International Journal for Multidisciplinary Research (IJFMR)

E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

A Parametrization of Theta Functions ϕ and ψ

Sabahat Parveen

Assistant professor, Department of Mathematics,
Maharashtra College of Arts, Sc. And Comm., Mumbai-08

Abstract

This paper defines a parameter of theta functions ϕ and ψ depending on two positive real numbers a and b. Some properties of which are also proved.

Keywords: Theta function, q- shifted factorial, transformation formula.

1.Introduction:

Jinhee Yi [2] has discovered several interesting theta function parametrizations. Inspired by Yi's work, a parametrization $I_{a,b}$ of theta functions ϕ and ψ for any positive real numbers a and b is introduced. Several features of this parameter are also explored.

The general definition of Ramanujan's theta function is:

For $|ab| < 1,\ f(a,b) = \sum_{n=-\infty}^{n=\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}$.

Now consider the following theta functions, each of which has a key role to play in this paper.

For $Im(z) > 0$ and $q = e^{2\pi iz}$,

$\phi(q) = f(q,q) = \sum_{n=-\infty}^{n=\infty} q^{n^2}$,

$\psi(q) = f(q,q^3) = \sum_{n=0}^{n=\infty} q^{\frac{n(n+1)}{2}}$,

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{n=\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty,$$

$\chi(q) = (-q;q^2)_\infty$.

Where $(a;q)_\infty$ is a q- shifted factorial defined as,

$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n), \quad |q| < 1.$

2.Preliminary Result:

From Ramanujan's notebook [1, Entry 24(iii), p. 39], we have

$\phi(q) = \chi(q) f(q)$.

$\psi(-q) = \frac{f(-q^2)}{\chi(q)}$.

Yi [2, 2.1.1, p. 11], has defined a following parameter for two positive real numbers n and k as:

$r_{k,n} = \frac{f(-q)}{k^{\frac{1}{4}} q^{\frac{(k-1)}{24}} f(-q^k)}, \quad q = e^{-2\pi\sqrt{\frac{n}{k}}}$.

Yi [2, 8.1.1, p. 133], has established parametrization of theta function ϕ for two positive real numbers n and k as:

$$h'_{k,n} = \frac{\phi\left(-e^{-2\pi\sqrt{\frac{n}{k}}}\right)}{k^{\frac{1}{4}}\,\phi\left(-e^{-2\pi\sqrt{nk}}\right)}. \tag{2.1}$$

Jinhee Yi in her thesis [2, Theorem 8.2.1(ii), p. 138], has discovered a following relation,

$$h'_{k,n} = \frac{r_{2,\frac{2n}{k}}}{r_{2,2nk}} \times r_{n,k}. \tag{2.2}$$

From Yi [2, Lemma 2.1.3(i), p. 13], we find that

$$r_{k,\frac{n}{m}} = r_{mk,n} \cdot r^{-1}{}_{nk,m}. \tag{2.3}$$

Yi [2, corollary 2.1.5(i), p. 14], has established the relation

$$r_{k^2,n} = r_{k,nk} \cdot r_{k,\frac{n}{k}}. \tag{2.4}$$

3.Main Results:

In this section, a parameter of theta function is defined. Following that certain properties are also established.

Definition: For any positive real numbers a and b, define a function $I_{a,b}$ as;

$$I_{a,b} = \frac{\phi(-q^2)}{\sqrt{2}\,a^{\frac{1}{4}}\,q^{\frac{a}{8}}\,\psi(q^a)}, \quad q = e^{-\pi\sqrt{\frac{b}{a}}}. \tag{3.1}$$

Remark 1:

Ramanujan [1, entry 27(ii), p.43], provides the transformation formula for ϕ and ψ functions as;

If $\alpha\beta = \pi$ then $2\sqrt{\alpha}\psi\left(e^{-2\alpha^2}\right) = \sqrt{\beta}e^{\frac{\alpha^2}{4}}\phi(-e^{-\beta^2})$.

