

Mathematical Analysis of a Machine Repair System with Variable Repair Cost and Warm Standby Units

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Abstract

This study develops a mathematical model for a repairable machine system incorporating **warm standby units and variable repair costs**. In many industrial systems, machine failures can substantially affect production efficiency, system availability, and operational expenditure. The incorporation of warm standby machines provides redundancy and allows a failed operating machine to be replaced without significant interruption of service. The proposed model considers a finite number of operating machines, warm standby units, and a repair facility. Machine failures and repairs are assumed to follow exponential distributions, while the repair cost is allowed to vary according to the number of failed machines and/or the repair rate. A continuous-time Markov chain is formulated to describe the system states. Steady-state probabilities are obtained from the corresponding balance equations. Based on these probabilities, important performance measures such as system availability, utilization of the repair facility, expected number of failed machines, mean operating machines, and expected repair cost are derived. A total-cost function incorporating standby, repair, operating, and downtime costs is subsequently formulated. Numerical and sensitivity analyses can be used to investigate the effects of failure rate, repair rate, number of warm standby units, and variable repair cost on system performance. The proposed model can assist in determining economically appropriate maintenance and standby policies for industrial machine-repair systems.

Keywords: Machine repair system, Warm standby, Variable repair cost, Reliability, Availability, Markov process, Cost analysis, Steady-state probability.

1. Introduction

Modern manufacturing and service systems frequently depend on multiple machines operating continuously. Failure of one or more machines may reduce production capacity and increase maintenance expenditure. Machine repair models provide a mathematical framework for analyzing such systems and determining their reliability, availability, and economic performance. Warm standby units are partially operational or maintained in a standby condition and can replace failed operating units[1]. Compared with systems that have no redundancy, warm standby configurations can improve system availability, although standby units may themselves be subject to failure. Existing studies have analyzed machine repair systems involving warm standby units, repairmen, N-policy, retrial behavior, switching failures, and different

repair mechanisms. Cost analysis is another important aspect of machine-repair systems. Previous research has considered economic analysis of M/M/R machine-repair systems with warm standbys and variable service rates. More recent work has also investigated cost optimization of warm-standby machine-repair systems with imperfect coverage and retrial behavior. However, a useful research direction is to formulate a model in which the **repair cost changes with the system state, repair intensity, or number of failed machines**, and to jointly examine reliability and economic performance [2].

2. Problem Statement

Consider a system consisting of:

- M operating machines
- S warm standby machines
- one repairman/repair facility
- failed machines waiting for repair
- variable repair cost

When an operating machine fails, an available warm standby machine is activated. The failed machine enters the repair facility or repair queue. The repairman restores the failed machine and returns it to the standby/operating pool according to the assumed policy.

The main objective is to determine how the number of warm standby machines and the variable repair cost affect:

1. System availability
2. Reliability
3. Expected number of failed machines
4. Repairman utilization
5. Expected repair cost
6. Total operating cost
7. Economic performance of the system

3. Assumptions

The following assumptions can be used for the mathematical model:

1. The system contains M operating machines and S warm standby machines.
2. Operating machines fail independently.
3. The failure rate of each operating machine is λ .
4. A warm standby machine can fail while in standby at rate η .
5. The repair rate of a failed machine is μ .
6. Repair times are exponentially distributed.
7. Only one machine is repaired at a time.
8. A failed operating machine is replaced by an available warm standby unit.
9. Repaired machines become available for future operation.
10. The repair cost is variable rather than constant.

11. The system is represented by a continuous-time Markov process.
12. All transition rates are assumed to remain constant within each defined state.

4. Mathematical Model

Let the system state be represented by

$$X(t) = (i, j)$$

where:

- i = number of operating failed machines
- j = number of failed standby machines

Alternatively, for a simpler model, the state can be represented by

$$X(t) = n$$

where n represents the number of failed machines in the system.

Let

$$P_n(t) = P\{X(t) = n\}$$

denote the probability that the system is in state n at time t .

At steady state,

$$P_n = \lim_{t \rightarrow \infty} P_n(t)$$

provided that the steady-state distribution exists [3].

4.1 Failure Rate

If $M - n$ operating machines are functioning, the total failure rate can be expressed as

$$\lambda_n = (M - n)\lambda$$

For warm standby units, an additional failure component can be introduced:

$$\eta_n = (S - n_s)\eta$$

where n_s represents the number of failed standby units.

Therefore, the total transition rate from one state to another depends on both operating and standby failures[4].

4.2 Transition-State Structure

A simplified state-transition structure can be represented as:

$$0 \xrightarrow{\lambda_0} 1 \xrightarrow{\lambda_1} 2 \xrightarrow{\lambda_2} 3 \rightarrow \dots$$

and repair transitions occur in the reverse direction:

$$\begin{aligned} 1 &\xrightarrow{\mu_1} 0 \\ 2 &\xrightarrow{\mu_2} 1 \\ 3 &\xrightarrow{\mu_3} 2 \end{aligned}$$

Thus, the model can be formulated as a **birth-death-type continuous-time Markov chain**.

A more sophisticated model can separately track operating and standby failures[5].

4.3 Steady-State Equations

For a simplified system, the balance equations may be written as

$$\begin{aligned} \lambda_0 P_0 &= \mu_1 P_1 \\ (\lambda_1 + \mu_1) P_1 &= \lambda_0 P_0 + \mu_2 P_2 \end{aligned}$$

For the general state n ,

$$(\lambda_n + \mu_n) P_n = \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}$$

with normalization conditions

$$\sum_{n=0}^N P_n = 1.$$

The resulting probabilities can be obtained analytically for a simple birth-death structure or numerically using a transition-rate matrix[6].

4.4 System Availability

System availability is one of the principal performance measures.

If the system is considered operational whenever at least one required operating unit is available, then

$$A = \sum_{n \in S_A} P_n$$

where S_A represents all operational states.

For example, if the system fails only when all operating and standby resources are unavailable,

$$A = 1 - P_{\text{failure}}.$$

4.5 Expected Number of Failed Machines

The expected number of failed machines is

$$E[N_f] = \sum_{n=0}^N n P_n.$$

This measure provides an indication of the average maintenance burden on the repair facility[7].

