Split Restrained Geodetic Number of Strong Product and Lexicographic Product of Graphs

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Abstract

Let 'G' be a graph. If $u, v \in V$, then a u-v geodetic of G is the shortest path between u and v. The closed interval I[u, v] consists of all vertices lying in some u-v geodetic of G . For $S \subseteq V(G)$ the set I[S] is the union of all sets I [u, v] for $u, v \in S$. A set S is a geodetic set of G if I[S]=V(G). The cardinality of minimum geodetic set of G is the geodetic number of G, denoted by g(G). A set S of vertices of a graph G is a split geodetic set if S is a geodetic set and $\langle V - S \rangle$ is disconnected, split geodetic number $g_s(G)$ of G is the minimum cardinality of a split geodetic set of G. In this paper I study split restrained geodetic number of strong product and lexicographic product of graphs. A set S of vertices of a graph G is a split restrained geodetic set if S is a geodetic set and the subgraph $\langle V - S \rangle$ is disconnected with no isolated vertices. The minimum cardinality of a split restrained geodetic set of G is the split restrained geodetic number of G and is denoted by $g_{sr}(G)$. The split restrained geodetic numbers of some standard strong product and the lexicographic product of graphs are determined.

Keywords: Geodetic set, Geodetic number, Split geodetic set, Split geodetic number, Split Restrained Geodetic set, Split Restrained Geodetic number, Strong product, Lexicographic product of graphs.

1. Introduction

In this paper, we follow the notations of [4]. The graphs considered here have at least one component which is not complete or at least two nontrivial components.

The distance d (u, v) between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. It is well known that this distance is a metric on the vertex set V(G). For a vertex v of G, the eccentricity e(v) is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is radius, rad G, and the maximum eccentricity is the diameter, diam G. A u-v path of length d (u, v) is called a u-v geodesic. We define I[u, v] to the set of all vertices lying on some u-v geodesic of G and for a nonempty subset S of V(G), $I[S] = \cup_{u,v \in S} I[u, v]$. A set S of vertices of G is called a geodetic set in G if I[S]=V(G), and a geodetic set of minimum cardinality is a minimum geodetic set. The cardinality of a minimum geodetic set in G is called the geodetic number of G, and we denote it by g(G). The geodetic number of a graph was introduced in [6,7] and further studied in [2,8,4].

A geodetic set S of a graph G=(V, E) is a split geodetic set if the induced subgraph $\langle V - S \rangle$ is disconnected. .The split geodetic number $g_s(G)$ of G is the minimum cardinality of a split geodetic set. The split geodetic number was introduced and studied in [9]. A set S of vertices of a graph G is a split restrained geodetic set if S is a geodetic set and the subgraph $\langle V - S \rangle$ is disconnected with no isolated vertices. The minimum

cardinality of a split restrained geodetic set of G is the split restrained geodetic number of G and is denoted by $g_{sr}(G)$. The split geodetic number was introduced and studied in [10].
The strong product of graphs G_1 and G_2, denoted by $G_1 \boxtimes G_2$, has vertex set $V(G_1) \times V(G_2)$, where two distinct vertices x_1, y_1 and x_2, y_2 are adjacent with respect to the strong product if (a) $x_1 = x_2$ and $y_1y_2 \in E(G_2)$ or (b) y1 = y2 and $x_1x_2 \in E(G_1)$ or (c) $x_1x_2 \in E(G_1)$ and $y_1y_2 \in E(G_2)$.
The lexicographic product of $G = G_1[G_2]$ has $V = V_1 \times V_2$ as its vertex set, and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever $[u_1 \text{ adjacent to } v_1]$ or $[u_1 = v_1 \text{ and } u_2 \text{ adjacent to } v_2]$.
For any undefined term in this paper, see [3] and [4].

2. Preliminary Notes

We need the following results to prove further results.
Theorem 2.1 [2] Every geodetic set of a graph contains its extreme vertices.
Theorem 2.2 [2] For any path P_n with n vertices, $g(P_n) = 2$.
Theorem 2.3 [2] For cycle C_n of order $n \geq 3$, $g(C_n) = \begin{cases} 2, \text{if n is even} \\ 3, \text{if n is odd} \end{cases}$

3. Main Results

3.1 For cycle C_n of order $n \geq 4$, $g_{sr}(K_2 \boxtimes C_n) = \begin{cases} 4 & \text{if n is even} \\ 6 & \text{if n is odd} \end{cases}$
Proof: Let $G = K_2 \boxtimes C_n$ be the graph formed from two copies G_1 and G_2 of C_n. Let $U = \{u_1, u_2, \dots, u_n\} \in V(G_1), W = \{w_1, w_2, \dots, w_n\} \in V(G_2)$ and $V(G) = U \cup W$. We have the following cases.
Case 1. Let n be even. Consider $S = \{u_i, u_j, w_i, w_j\}$ be the split restrained geodetic set, where $d(u_i, u_j) = diam(K_2 \boxtimes C_n) = d(w_i, w_j)$ and $\{(u_i, w_i), (u_j, w_j)\} \in E(K_2 \boxtimes C_n)$, such that $I[S] = V(K_2 \boxtimes C_n)$ and $\langle V(K_2 \boxtimes C_n) - S \rangle$ is disconnected with no isolated vertices. If possible suppose $S' = \{u_i, u_j, w_i\} \subseteq S$ be such that every two vertices $u, v \in S'$ there exist a vertex $w_k \neq u, v$ of G that lies in $u - v$ geodesic. Thus S is the minimum split restrained geodetic set of $K_2 \boxtimes C_n$, therefore $g_{sr}(K_2 \boxtimes C_n) = 4$.