Substitute $\beta^2 = \frac{2\pi}{\sqrt{n}}$, for any real n, then we get

$$\phi\left(-e^{\frac{-2\pi}{\sqrt{n}}}\right) = \sqrt{2}n^{\frac{1}{4}}\,e^{\frac{-\pi\sqrt{n}}{8}}\,\psi\left(e^{-\pi\sqrt{n}}\right). \tag{3.2}$$

Setting $q = e^{-\pi}$ and $n = 4$ in equation (3.2), we get

$$\phi(-q) = 2\,q^{1/4}\,\psi(q^2). \tag{3.3}$$

Theorem 1: For all positive real numbers a and b**,** we have

1. $I_{a,1} = 1$,
2. $I_{a,\frac{1}{b}} = I_{a,b}^{-1} = \frac{1}{I_{a,b}}$,
3. $I_{a,b} = r_{4,\frac{b}{a}} \cdot I_{b,a}$.

Proof:

1. By putting $b = 1$ in the definition (3.1) and using the identity (3.2), we get the required result.
2. From the definition (3.1), we get

$$I_{a,b} \cdot I_{a,\frac{1}{b}} = \frac{\phi\left(-e^{-2\pi\sqrt{b/a}}\right)\phi\left(-e^{-2\pi\sqrt{1/ab}}\right)}{2\cdot a^{1/4}\cdot e^{\frac{-\pi\sqrt{ab}}{8}}\cdot e^{\frac{-\pi\sqrt{a/b}}{8}}\cdot\psi\left(e^{-\pi\sqrt{ab}}\right)\cdot\psi\left(e^{-\pi\sqrt{a/b}}\right)}. \tag{3.4}$$

When $n = ab$ and $n = a/b$ are substituted in the equation (3.2) then the equation (3.4) becomes

$I_{a,b} \cdot I_{a,\frac{1}{b}} = 1$ and hence the result follows.

3. Again, by using the definition (3.1), we find that

$$\frac{I_{a,b}}{I_{b,a}} = \frac{b^{1/4}\cdot\phi\left(-e^{-2\pi\sqrt{b/a}}\right)}{a^{1/4}\cdot\phi\left(-e^{-2\pi\sqrt{a/b}}\right)}. \tag{3.5}$$

Applying (2.1) and (2.2) to the ratio (3.5), we get

$\frac{I_{a,b}}{I_{b,a}} = \frac{h'_{a,b}}{h'_{b,a}} = \frac{r_{2,\frac{2b}{a}}}{r_{2,\frac{2a}{b}}}$. We complete the proof by substituting $k = 2$ and $n = b/a$ in (2.4).

Remark 2:

1. From the theorem 1(2), it is clear that $a > 1$ then $I_{a,b} < 1, \forall\, b < 1$.
2. From the theorem 1(3), we find that $I_{a,b} > 1, \forall\, a, b > 1$.

Lemma: Let a, b and c are positive real numbers, then $I_{a,\frac{b}{c}} = I_{ca,b} \cdot I^{-1}_{ab,c}$.

Proof: From the definition (3.1), we get

$$I_{a,\frac{b}{c}} = \frac{\phi\left(-e^{-2\pi\sqrt{b/ca}}\right)}{\sqrt{2}\, a^{\frac{1}{4}}\, e^{-\pi\frac{\sqrt{ab/c}}{8}}\, \psi\left(e^{-\pi\sqrt{ab/c}}\right)}.$$

Using the identity (3.2) for $n = ab/c$, we find that

$$I_{a,\frac{b}{c}} = \frac{b^{\frac{1}{4}}\, \phi\left(-e^{-2\pi\sqrt{b/ca}}\right)}{c^{\frac{1}{4}}\, \phi\left(-e^{-2\pi\sqrt{c/ab}}\right)}.$$

Then the lemma follows by multiplying and dividing the above equation with $\sqrt{2}\, a^{\frac{1}{4}}\, e^{-\pi\frac{\sqrt{abc}}{8}}\, \psi\left(e^{-\pi\sqrt{abc}}\right)$.