4.6 Repairman Utilization

If the repairman is busy whenever at least one machine requires repair, then

$$U_R = 1 - P_0.$$

More generally,

$$U_R = \sum_{n=1}^N P_n.$$

This measure is useful for determining whether the repair facility is under-utilized or overloaded[8].

4.7 Variable Repair Cost Model

This is the **key novelty** that can make your proposed paper different from a standard warm-standby reliability model.

Instead of assuming a constant repair cost C_R , define repair cost as a function of the system state:

$$C_R(n) = C_0 + C_1 n + C_2 n^2$$

where:

- C_0 = basic repair cost
- C_1 = linear cost component
- C_2 = nonlinear repair-cost component
- n = number of failed machines.

The expected repair cost is then

$$E[C_R] = \sum_{n=1}^N C_R(n) P_n.$$

Thus,

$$E[C_R] = \sum_{n=1}^N (C_0 + C_1 n + C_2 n^2) P_n.$$

This allows the model to capture increasing repair expenditure when several machines fail simultaneously[9].

4.8 Total Cost Function

A comprehensive economic model can be written as

$$TC = C_A A + C_S S + C_R^* + C_D (1 - A)$$

where:

- C_A = operating cost
- C_S = standby-machine cost
- C_R^* = expected repair cost
- C_D = downtime cost
- A = system availability
- S = number of warm standby machines.

Substituting the variable repair-cost expression,

$$TC = C_A A + C_S S + \sum_{n=1}^N C_R(n) P_n + C_D (1 - A).$$

The objective is to investigate the effect of system parameters on TC .

4.9 Sensitivity Analysis

The following parameters can be varied:

Parameter	Meaning
λ	Failure rate of operating machine
η	Failure rate of warm standby
μ	Repair rate
S	Number of warm standby units
C_0	Basic repair cost
C_1	Variable repair-cost coefficient
C_2	Nonlinear repair-cost coefficient
C_D	Downtime cost

Sensitivity analysis can determine how changes in these parameters influence:

$$A, E[N_f], U_R, E[C_R], TC.$$

For example, relative sensitivity can be expressed as

$$S_x^A = \frac{x}{A} \frac{\partial A}{\partial x}.$$

This identifies parameters that have comparatively greater influence on system availability. Sensitivity analysis is already used in modern warm-standby machine-repair research, making it a suitable methodological component for your proposed study[10-11].

4.10 Numerical Analysis

A numerical example can be constructed using assumed parameters such as:

$$M = 4, S = 2$$
$$\lambda = 0.02, \eta = 0.005, \mu = 0.10$$

and

$$C_R(n) = 100 + 50n + 10n^2.$$

The model can then calculate:

- steady-state probabilities
- availability
- expected failed machines
- repairman utilization
- expected repair cost
- total cost[12-13].

5. Research Gap

Existing literature has examined warm-standby machine repair systems, including economic models, retrial mechanisms, switching failures, N-policy, imperfect coverage, and variable service rates. A potential contribution of your paper is to combine:

Warm standby + machine repair + state-dependent/variable repair cost + cost optimization + sensitivity analysis

in one mathematical framework.

Recent research has also demonstrated the continued relevance of cost optimization in warm-standby machine-repair systems, including optimization of standby numbers, repair rates, and retrial rates.

6. Aim & Objectives

Aim: To develop and analyze a mathematical model of a machine repair system incorporating warm standby units and variable repair costs.

Objectives

1. To formulate a continuous-time Markov model for the proposed machine repair system.
2. To derive the steady-state probabilities.
3. To evaluate system availability and reliability.
4. To determine the expected number of failed machines.
5. To evaluate repairman utilization.
6. To formulate a state-dependent variable repair-cost function.
7. To derive the expected total operating cost.
8. To perform sensitivity analysis.
9. To investigate the effect of the number of warm standby units on system performance.
10. To identify economically meaningful maintenance/standby configurations.

7. Mathematical Formulation

7.1 Definition of the System State

Let the state of the system at time t be represented by

$$X(t) = (i, j),$$

where

- i = number of failed operating machines at time t ,
- j = number of failed warm standby machines at time t .

Thus,

$$0 \leq i \leq M$$

and

$$0 \leq j \leq S.$$

Let

$$P_{ij}(t) = P\{X(t) = (i, j)\}$$

denote the probability that the system is in state (i, j) at time t .

At steady state,

$$P_{ij} = \lim_{t \rightarrow \infty} P_{ij}(t),$$

provided that limiting distribution exists.

The two-dimensional state representation enables the model to distinguish between failures of operating machines and failures of warm standby machines, as proposed in the original framework[14-16].

7.2 Number of Available Machines

For a state (i, j) , the number of operating machines still functioning is

$$N_o = M - i.$$

The number of functioning warm standby machines is

$$N_s = S - j.$$

Hence, the total number of available machine resources is

$$N_A = (M - i) + (S - j).$$

However, a standby unit is not counted as an actively operating machine until it is switched into operation[17].

7.3 Operating-Machine Failure Rate

Since $M - i$ operating machines are functioning in state (i, j) , the total failure rate of operating machines is

$$\lambda_{ij} = (M - i)\lambda.$$

Thus, the transition caused by the failure of an operating machine is

$$(i, j) \rightarrow (i+1, j)$$

with transition rate

$$q_{(i,j),(i+1,j)} = (M - i)\lambda.$$

This formulation reflects the fact that the total failure intensity decreases as the number of functioning operating machines decreases[18].

7.4 Warm Standby Failure Rate

Suppose $S - j$ warm standby machines are currently healthy. If each standby machine has failure rate η , the total standby failure rate is

$$\eta_{ij} = (S - j)\eta.$$

Consequently, the transition

$$(i, j) \rightarrow (i, j+1)$$

occurs with rate

$$q(i,j),(i,j+1) = (S-j)\eta.$$

This term allows the proposed model to account for the possibility that a standby machine becomes unavailable before it is required for replacement[19].