Case 2: Let n be odd. Consider $S = \{u_i, u_j, u_k, w_i, w_j, w_k\}$ be the split restrained geodetic set , where $d(u_i, u_j) = d(u_j, u_k) = diam(K_2 \boxtimes C_n) = d(w_i, w_j) = d(w_j, w_k)$ and $\{(u_i, w_i), (u_j, w_j), (u_k, w_k)\} \in E(K_2 \boxtimes C_n)$, such that $I[S] = V(K_2 \boxtimes C_n)$ and $\langle V(K_2 \boxtimes C_n) - S \rangle$ is disconnected. If possible suppose $S' = \{u_i, u_j, u_k, w_i, w_k\} \subseteq S$ be such that every two vertices $u, v \in S'$ there exist a vertex $w_l \neq u, v$ of G that lies in $u - v$ geodesic. Thus S is the minimum split restrained geodetic set of $K_2 \boxtimes C_n$, there fore $g_{sr}(K_2 \boxtimes C_n) = 6$.

Theorem 3.2 For any path P_n, $n \geq 5$, $g_{sr}(K_2 \boxtimes P_n) = 6$.
Proof: Let $K_2 \boxtimes P_n$ be the graph formed from two copies G_1 and G_2 of P_n. Let $U = \{u_1, u_2, \dots, u_n\} \in V(G_1), W = \{w_1, w_2, \dots, w_n\} \in V(G_2)$ and $V = U \cup W$. Let $S = \{H_1 \cup H_2\}$,where $H_1 = \{u_1, u_n, w_1, w_n\} \subseteq V(K_2 \boxtimes P_n)$ and $H_2 = \{u_i, w_i\} \in E(K_2 \boxtimes P_n) \subseteq V(K_2 \boxtimes P_n) - H_1$, u_i and w_i are the vertices having maximum degree that is $deg(u_i) = deg(w_i) = 5$. Now S be the set of vertices, such that $I[S] = V(K_2 \boxtimes P_n)$and $\langle V(K_2 \boxtimes P_n) - S \rangle$ has more than one component, which does not contain any isolated vertices. Then by the above argument, S is the minimum split restrained geodetic set of $K_2 \boxtimes P_n$. Clearly it follows that $|S| = |H_1 \cup H_2| = 4 + 2 = 6$. Therefore $g_{sr}(K_2 \boxtimes P_n) = 6$.

Theorem 3.3 For any integers $r, s \geq 2$, $g_{sr}(K_2 \boxtimes K_{r,s}) = 2\min(r, s)$.

Proof : Let $G = K_{r,s}$, such that $U = \{u_1, u_2, \dots, u_r\}$, $W = \{w_1, w_2, \dots, w_s\}$ are the partite sets of G, where $r \leq s$ and also $V(K_{r,s}) = U \cup W$. $K_2 \boxtimes K_{r,s}$ be the graph formed from two copies G_1 and G_2 of $K_{r,s}$. Let $U_1 = \{(a_1, u_1), (a_1, u_2), \dots, (a_1, u_r), (a_1, w_1), (a_1, w_2), \dots, (a_1, w_s)\} \in V(G_1)$,
$W_1 = \{(b_1, u_1), (b_1, u_2), \dots, (b_1, u_r), (b_1, w_1), (b_1, w_2), \dots, (b_1, w_s)\} \in V(G_2)$, where $a_1, b_1 \in K_2$ and $V(K_2 \boxtimes K_{r,s}) = U_1 \cup W_1$.

Consider $S = \{(a_1, u_1), (a_1, u_{2)}, \dots, (a_1, u_r), (b_1, u_1), (b_1, u_2), \dots, (b_1, u_r)\}$, for every $(a_1, w_k), (b_1, w_k)$, $1 \leq k \leq s$ lies on the $(a_1, u_i) - (b_1, u_j)$ geodesic for $1 \leq i \neq j \leq r$. Since $\langle V(K_2 \boxtimes K_{r,s}) - S\rangle$ is disconnected with no isolated vertices, we have S is a split restrained geodetic set of $K_2 \boxtimes K_{r,s}$

Let $X = \{(a_1, u_1), (a_1, u_{2)}, \dots, (a_1, u_{r-1}), (b_1, u_1), (b_1, u_2), \dots, (b_1, u_r)\}$ be any set of vertices such that $|X| < |S|$, then X is not a geodetic set of $K_2 \boxtimes K_{r,s}$, since $(a_1, u_r) \notin I[X]$. Also let $Y = \{(a_1, u_1), (a_1, u_{2)}, \dots, (a_1, u_r), (b_1, u_1), (b_1, u_2), \dots, (b_1, u_{r-1})\}$ be any set of vertices such that $|Y| < |S|$, then Y is not a geodetic set of $K_2 \boxtimes K_{r,s}$, since $(b_1, u_r) \notin I[Y]$. It is clear that S is a minimum split restrained geodetic set of $K_2 \boxtimes K_{r,s}$. Hence $g_{sr}(K_2 \boxtimes K_{r,s}) = 2\min(r, s)$=2r.