Theorem 2: If a, b and c are positive real numbers, then $I_{\frac{a}{b},\frac{c}{d}} = \frac{I_{ad,bc}}{I_{ac,bd}}$.

Proof: From the lemma, we found that

$$I_{n,\frac{a}{b}} = I_{nb,a} \cdot I^{-1}_{an,b}. \tag{3.6}$$

Put $n = c/d$ in the equation (3.6) and use the theorem 1(3), we get

$$I_{\frac{c}{d},\frac{a}{b}} = I_{ad,bc} \cdot I^{-1}_{bd,ac} \cdot r_{4,\frac{ad}{bc}} \cdot r^{-1}_{4,\frac{bd}{ac}}. \tag{3.7}$$

Substituting $a = c/d$ and $b = a/b$ in the theorem 1(3), we get

$$I_{\frac{a}{b},\frac{c}{d}} = r_{4,\frac{c/d}{a/b}} \cdot I_{\frac{c}{d},\frac{a}{b}}. \tag{3.8}$$

With the help of the identity (2.3), we simplify the equation (3.7) and (3.8) to get the main result.

Corollary 1: $I_{a^2,b} = I_{ab,a} \cdot I_{a,b/a}$, where a and b are positive real numbers.

Proof: We complete the proof by substituting $b = \frac{1}{a}, c = b$ and $d = 1$ in the theorem 2.

Corollary 2: For all positive real numbers a and b,

1. $I_{\frac{a}{b},\frac{a}{b}} = I_{b,b} \cdot I_{a,\frac{a}{b^2}}$,
2. $I_{a,a} \cdot I_{a,\frac{b^2}{a}} = r_{4,\frac{b^2}{a^2}} \cdot I_{b,b} \cdot I_{b,\frac{a^2}{b}}$,

3. $I_{a,a} \cdot I_{a^2b,b} = I_{b,b} \cdot I_{ab^2,a}$.

Proof:

1. When we substitute $c = 1$ and $d = b/a$ in the theorem 2, we get that $I_{\frac{a}{b},\frac{a}{b}} = \frac{I_{b,b}}{I_{a,b^2/a}}$. Then use of the theorem 1(2) completes the proof.
2. By changing the position of a and b in the corollary 2(1), it is found that
$$I_{\frac{b}{a},\frac{b}{a}} = I_{a,a} \cdot I_{b,\frac{b}{a^2}}. \quad (3.9)$$
Put $c = a$ and $d = b$ in the theorem 2, we get
$$I_{\frac{a}{b},\frac{a}{b}} = \frac{I_{ab,ab}}{I_{a^2,b^2}}. \quad (3.10)$$
Change the position of a and b in (3.10), we found that
$$I_{\frac{b}{a},\frac{b}{a}} = \frac{I_{ab,ab}}{I_{b^2,a^2}}. \quad (3.11)$$
Comparing the equations (3.10) and (3.11), we get
$$I_{\frac{a}{b},\frac{a}{b}} \cdot I_{a^2,b^2} = I_{\frac{b}{a},\frac{b}{a}} \cdot I_{b^2,a^2}.$$

Now by using corollary 2(1), equation (3.9), the theorem 1(3) for I_{a^2,b^2}, we discover that
$$I_{b,b} \cdot I_{a,\frac{a}{b^2}} \cdot r_{4,\frac{b^2}{a^2}} = I_{a,a} \cdot I_{b,\frac{b}{a^2}}.$$
We complete the proof by making use of the theorem 1(2).

3. Put $b = 1, c = b^2, d = a$ and then $a = b, b = 1, c = a^2, d = b$ in the theorem 2, we get
$$I_{a,\frac{b^2}{a}} = \frac{I_{a^2,b^2}}{I_{ab^2,a}}, \qquad I_{a,\frac{a^2}{b}} = \frac{I_{b^2,a^2}}{I_{a^2b,b}}.$$

We reach the result by substituting the above values in the corollary 2(2) and the applying the theorem 1(3).