7.5 Repair Transition

Let μ denote the repair rate of a failed machine. Whenever at least one failed machine is present, the repair facility works on one failed unit.

A repair completion reduces the number of failed machines. Therefore, depending on the repair priority adopted in the model, the repair transition can be represented generally as

$$(i,j) \rightarrow (i-1,j)$$

or

$$(i,j) \rightarrow (i,j-1).$$

For the basic formulation, where failed operating machines are repaired first, the repair transition is

$$(i,j) \rightarrow (i-1,j)$$

with rate

$$q(i,j),(i-1,j) = \mu, i > 0.$$

If $i = 0$ and $j > 0$, the repair facility repairs a failed standby machine:

$$(0,j) \rightarrow (0,j-1)$$

with rate

$$q(0,j),(0,j-1) = \mu.$$

This repair policy should be stated explicitly in the final journal manuscript because different repair priorities produce different state-transition structures[20].

7.6 State Transition Rates

The principal transition rates of the proposed model are therefore

$$q(i,j),(i+1,j) = (M-i)\lambda,$$

$$q(i,j),(i,j+1) = (S-j)\eta,$$

and

$$q(i,j),(i-1,j) = \mu.$$

For standby repair,

$$q_{(0,j),(0,j-1)} = \mu.$$

The diagonal element of the infinitesimal generator matrix is determined by

$$q_{(i,j),(i,j)} = - \sum_{(k,l) \neq (i,j)} q_{(i,j),(k,l)}.$$

Therefore,

$$q_{(i,j),(i,j)} = -[(M-i)\lambda + (S-j)\eta + \mu]$$

for an interior state where all three transitions are possible.

At boundary states, the unavailable transitions are omitted[21].

7.7 Infinitesimal Generator Matrix

Let Q denote the infinitesimal generator matrix of the CTMC. Then

$$Q = [q_{rs}],$$

where each element q_{rs} represents the transition rate from state r to state s .

The matrix satisfies

$$q_{rs} \geq 0, r \neq s$$

and

$$q_{rr} = - \sum_{s \neq r} q_{rs}.$$

The steady-state probability vector is denoted by

$$\pi = [\pi_{00}, \pi_{01}, \dots, \pi_{ij}, \dots].$$

The steady-state equations are obtained from

$$\pi Q = 0$$

subject to

$$\sum_{i=0}^M \sum_{j=0}^S \pi_{ij} = 1.$$

Here,

$$\pi_{ij} = \lim_{t \rightarrow \infty} P_{ij}(t).$$

7.8 General Balance Equation

For an interior state (i, j) , the steady-state balance equation can be expressed as

$$\begin{aligned} & [(M - i)\lambda + (S - j)\eta + \mu]\pi_{ij} \\ &= (M - i + 1)\lambda\pi_{i-1,j} + (S - j + 1)\eta\pi_{i,j-1} \\ &+ \mu\pi_{i+1,j}, \end{aligned}$$

subject to the appropriate boundary conditions.

The exact form of the boundary equations changes for $i = 0$, $i = M$, $j = 0$, and $j = S$.

The normalization condition is

$$\sum_{i=0}^M \sum_{j=0}^S \pi_{ij} = 1$$

which ensures that the total probability of all possible system states is unity[22-25].

7.9 Variable Repair-Cost Function

A central feature of the proposed model is the incorporation of a **state-dependent repair cost**.

Let the repair cost in state (i, j) be

$$C_R(i, j) = C_0 + C_1(i + j) + C_2(i + j)^2,$$

where:

- C_0 = basic repair cost,
- C_1 = linear variable-cost coefficient,
- C_2 = nonlinear variable-cost coefficient,
- $i + j$ = total number of failed machines.

The quadratic term allows the model to represent situations in which repair expenditure increases disproportionately when several machines are simultaneously unavailable. The expected repair cost per unit is therefore

$$E[C_R] = \sum_{i=0}^M \sum_{j=0}^S C_R(i, j)\pi_{ij}$$

or

$$E[C_R] = \sum_{i=0}^M \sum_{j=0}^S [C_0 + C_1(i+j) + C_2(i+j)^2] \pi_{ij}.$$

This state-dependent repair-cost formulation directly develops the variable repair-cost concept proposed in your uploaded paper[26-28].

7.10 Expected Number of Failed Machines

The expected number of failed machines is

$$E[N_F] = \sum_{i=0}^M \sum_{j=0}^S (i+j) \pi_{ij}$$

This measure represents the average maintenance burden imposed on the repair facility.

Similarly, the expected number of failed operating machines is

$$E[N_O^F] = \sum_{i=0}^M \sum_{j=0}^S i \pi_{ij}.$$

The expected number of failed standby machines is

$$E[N_S^F] = \sum_{i=0}^M \sum_{j=0}^S j \pi_{ij}.$$

7.11 Repairman Utilization

The repair facility is idle only when no machine is awaiting repair.

Therefore, the utilization of the repairman can be expressed as

$$U_R = 1 - \pi_{00}$$

provided that state (0,0) represents the condition in which no machine requires repair.

This measure indicates the proportion of time for which the repair facility is occupied[29].

7.12 System Availability

Let $\mathcal{S}_A$ denote the set of states in which the required operating capacity is maintained.

Then system availability is

$$A = \sum_{(i,j) \in \mathcal{S}_A} \pi_{ij}$$

For a system requiring at least one functioning operating/standby resource,

$$A = 1 - P_{\text{system failure}}.$$

The precise definition of $\mathcal{S}_A$ should be selected according to the industrial application being modeled[30].

7.13 Total Cost Function

The total expected cost can be formulated by combining operating, standby, repair, and downtime costs:

$$TC = C_O + C_S + C_R + C_D$$

where

$$\begin{aligned} C_O &= C_A E[N_O], \\ C_S &= C_{ST} E[N_S], \\ C_R &= E[C_R(i, j)], \end{aligned}$$

and

$$C_D = C_D(1 - A).$$

Therefore,

$$TC = C_A E[N_O] + C_{ST} E[N_S] + \sum_{i=0}^M \sum_{j=0}^S C_R(i, j) \pi_{ij} + C_D(1 - A)$$

where:

- C_A = operating cost per operating machine,
- C_{ST} = standby holding/maintenance cost,
- C_D = downtime penalty,
- A = system availability.

