Non-Linear Diophantine Equation

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ABSTRACT
This paper is an important study about these Non-Linear Equation which have integer solution exists. i.e., Non-Linear Diophantine Equation, which have named by the famous Greek Mathematician Diophantus of Alexandria. In this paper we have focused to solve Non-Linear Diophantine Equations.

Key Words: -Non-linear Diophantine equations, factor method, some algebric rules.

Mathematics Subject Classification: -97F10,97F20,97F60.

Introduction: Let we have a linear equation \( ax + by = c \),
where \( a, b, c \in \mathbb{Z} \); \( a \neq 0, b \neq 0 \)
and \( x, y \in \mathbb{Z} \) with \( x \geq 0, y \geq 0 \), then this equation is called LinearDiophantine Equation.
Ex: \( 10x + 6y = 110 \)
Here \( a = 10, b = 6, c = 110 \)
Clearly \( a \neq 0, b \neq 0 \)

Now we consider Non- linear Diophantine equation.
Some rules to find out the solution of Non-Linear Diophantine Equation
Factor Method: -
Any Non-Linear Diophantine Equation can be solved by this method.
Rules: -
(1) Factorized the equation into two parts (If possible).
(2) Consider such cases.
(3) Find values of \( (x, y) \)
(4) Satisfied the condition.

SOME EXAMPLES:-
(1) Solve \((x^2 + 1)(y^2 + 1) + 2(x - y)(1 - xy) = 4(1 + xy)\).
Sol\textsuperscript{a}:-
\[(x^2 + 1)(y^2 + 1) + 2(x - y)(1 - xy) = 4 + 4xy.
\]
\[x^2y^2 + x^2 + y^2 + 1 + 2(x - y)(1 - xy) - 2xy - 2xy = 4 \]
\[x^2 - 2xy + y^2 + x^2y^2 - 2xy + 1 + 2(x - y)(1 - xy) = 4 \]
\[(x - y)^2 + (xy - 1)^2 + 2(x - y)(1 - xy) = 4 \]
\[((x - y) - (xy - 1))^2 = 4 \]
\[x(1 - y) + 1(1 - y))^2 = 4 \]
\[(1 - y)(x + 1) = \pm 2\]
\[
\begin{align*}
(1 - y) &= 2 \\
(x + 1) &= 1
\end{align*}
\begin{align*}
(1 - y) &= 2 \\
(x + 1) &= -1
\end{align*}
\begin{align*}
(1 - y) &= 1 \\
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\end{align*}

We get values from above \((0,-1), (-2,-1), (1,0), (1,2), (-3,0), (0,3), (-2,3), (-3,2)\).

Above are the required solution.

(2) Determine all non-negative pairs \((x, y)\) for which \((xy - 7)^2 = x^2 + y^2\)

Sol\(^n\): -

\[ (xy - 7)^2 = x^2 + y^2 \]
\[ x^2y^2 - 14xy + 49 = x^2 + y^2 \]
\[ x^2y^2 - 12xy + 49 = x^2 + 2xy + y^2 \]
\[ x^2y^2 - 2. xy. 6 + 36 + 13 = (x + y)^2 \]
\[ (xy - 6)^2 + 13 = (x + y)^2 \]
\[ (xy - 6)^2 - (x + y)^2 = -13 \]
\[ (x + y)^2 - (xy - 6)^2 = 13 \]
\[ (x + y + xy - 6)(x + y - xy + 6) = 13 \]

So,
\[ x + y + xy - 6 = 13 \] \[ (x + y - xy + 6) = 1 \]

(1)-(2) we get
\[ x + y + xy - 6 - x - y + xy - 6 = 12 \]
\[ 2xy - 12 = 12 \]
\[ 2xy = 24 \]
\[ xy = 12 \]

(3) Solve the following equation in integer \(x, y\):

\[ x^2(y - 1) + y^2(x - 1) = 1 \]

Sol\(^n\): -

\[ x^2(y - 1) + y^2(x - 1) = 1 \]

Put \((x - 1) = u, \ (y - 1) = v\)
\[ x = u + 1, \ y = v + 1 \]

So, \((u + 1)^2v + (v + 1)^2u = 1\)
\[ (u^2 + 2uv + v + (v^2 + 2v + 1)u = 1 \]
\[ u^2v + 2uv + v + v^2u + 2uv + u = 1 \]
\[ u^2v + v^2u + 4uv + u + v = 1 \]
\[ uv(u + v + 4) + 1(u + v + 4) = 5 \]
\[ (u + v + 4)(1 + uv) = 5 \]

So, condition(1) will be
\[ u + v + 4 = 5 \]

1. ** Condition (2) will be**
   \[ u + v + 4 = -5 \]

2. ** Condition (3) will be**
   \[ u + v + 4 = 1 \]
   \[ 1 + uv = 1 \]

3. ** Condition (4) will be**
   \[ u + v + 4 = -1 \]
   \[ 1 + uv = -5 \]

So, from equ (1) we get,
\[ uv = 0 \]
\[ u + v = 1 \]
\[ uv + v^2 = v \]
\[ v(v - 1) = 0 \]
\[ v = 0 \ or \ 1, \ u = 0 \ or \ 1 \]
\[ v^2 + 9v - 2 = 0 \]
\[ v = \frac{-9 \pm \sqrt{81 + 4}}{2} \]
\[ v = \frac{-9 \pm \sqrt{85}}{2} \notin \mathbb{Z} \]

So not possible.

Now from equation (3) we get,
\[ u + v = -3 \]
\[ uv = 4 \]
\[ uv + v^2 = -3v \]
\[ 4 + v^2 = -3v \]
\[ v^2 + 3v + 4 = 0 \]

Here we see No integer solution.

So case is not possible.

Now, from equation (4) we get,
\[ u + v = -5, \ uv = -6 \]

So,
\[ uv + v^2 = -5v \]
\[ -6 + v^2 = -5v \]
\[ v^2 + 5v - 6 = 0 \]
\[ v^2 + (6 - 1)v - 6 = 0 \]
\[ v^2 + 6v - v - 6 = 0 \]
\[ v(v + 6) - 1(v + 6) = 0 \]
\[ v = 1, -6 \ and \ u = -6 \ or \ 1 \]

So, required pairs are: 
\[ -(1, 2), (2, 1), (-5, 2), (2, -5). \]
(4) Find +ve Z solution to the equation \( x^3 - y^3 = xy + 61 \)

\[
\text{Sol}^n: \quad x^3 - y^3 = xy + 61 \\
(x - y)(x^2 + xy + y^2) = xy + 61 \\
\text{Clearly, } (x^2 + y^2) = 61 \ldots \ldots \ldots (1) \\
(x - y) = 1 \ldots \ldots \ldots \ldots \ldots \ldots (2)
\]

Now,
\[
(x - y)^2 = (1)^2 \\
x^2 - 2xy + y^2 = 1 \\
61 - 2xy = 1 \\
2xy = 60 \\
xy = 30 \ldots \ldots \ldots \ldots \ldots \ldots (3)
\]

From equation (2) we get, \( xy = 30 \) \text{ and } \( x - y = 1 \)

So, \( (x, y) = (15, 2) \) or \( (6, 5) \)

But \( 15 - 2 = 13 \neq 1 \)

So above pair is not acceptable.
So, our required pair is, \((x, y) = (6, 5)\).

**Advantages:**

(i) This method is very much easy to be computed.

**Disadvantages:**

(i) Convert to any Non-Linear Diophantine Equation to factor form is not easy to compute.

(ii) If we cannot convert it to be factor form, then the method will be failed.

**REFERENCES**