Theorem 3: Let a, b, c, d and k be positive real numbers such that $ab = cd$, then
$$I_{a,b} \cdot I_{kc,kd} = I_{ka,kb} \cdot I_{c,d}.$$

Proof: From the definition (3.1), we get
$$\frac{I_{ka,kb}}{I_{a,b}} = \frac{\psi\left(e^{-2\pi\sqrt{ab}}\right)}{k^{1/4}\psi\left(e^{-\pi k\sqrt{ab}}\right)} = \frac{\psi\left(e^{-2\pi\sqrt{cd}}\right)}{k^{1/4}\psi\left(e^{-\pi k\sqrt{cd}}\right)} = \frac{I_{kc,kd}}{I_{c,d}}.$$
And hence the theorem follows.

Corollary 3: Let n and p be two positive real numbers then
$$I_{np,np} = I_{n,np^2} \cdot I_{p,p} \cdot r_{4,p^2}^{-1}.$$

Proof: Substitute $a = p^2, b = 1, k = n$ and $c = d = p$ in the theorem 3, we get
$$I_{p^2,1} \cdot I_{np,np} = I_{np^2,n} \cdot I_{p,p}.$$
We derive the result by using the theorem 1(1) and (3).

Theorem 4: For all positive real numbers n,
$$\frac{\phi\left(-e^{-2^n\pi}\right)}{\psi(e^{-2\pi})} = 2^{\frac{4-n}{4}}\, e^{-\pi/4} \prod_{i=1}^{n} I_{2,2^{2i-1}}.$$

Proof: We use mathematical induction to prove the theorem.

- Step I: Let $n = 1$.
  From definition (3.1), we have
  $$I_{2,2^{2-1}} = I_{2,2} = \frac{\phi(-e^{-2\pi})}{2^{3/4}\, e^{-\pi/4}\, \psi(e^{-2\pi})}.$$

By rearranging the above equation, we get

$$2^{\frac{4-1}{4}} e^{-\pi/4} I_{2,2} = \frac{\phi(-e^{-2\pi})}{\psi(e^{-2\pi})},$$

Hence result is true for $n = 1$.

- Step 2: Suppose result is true for $n = k$.

$$\frac{\phi\left(-e^{-2^k\pi}\right)}{\psi(e^{-2\pi})} = 2^{\frac{4-k}{4}} e^{-\pi/4} \prod_{i=1}^{k} I_{2,2^{2i-1}}.$$

- Step 3: To prove for $n = k + 1$.

$$\prod_{i=1}^{k+1} I_{2,2^{2i-1}} = \prod_{i=1}^{k} I_{2,2^{2i-1}} \cdot I_{2,2^{k+1}}.$$

Now by using step 2 and the definition (3.1), we get

$$\prod_{i=1}^{k+1} I_{2,2^{2i-1}} = \frac{\phi\left(-e^{-2^k\pi}\right)\phi\left(-e^{-2^{(k+1)}\pi}\right)}{2^{\frac{7-k}{4}} e^{-\pi/4} \psi(e^{-2\pi}) \psi\left(e^{-2^{(k+1)}\pi}\right)}.$$

Using the equation (3.3) for $q = e^{-2^k\pi}$, we get

$$\frac{\phi\left(-e^{-2^{(k+1)}\pi}\right)}{\psi(e^{-2\pi})} = 2^{\frac{4-(k+1)}{4}} e^{-\pi/4} \prod_{i=1}^{k+1} I_{2,2^{2i-1}}.$$

The above equation demonstrate that result is true for $n = k + 1$.

References:

1. B. C. Berndt, *Ramanujan's notebook:* Part III. Springer Science and Business Media, 2012.
2. Yi, Jinhee. The construction and applications of modular equations. University of Illinois at Urban-Champaign,2001.