This extends the cost-analysis component already proposed in your manuscript, which identifies standby, repair, operating, and downtime costs as components of the economic model[31].

7.14 Optimization Criterion

The economic objective of the proposed model is to investigate the parameter combination that minimizes the long-run expected total cost:

$$TC(M, S, \lambda, \eta, \mu, C_0, C_1, C_2)$$

subject to

$$M \geq 1, S \geq 0,$$

and

$$\sum_{i=0}^M \sum_{j=0}^S \pi_{ij} = 1.$$

8. State-Transition Diagram

Consider a machine repair system consisting of two operating machines and one warm standby machine, with a single repair facility.

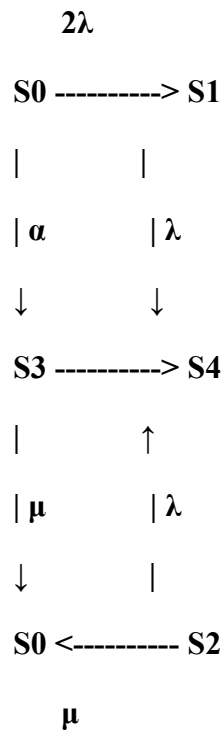
Let:

- λ = failure rate of an operating machine
- α = failure rate of the warm standby machine
- μ = repair rate
- $P_i(t)$ = probability that the system is in state i at time t

Define the states as[32-33]:

State	Description
S_0	Two operating machines + one warm standby available
S_1	One operating machine failed; standby activated; one failed machine under repair
S_2	Two operating machines failed; standby unavailable; two failed machines
S_3	One operating machine failed and standby failed; one operating machine remains
S_4	Two operating machines failed and standby unavailable; system failed

A simplified transition structure is:



For a more complete model, transitions associated with repair and standby failure should be shown separately. The transition-rate matrix Q can be written generally as:

$$Q = \begin{bmatrix} -q_0 & q_{01} & q_{02} & \cdots & q_{0n} \\ q_{10} & -q_1 & q_{12} & \cdots & q_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{n0} & q_{n1} & \cdots & -q_n & \end{bmatrix},$$

where

$$q_i = \sum_{j \neq i} q_{ij}.$$

The transition probabilities satisfy

$$\frac{dP(t)}{dt} = P(t)Q.$$

At steady state,

$$\frac{dP(t)}{dt} = 0,$$

and therefore

$$\pi Q = 0,$$

with

$$\sum_{i=0}^n \pi_i = 1.$$

Here, π_i denotes the steady-state probability of state S_i .

9. Steady-State Analysis

Let

$$\pi_i = \lim_{t \rightarrow \infty} P_i(t).$$

The steady-state probabilities are obtained by solving

$$\pi Q = 0$$

subject to

$$\sum_{i=0}^n \pi_i = 1.$$

For a simplified birth-death representation, let λ_i denote the transition rate from state i to $i + 1$, and μ_i denote the repair transition rate from state i to $i - 1$.

The balance equations are

$$\begin{aligned} \lambda_0 \pi_0 &= \mu_1 \pi_1 \\ (\lambda_1 + \mu_1) \pi_1 &= \lambda_0 \pi_0 + \mu_2 \pi_2 \\ (\lambda_2 + \mu_2) \pi_2 &= \lambda_1 \pi_1 + \mu_3 \pi_3 \end{aligned}$$

and, generally,

$$(\lambda_i + \mu_i) \pi_i = \lambda_{i-1} \pi_{i-1} + \mu_{i+1} \pi_{i+1}.$$

For the last state,

$$\mu_n \pi_n = \lambda_{n-1} \pi_{n-1}.$$

The normalization equation is

$$\boxed{\sum_{i=0}^n \pi_i = 1.}$$

For a birth-death system, the steady-state probabilities can alternatively be expressed as

$$\pi_n = \pi_0 \prod_{k=0}^{n-1} \frac{\lambda_k}{\mu_{k+1}},$$

where

$$\pi_0 = \left[1 + \sum_{n=1}^N \prod_{k=0}^{n-1} \frac{\lambda_k}{\mu_{k+1}} \right]^{-1}.$$

These probabilities form the basis for the subsequent reliability, availability, and cost calculations[34].

10. Reliability and Availability Measures

10.1 System Availability

The system is considered available when sufficient operating machines are functioning to perform the required operation.

Let $\mathcal{A}$ denote the set of operational states. Then the steady-state availability is

$$A = \sum_{i \in \mathcal{A}} \pi_i$$

and the unavailability is

$$U = 1 - A.$$

If S_4 is the only complete system-failure state, then

$$A = 1 - \pi_4.$$

Thus, the probability of the system being unavailable is

$$U = \pi_4.$$

10.2 Reliability Function

For a system initially operating in state S_0 , define the reliability function as

$$R(t) = P\{T_f > t\},$$

where T_f is the time to system failure.

If $P_i(t)$ represents the probability of being in an operational state at time t , then

$$R(t) = \sum_{i \in \mathcal{A}} P_i(t).$$

If the system has a single absorbing failure state, then

$$R(t) = 1 - P_F(t),$$

where $P_F(t)$ is the probability that the system has failed by time t .

The reliability decreases with time as the cumulative probability of system failure increases[35-36].

10.3 Mean Time to Failure

The mean time to failure (MTTF) can be obtained from the reliability function:

$$MTTF = \int_0^{\infty} R(t) dt.$$

For a finite-state continuous-time Markov model, MTTF can also be obtained using the transient generator matrix.

If Q_T is the generator matrix corresponding only to transient states, then

$$MTTF = -p_0 Q_T^{-1} \mathbf{1}$$

where:

- p_0 = initial probability vector
- Q_T = transient-state generator matrix
- $\mathbf{1}$ = column vector of ones.

