1.1) and (2.31), we have
\[a^r P_1(T)C(X, \bar{Y}, Z) = a^r P_1(T)L(X, \bar{Y}, Z) + \frac{n a^r}{(n-2)} P_1(T)\{K(X, \bar{Y}, Z) - V(X, \bar{Y}, Z)\} \]

(2.32)

Differentiating equation (2.31) with respect to \( T \), using equation (2.31) and then barring \( X \) in the resulting equation, we get
\[-a^r (\nabla C)(X, \bar{Y}, \bar{Z}, T) + C((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z}) - a^r C(X, (\nabla F)(Y, T), \bar{Z}) - a^r C(X, \bar{Y}, (\nabla F)(Z, T))
= a^r ((\nabla L)(X, \bar{Y}, \bar{Z}, T) + L((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z}) - a^r L(X, (\nabla F)(Y, T), \bar{Z})
-a^r K(X, (\nabla F)(Y, T), \bar{Z}) - (X, \bar{Y}, (\nabla F)(Z, T)) + a^r (\nabla V)(X, \bar{Y}, \bar{Z}, T) - V((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z})
+ a^r V(X, (\nabla F)(Y, T), \bar{Z}) + a^r V(X, \bar{Y}, (\nabla F)(Z, T))) \]

(2.33)

Adding equations (2.32) and (2.33), we get
\[-a^r (\nabla C)(X, \bar{Y}, \bar{Z}, T) + C((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z}) - a^r C(X, (\nabla F)(Y, T), \bar{Z})
-a^r C(X, \bar{Y}, (\nabla F)(Z, T)) + a^r P_1(T)C(X, \bar{Y}, \bar{Z})
= -a^r ((\nabla L)(X, \bar{Y}, \bar{Z}, T) + L((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z}) - a^r L(X, (\nabla F)(Y, T), \bar{Z})
-a^r L(X, \bar{Y}, (\nabla F)(Z, T)) + a^r P_1(T)L(X, \bar{Y}, \bar{Z})
+ \frac{n}{(n-1)} (-a^r (\nabla K)(X, \bar{Y}, \bar{Z}, T)
+ K((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z}) - a^r K(X, (\nabla F)(Y, T), \bar{Z}) - a^r K(X, \bar{Y}, (\nabla F)(Z, T))
+ a^r P_1(T)K(X, \bar{Y}, \bar{Z}) + a^r (\nabla V)(X, \bar{Y}, \bar{Z}, T) - V((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z})
+ a^r V(X, (\nabla F)(Y, T), \bar{Z}) - a^r P_1(T)\bar{V}(X, \bar{Y}, \bar{Z})) \]

(2.34)

If a (1,2,3)-recurrent H-HSU-Structure manifold is conformal-(1,2,3) recurrent and conharmonic-(1,2,3)-recurrent for the same recurrence parameter then from equation (2.34), we get
\[a^r (\nabla V)(X, \bar{Y}, \bar{Z}, T) - V((\nabla F)(\bar{X}, T), \bar{Y}, \bar{Z}) + a^r V(X, (\nabla F)(Y, T), \bar{Z}) + a^r V(X, \bar{Y}, (\nabla F)(Z, T))
= a^r P_1(T)\bar{V}(X, \bar{Y}, \bar{Z}). \]

Which shows that the manifolds is concircular -(1,2,3)-recurrent.

Similarly, it can be shown that if the recurrent manifold is either conformal-(1,2,3)-recurrent and concircular-(1,2,3)-recurrent or conharmonic-(1,2,3)-recurrent and concircular-(1,2,3)-recurrent then it is either conharmonic-(1,2,3)-recurrent or conformal-(1,2,3)-recurrent for same recurrence parameter.

References


