Non-Linear Diophantine Equation

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ABSTRACT
This paper is an important study about these Non-Linear Equation which have integer solution exists. i.e., Non-Linear Diophantine Equation, which have named by the famous Greek Mathematician Diophantus of Alexandria. In this paper we have focused to solve Non-Linear Diophantine Equations.

Key Words: -Non-linear Diophantine equations, factor method, some algebric rules.

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Introduction: Let we have a linear equation $ax + by = c$,

where $a, b, c\epsilon z$; $a \neq 0, b \neq 0$

and $x, y\epsilon z$ with $x \geq 0, y \geq 0$, then this equation is called LinearDiophantine Equation.

Ex: $10x + 6y = 110$

Here $a = 10, b = 6, c = 110$

Clearly $a \neq 0, b \neq 0$

Now we consider Non- linear Diophantine equation.
Some rules to find out the solution of Non-Linear Diophantine Equation
Factor Method: -
Any Non-Linear Diophantine Equation can be solved by this method.
Rules: -

(1) Factorized the equation into two parts (If possible).
(2) Consider such cases.
(3) Find values of (x, y)
(4) Satisfied the condition.

SOME EXAMPLES:-

(1) **Solve**$(x^2 + 1)(y^2 + 1) + 2(x - y)(1 - xy) = 4(1 + xy)$.

Sol^n :-

$$(x^2 + 1)(y^2 + 1) + 2(x - y)(1 - xy) = 4 + 4xy.$$

$$x^2y^2 + x^2 + y^2 + 1 + 2(x - y)(1 - xy) - 2xy - 2xy = 4$$

$$x^2 - 2xy + y^2 + x^2y^2 - 2xy + 1 + 2(x - y)(1 - xy) = 4$$

$$(x - y)^2 + (xy - 1)^2 + 2(x - y)(1 - xy) = 4$$

$$\{(x - y) - (xy - 1)\}^2 = 4$$

$$\{x(1 - y) + 1(1 - y)\}^2 = 4$$

$$(1-y)(x+1) = \pm 2$$

$$\begin{cases}(1-y)=2\\(x+1)=1\end{cases}\quad\begin{cases}(1-y)=2\\(x+1)=-1\end{cases}\quad\begin{cases}(1-y)=1\\(x+1)=2\end{cases}\quad\begin{cases}(1-y)=-1\\(x+1)=2\end{cases}\quad\begin{cases}(x+1)=-2\\(1-y)=1\end{cases}\quad\begin{cases}(x+1)=1\\(1-y)=-2\end{cases}$$

We get values from above$(0,-1),(-2,-1),(1,0),(1,2),(-3,0),(0,3),(-2,3),(-3,2)$.
Above are the required solution.

(2) Determine all non-negative pairs (x,y) for which$(xy-7)^2 = x^2+y^2$

Sol^n: - $(xy-7)^2 = x^2+y^2$

$x^2y^2 - 14xy + 49 = x^2 + y^2$

$x^2y^2 - 12xy + 49 = x^2 + 2xy + y^2$

$x^2y^2 - 2.xy.6 + 36 + 13 = (x+y)^2$

$(xy-6)^2 + 13 = (x+y)^2$

$(xy-6))^2 - (x+y)^2 = -13$

$(x+y)^2 - (xy-6)^2 = 13$

$(x+y+xy-6)(x+y-xy+6) = 13$

So, $(x+y+xy-6) = 13$…………………………..(1)

$(x+y-xy+6) = 1$……………………………..(2)

(1)-(2) we get

$x+y+xy-6-x-y+xy-6 = 12$

$2xy - 12 = 12$

$2xy = 24$

$xy = 12$…………………………….(3)

Again (1)+(2) we get

$x+y+xy-6+x+y-xy+6 = 14$

$(x+y) = 7$

From equation (1) we get

$x+y+6 = 13$

$(x+y) = 7$………………………………….(4)

Now from equation (4) we get

$(x,y) = (1,6),(2,5),(3,4),(4,3),(5,2),(6,1),(7,0),(0,7)$.

Now from eq 3 we get actual ans is (3,4) and (4,3)

(3) Solve the following equation in integer x, y:

$$x^2(y-1) + y^2(x-1) = 1$$

Sol^n:- $x^2(y-1) + y^2(x-1) = 1$

Put

$(x-1) = u, \quad (y-1) = v$

$x = u+1, \quad y = v+1$

So, $(u+1)^2v + (v+1)^2u = 1$

$(u^2+2u+1)v + (v^2+2v+1)u = 1$

$u^2v + 2uv + v + v^2u + 2uv + u = 1$

$u^2v + v^2u + 4uv + u + v = 1$

$uv(u+v+4)+1(u+v+4)=5$

$(u+v+4)(1+uv)=5$

So, condition(1) will be

$$u+v+4=5$$

$1+uv=1$

Condition (2) will be

$u+v+4=-5$

$1+uv=-1$

Condition (3) will be

$u+v+4=1$

$1+uv=5$

Condition (4) will be

$u+v+4=-1$

$1+uv=-5$

So, from equ (1) we get,

$uv=0$

$u+v=1$

$uv+v^2=v$

$v(v-1)=0$

$v=0 \; or \; 1, \quad u=0 \; or \; 1$

$v^2+9v-2=0$

$v=\frac{-9\pm\sqrt{81+4}}{2}$

$v=\frac{-9\pm\sqrt{85}}{2} \quad \notin Z$

So not possible.

Now from equation (3) we get,

$u+v=-3 \qquad uv=4$

$uv+v^2=-3v$

$4+v^2=-3v$

$v^2+3v+4=0$

Here we see No integer solution.

So case is not possible.

Now, from equation (4) we get,

$u+v=-5, \qquad uv=-6$

So, $uv+v^2=-5v$

$-6+v^2=-5v$

$v^2+5v-6=0$

$v^2+(6-1)v-6=0$

$v^2+6v-v-6=0$

$v(v+6)-1(v+6)=0$

$v=1,-6 \; and \; u=-6 \; or \; 1$

So, required pairs are: -$(1,2),(2,1),(-5,2),(2,-5)$.

(4) Find +ve Z solution to the equation $x^3 - y^3 = xy + 61$

Sol^n : $x^3 - y^3 = xy + 61$

$(x-y)(x^2 + xy + y^2 = xy + 61$

Clearly, $(x^2 + y^2) = 61 \dots\dots\dots\dots\dots..(1)$

$(x-y) = 1 \dots\dots\dots\dots\dots\dots\dots (2)$

Now,

$(x-y)^2 = (1)^2$

$x^2 - 2xy + y^2 = 1$

$61 - 2xy = 1$

$2xy = 60$

$xy = 30 \dots\dots\dots\dots\dots\dots\dots.(3)$

From equation (2) we get, $xy = 30 \quad and \quad (x-y) = 1$

So, $(x,y) = (15,2) \quad or\ (6,5)$

But$(15-2) = 13 \neq 1$

So above pair is not acceptable.

So, our required pair is,$(x,y) = (6,5)$.

Advantages:

(i) This method is very much easy to be computed.

Disadvantages:

(i) Convert to any Non-Linear Diophantine Equation to factor form is not easy to compute.

(ii) If we cannot convert it to be factor form, then the method will be failed.

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