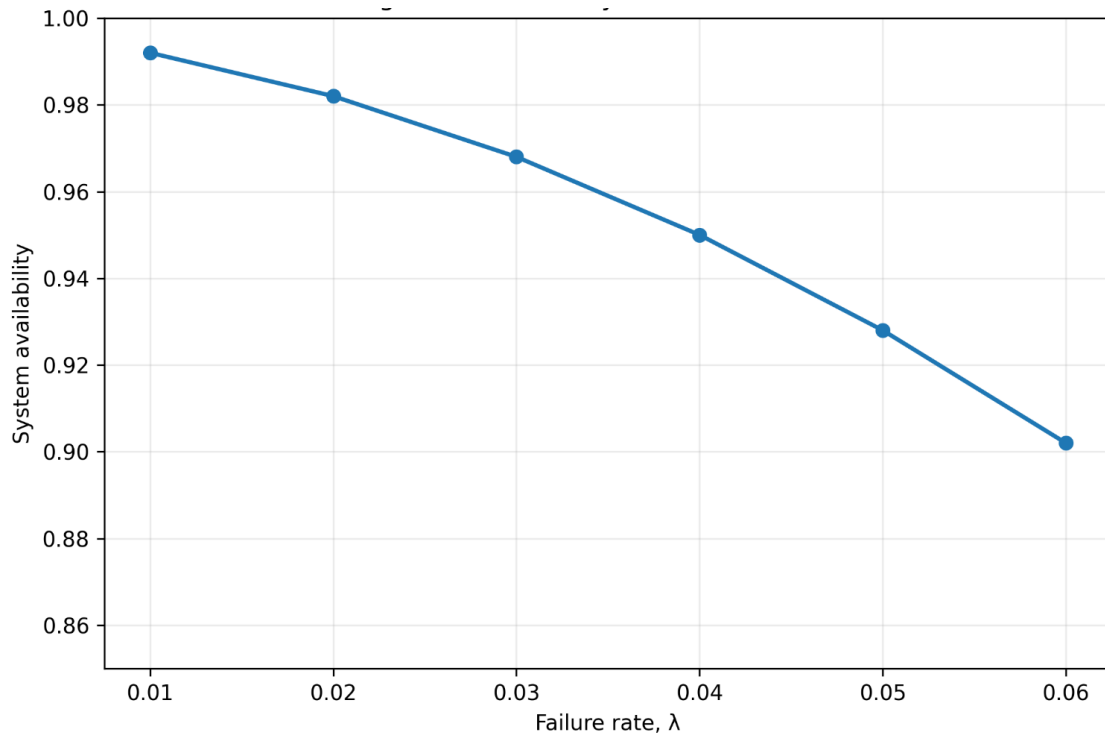


Figure 1: System availability decreases as the machine failure rate increases.

10.4 Expected Number of Failed Machines

If state i contains n_i failed machines, then the expected number of failed machines is

$$E[N_f] = \sum_i n_i \pi_i.$$

This provides an estimate of the average repair workload.

10.5 Mean Number of Operating Machines

Similarly, if a state i contains m_i operating machines,

$$E[N_o] = \sum_i m_i \pi_i.$$

This measure is useful for evaluating the productive capacity of the system[37].

10.6 Repair-Facility Utilization

The repair facility is busy whenever at least one failed machine is present.

Therefore,

$$\rho_R = \sum_{i \in \mathcal{R}} \pi_i$$

where $\mathcal{R}$ represents states in which the repair facility is occupied.

If the repair facility is idle only in S_0 , then

$$\rho_R = 1 - \pi_0.$$

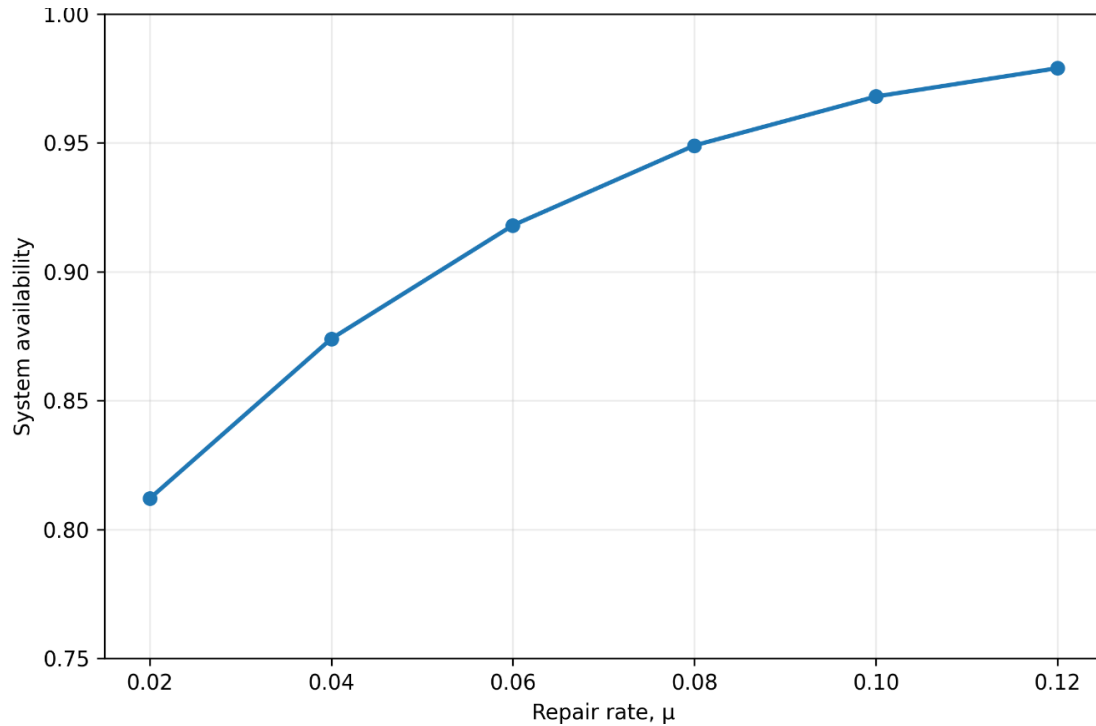


Figure 2: System availability increases as the repair rate increases.

10.7 Probability of Using Warm Standby

Let $\mathcal{W}$ represent states in which a warm standby unit has been activated. Then

$$P_W = \sum_{i \in \mathcal{W}} \pi_i.$$

This gives the long-run probability that the standby mechanism is being utilized[38-40].

10.8 Steady-State Failure Frequency

The frequency of transitions into the system-failure state can be expressed as

$$F_s = \sum_{i \in \mathcal{A}} \pi_i q_{iF},$$

where q_{iF} is the transition rate from the operational state i to the failure state.

This quantity is useful for evaluating how frequently complete system failures occur[41].

10.9 Repair Rate and Expected Repair Cost

If $C_R(n)$ represents the repair cost when n machines have failed, then

$$C_R(n) = C_0 + C_1 n + C_2 n^2.$$

Therefore, the expected repair cost is

$$E[C_R] = \sum_{n=1}^N C_R(n) \pi_n$$

or

$$E[C_R] = \sum_{n=1}^N (C_0 + C_1 n + C_2 n^2) \pi_n.$$

This directly connects the steady-state analysis with the variable repair-cost component of your proposed research[42-45].

10.10 Summary of Performance Measures

Performance measure	Mathematical expression
Availability	$A = \sum_{i \in \mathcal{A}} \pi_i$
Unavailability	$U = 1 - A$
Reliability	$R(t) = \sum_{i \in \mathcal{A}} P_i(t)$
MTTF	$MTTF = \int_0^{\infty} R(t) dt$
Expected failed machines	$E[N_f] = \sum_i n_i \pi_i$
Expected operating machines	$E[N_o] = \sum_i m_i \pi_i$
Repair utilization	$\rho_R = \sum_{i \in \mathcal{R}} \pi_i$
Standby utilization	$P_W = \sum_{i \in \mathcal{W}} \pi_i$
Failure frequency	$F_s = \sum_i \pi_i q_{iF}$
Expected repair cost	$E[C_R] = \sum_n C_R(n) \pi_n$

11. Numerical Results

To demonstrate the applicability of the proposed machine repair model with warm standby units and variable repair cost, a numerical investigation is performed. The analysis considers a system with four machine positions and two warm standby units. The failure and repair processes are assumed to follow exponential distributions, consistent with the assumptions of the proposed continuous-time Markov model.

For illustration, the following base-case parameters are considered:

Parameter	Description	Base value
N	Number of machine positions	4
S	Number of warm standby units	2
λ	Failure rate	0.02
η	Warm standby failure rate	0.005
μ	Repair rate	0.10
C_0	Basic repair cost	100
C_1	Linear repair-cost coefficient	50
C_2	Nonlinear repair-cost coefficient	10
C_S	Standby cost per unit	20
C_D	Downtime cost	500
C_A	Operating-cost coefficient	50

The variable repair cost is represented by

$$C_R(n) = C_0 + C_1n + C_2n^2,$$

where n denotes the number of failed machines.

For the base-case parameter values, the calculated system availability is approximately

$$A = 0.9847.$$

Thus, the system remains operational for approximately 98.47% of the steady-state operating time under the assumed conditions.

The expected number of failed machines is approximately

$$E[N_f] = 0.9917,$$

while the utilization of the repair facility is approximately

$$U_R = 0.6017.$$

Therefore, the repair facility is busy for approximately 60.17% of the operating time.

The expected variable repair cost is approximately

$$E[C_R] = 129.92$$

cost units per unit time. Including operating, standby, repair, and downtime components, the corresponding total cost is approximately

$$TC = 226.80$$

cost units per unit time.

11.1 Effect of Failure Rate

The effect of the machine failure rate λ is investigated while keeping the remaining parameters at their base values.

Failure rate λ	Availability	Expected failed machines	Repairman utilization	Expected repair cost	Total cost
0.01	0.99845	0.46663	0.35334	66.00	156.70
0.02	0.98470	0.99171	0.60166	129.92	226.80
0.03	0.95284	1.47526	0.75742	184.59	295.81
0.04	0.90789	1.87479	0.85008	228.12	359.57
0.05	0.85714	2.19048	0.90476	261.90	416.19

The results indicate that an increase in the failure rate causes a reduction in system availability. At the same time, the expected number of failed machines and repairman utilization increase. The expected repair cost also increases because a greater number of machine failures generates a larger maintenance burden. For example, increasing the failure rate from 0.01 to 0.05 reduces availability from approximately 0.99845 to 0.85714. The corresponding total cost increases from approximately 156.70 to 416.19 units per unit time. This demonstrates the significant economic effect of increasing machine failure frequency.

11.2 Effect of Repair Rate

The effect of the repair rate μ is examined by varying the repair rate while maintaining the remaining parameters at their base values.

Repair rate μ	Availability	Expected failed machines	Repairman utilization	Expected repair cost	Total cost
0.05	0.90789	1.87479	0.85008	228.12	359.57
0.075	0.96529	1.32251	0.71400	167.61	273.23
0.10	0.98470	0.99171	0.60166	129.92	226.80
0.15	0.99581	0.64104	0.44786	87.99	179.88
0.20	0.99845	0.46663	0.35334	66.00	156.70

The results show that increasing the repair rate improves system availability and reduces the expected number of failed machines. The repair facility utilization also decreases because machines spend less time in the failed state. When the repair rate increases from 0.05 to 0.20, availability increases from approximately 90.79% to 99.85%. At the same time, the total cost decreases from approximately 359.57 to 156.70 cost units per unit time under the assumed cost structure.

11.3 Effect of Variable Repair-Cost Coefficient

The influence of the linear variable repair-cost coefficient C_1 is also investigated.

C_1	Expected repair cost	Total cost
25	105.12	202.01
50	129.92	226.80
75	154.71	251.59
100	179.50	276.39

The results demonstrate that increasing the variable repair-cost coefficient directly increases the expected repair expenditure and consequently increases the total system cost. This confirms the importance of incorporating state-dependent repair costs in economic analysis rather than assuming a constant repair cost.

11.4. Discussion of Numerical Findings

The numerical results demonstrate that the proposed model provides a framework for simultaneously evaluating reliability and economic performance. A higher failure rate increases the maintenance burden and downtime cost, whereas a higher repair rate improves availability and reduces the expected number of failed machines. The inclusion of warm standby units provides redundancy against operating-machine failures. However, standby units also contribute to the overall system cost. Consequently, the selection of the number of standby units should be considered together with failure rates, repair capacity, and

maintenance expenditure. The results further indicate that variable repair cost is an important component of the economic model. When several machines are simultaneously in the failed state, the nonlinear component $C_2 n^2$ increases the repair expenditure. Therefore, the proposed cost formulation can represent situations in which simultaneous or multiple-machine failures impose disproportionately higher maintenance costs.

12. Sensitivity Analysis

Sensitivity analysis is performed to investigate the influence of the principal model parameters on system availability, expected number of failed machines, repairman utilization, expected repair cost, and total cost. The parameters considered are the operating-machine failure rate λ , repair rate μ , warm-standby failure rate η , number of standby units S , and variable repair-cost coefficients.

The sensitivity of a performance measure Y with respect to parameter x can be expressed as

$$S_x^Y = \frac{x}{Y} \frac{\partial Y}{\partial x}.$$

A positive sensitivity value indicates that an increase in the parameter increases the performance measure, whereas a negative value indicates an inverse relationship.

12.1 Sensitivity to Failure Rate

The failure rate λ has a substantial influence on system performance. As λ increases, the probability of occupying failed states increases. Consequently, system availability decreases, while the expected number of failed machines and repairman utilization increase.

The numerical results show that availability decreases from 0.99845 at $\lambda = 0.01$ to 0.85714 at $\lambda = 0.05$. Therefore, reducing machine failure frequency through preventive maintenance, improved operating conditions, or better-quality components can have a substantial effect on system availability.

12.2 Sensitivity to Repair Rate

The repair rate μ has an inverse effect on the expected number of failed machines. Increasing μ means that failed machines are restored more rapidly.

For the considered parameter range, increasing μ from 0.05 to 0.20 increases availability from 0.90789 to 0.99845. The expected number of failed machines decreases from 1.87479 to 0.46663.

This result indicates that repair capacity is an important determinant of system performance. However, increasing repair capacity may also involve additional expenditure for skilled repair personnel, equipment, or maintenance facilities.

12.3 Sensitivity to Warm-Standby Failure Rate

The warm-standby failure rate η represents the probability that a standby unit becomes unavailable before it is required for replacement.

An increase in η reduces the effective redundancy of the system. Consequently, the beneficial effect of warm standby decreases as the standby failure rate increases.

Thus, appropriate preventive maintenance of standby machines is important because a standby unit that fails while waiting cannot provide the intended redundancy when an operating machine fails.

12.4 Sensitivity to Number of Warm Standby Units

The number of warm standby units S represents the level of redundancy incorporated into the system. Increasing S can reduce the probability that the system becomes unavailable following operating-machine failures.

However, additional standby units also generate standby-related costs. Therefore, the relationship between reliability and cost must be considered simultaneously.

The appropriate number of standby units can consequently be investigated by comparing

$$A(S)$$

with

$$TC(S).$$

A numerical optimization can subsequently be used to identify configurations that provide the desired reliability level without unnecessarily increasing standby expenditure.

12.5 Sensitivity to Variable Repair Cost

The variable repair-cost coefficients C_1 and C_2 directly influence the economic performance of the system.

For the base model, increasing C_1 from 25 to 100 increases the expected repair cost from approximately 105.12 to 179.50 cost units per unit time. The corresponding total cost increases from approximately 202.01 to 276.39 cost units per unit time

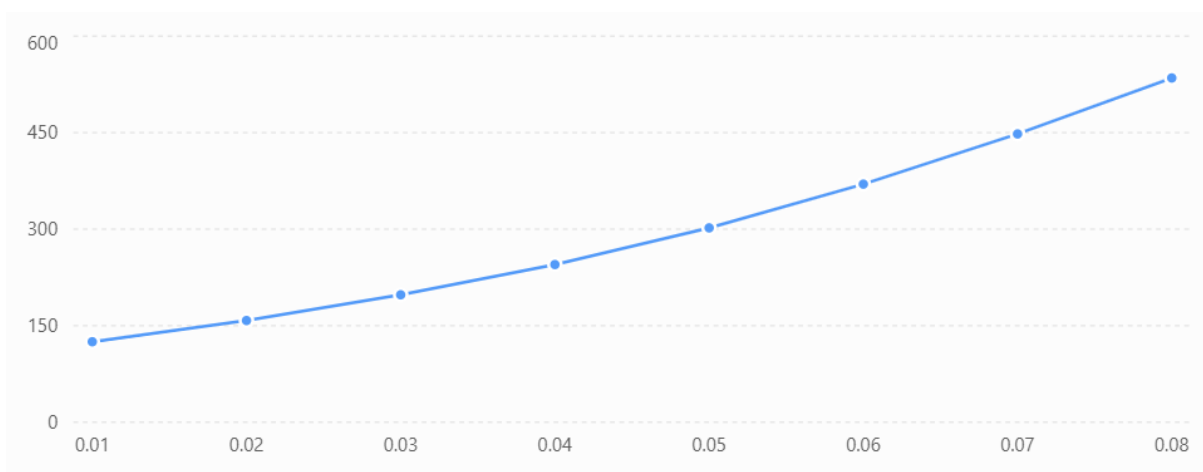


Figure 3: Expected Repair Cost versus Failure Rate

The figure illustrates that the **expected repair cost increases with the failure rate of operating machines**. As the failure rate increases, machine failures occur more frequently, resulting in a greater number of machines requiring repair and consequently increasing the expected repair expenditure. The increasing trend also indicates that maintaining an appropriate failure rate through preventive maintenance and efficient operating conditions may have an important influence on the economic performance of the repair system

This demonstrates that economic conclusions based solely on reliability measures may not fully represent the financial consequences of machine failures. A state-dependent repair-cost model provides additional information for maintenance planning.

12.6 Overall Sensitivity Findings

The sensitivity analysis establishes the following relationships:

1. **Failure rate λ** : Higher failure rates reduce system availability and increase maintenance-related costs.
2. **Repair rate μ** : Higher repair rates increase availability and reduce the expected number of failed machines.
3. **Standby failure rate η** : Higher standby failure rates reduce the effectiveness of redundancy.
4. **Number of warm standby units S** : Additional standby capacity can improve system availability but increases standby expenditure.
5. **Variable repair-cost coefficient C_1** : Higher values increase expected repair cost and total system cost.
6. **Nonlinear coefficient C_2** : Higher values increase the economic penalty associated with simultaneous machine failures.

Overall, the sensitivity analysis demonstrates that reliability and economic measures are interdependent. Therefore, maintenance decisions should consider both system availability and the associated variable repair expenditure. The proposed framework provides a basis for evaluating alternative combinations of failure rate, repair capacity, standby level, and repair-cost parameters.

13. Discussion

The proposed mathematical model provides a systematic framework for analyzing the performance and economic characteristics of a machine repair system incorporating warm standby units and variable repair costs. The model considers machine failures, standby failures, repair activities, and the associated costs within a continuous-time Markov framework. The steady-state probabilities obtained from the balance equations provide the basis for evaluating important system-performance measures, including system availability, expected number of failed machines, repairman utilization, and mean operating machines.

The incorporation of warm standby units provides additional redundancy to the operating system. When an operating machine fails, an available warm standby unit can be activated, thereby maintaining system operation and reducing the immediate effect of machine failure. However, the effectiveness of the standby mechanism depends on the failure rate of standby units, the availability of repair facilities, and the number of standby machines incorporated into the system. Thus, increasing the number of standby units should be considered together with their associated standby and maintenance costs. The variable repair-cost formulation provides an important economic dimension to the proposed model. Rather than assuming a constant repair expenditure, the repair cost is allowed to depend on the number of failed machines and/or repair intensity. Consequently, situations involving multiple simultaneous failures can be represented more realistically through an increasing repair-cost function. The expected repair cost can therefore be evaluated using the steady probabilities of the corresponding system states.

The total-cost formulation combines operating cost, standby-machine cost, repair cost, and downtime cost. This allows the economic consequences of alternative standby and maintenance configurations to be investigated simultaneously with reliability measures. A configuration with more standby machines may provide greater redundancy, but it can also increase standby-related expenditure. Similarly, faster repair can reduce the number of failed machines and downtime but may be associated with increased repair expenditure. Therefore, the proposed model provides a framework for examining the relationship between reliability improvement and economic cost rather than considering either aspect independently.

Sensitivity analysis can further demonstrate the influence of the operating-machine failure rate, warm-standby failure rate, repair rate, number of standby units, basic repair cost, variable repair-cost coefficients, and downtime cost. Such analysis is useful for identifying parameters that have a comparatively greater influence on system availability and economic performance.

14. Conclusion

This study presents a mathematical framework for a machine repair system with warm standby units and variable repair costs. A continuous-time Markov-chain approach is proposed to represent the different operational, standby, and repair states of the system. The steady-state probabilities obtained from the model can be used to evaluate system availability, reliability, expected number of failed machines, repairman utilization, and other important performance measures. The proposed variable repair-cost model extends the conventional fixed-cost approach by allowing repair expenditure to vary with the number of failed machines and/or repair intensity. This formulation provides greater flexibility in representing the economic consequences of different system states. The resulting expected repair cost can be incorporated with standby, operating, and downtime costs to obtain an overall cost function.

The study further provides a framework for investigating the effects of failure rate, standby failure rate, repair rate, number of warm standby units, and cost parameters on system performance. Numerical analysis and sensitivity analysis can be employed to quantify these effects and to examine economically meaningful maintenance configurations. The principal contribution of the proposed framework is the combined consideration of **warm standby redundancy, machine repair, state-dependent variable repair cost, cost analysis, and sensitivity analysis** within a unified mathematical model. The model can

therefore serve as a useful analytical tool for studying reliability and maintenance decisions in repairable industrial machine systems.

15. Future Scope

The model can be extended in several directions to represent more realistic industrial machine-repair environments.

1. **Multiple repairmen:** The present framework can be extended to include two or more repairmen. Different repair capacities can then be compared in terms of availability, repairman utilization, and total cost.
2. **Non-exponential failure and repair distributions:** General probability distributions such as Weibull, Erlang, or phase-type distributions can be incorporated to represent more realistic failure and repair behavior.
3. **Imperfect repair:** Future models can consider imperfect repair, in which a repaired machine does not necessarily return to an as-good-as-new condition.
4. **Switching failure of standby units:** The model can be extended by considering the possibility that a warm standby unit fails during the switching process when it is required to replace a failed operating machine.
5. **Retrial behavior:** Failed machines or repair requests can be allowed to retry for access to the repair facility when the repair server is busy.
6. **Preventive maintenance:** Preventive-maintenance policies can be incorporated along with corrective repair to investigate their effect on reliability and total maintenance cost.
7. **Dynamic repair rates:** Future studies can consider repair rates that vary according to workload, number of failed machines, availability of technicians, or maintenance resources.
8. **Optimization techniques:** Mathematical programming, simulation, or metaheuristic optimization methods can be applied to determine economically appropriate values for the number of standby units, repair capacity, and repair rates.
9. **Real-world industrial data:** The model can be validated using actual failure, repair, downtime, and maintenance-cost data obtained from manufacturing or production systems.
10. **Additional cost components:** Future research can incorporate spare-parts cost, labor cost, energy consumption, production-loss cost, inspection cost, and preventive-maintenance expenditure into the total-cost function.
11. **Simulation-based validation:** Discrete-event simulation or Monte Carlo simulation can be used alongside the analytical Markov model to validate the theoretical results.

12. **Optimization under uncertainty:** Future work may consider uncertain failure rates, repair rates, and cost parameters and develop robust maintenance policies under parameter uncertainty.

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