Theorem 3.4 For any Tadpole graph for $n > 2$ and $m > 3$, $g_{sr}(K_2 \boxtimes T_{m,n}) = \begin{cases} 6 & \text{for even cycle} \\ 8 & \text{for odd cycle.} \end{cases}$

Proof: Tadpole graph is a special type of graph consisting of cycle graph of m vertices and a path graph of n vertices connected with a bridge.

$V = \{c_1, c_2, \dots, c_m\}$ are the vertices of C_m and $U = \{p_1, p_2, \dots, p_n\}$ are the vertices of P_n. $W = V \cup U$ are the vertices of tadpole graph.

$K_2 \boxtimes T_{m,n}$ be the graph formed from two copies G_1 and G_2 of $T_{m,n}$. Let $U_1 = \{(a_1, c_1), (a_1, c_2), \dots, (a_1, c_m), (a_1, p_1), (a_1, p_2), \dots, (a_1, p_n)\} \in V(G_1)$, $W_1 = \{(b_1, c_1), (b_1, c_2), \dots, (b_1, c_m), (b_1, p_1), (b_1, p_2), \dots, (b_1, p_n)\} \in V(G_2)$ where $a_1, b_1 \in K_2$ and $V(K_2 \boxtimes T_{m,n}) = U_1 \cup W_1$.

We have the following cases.

Case 1: For even cycle

Let $S = \{(a_1, c_1), (a_1, p_n), (b_1, c_1), (b_1, p_n), (a_1, c_n), (b_1, c_n)\}$ be a split restrained geodetic set of $K_2 \boxtimes T_{m,n}$, where $d\{(a_1, c_1), (a_1, p_n)\} = d\{(b_1, c_1), (b_1, p_n)\} = diam(K_2 \boxtimes T_{m,n})$. Suppose $S' = \{(a_1, c_1), (a_1, p_n), (b_1, c_1), (b_1, p_n)\}$, $|S'| < |S|$, which is a geodetic set and $V - S'$ is connected. Hence S is a minimum split restrained geodetic set. Also for all $x, y \in \langle V(K_2 \boxtimes T_{m,n}) - S\rangle$, it follows that $\langle V(K_2 \boxtimes T_{m,n}) - S\rangle$ is disconnected with no isolated vertices. Thus $g_{sr}(K_2 \boxtimes T_{m,n}) = 6$.

Case 2: For odd cycle.

Let $S = \{(a_1, c_1), (a_1, p_n), (b_1, c_1), (b_1, p_n), (a_1, c_2), (b_1, c_2), (a_1, c_n), (b_1, c_n)\}$ be a split restrained geodetic set of $K_2 \boxtimes T_{m,n}$. Suppose $S' = \{(a_1, c_1), (a_1, p_n), (b_1, c_1), (b_1, p_n), (a_1, c_2), (b_1, c_2)\}$, $|S'| < |S|$, which is a geodetic set and $V - S'$ is connected. Hence S is a minimum split restrained geodetic set. Also for all $x, y \in \langle V(K_2 \boxtimes T_{m,n}) - S\rangle$, it follows that $\langle V(K_2 \boxtimes T_{m,n}) - S\rangle$ is disconnected with no isolated vertices. Thus $g_{sr}(K_2 \boxtimes T_{m,n}) = 8$.

4. Adding an End-Edge

For an edge $e = (u, v)$ of a graph G with $\deg(u) = 1$ and $\deg(v) > 1$, we call e an end-edge and u an end –vertex.

Theorem 4.1 Let G' be the graph obtained by adding an end-edge (x, y) to a cycle each $C_n = G$ of order n >3, with $x \in G$ and $y \notin G$ then $g_{sr}(K_2 \boxtimes G') = 6$.
Proof: Let $K_2 \boxtimes G'$ be the graph formed from two copies G_1' and G_2' of G'.Let $U = \{(a_1, u_1), (a_1, u_2), \dots, (a_1, u_{n-1}), (a_1, x), (a_1, y)\} \in V(G_1')$,
$W\{(b_1, w_1), (b_1, w_2), \dots, (b_1, w_{n-1}), (b_1, x), (b_1, y)\} \in V(G_2')$ and $V(K_2 \boxtimes G') = U \cup W$.
We have the following cases.
Case 1. For even cycle
Let $S = \{(a_1, u_i), (a_1, y), (b_1, w_i), (b_1, y), (a_1, x), (b_1, x)\}$ be the split restrained geodetic set ,where $d\{(a_1, u_i), (a_1, y)\} = d\{(b_1, w_i), (b_1, y)\} = \operatorname{diam}(K_2 \boxtimes G')$, such that $I[S] = V(K_2 \boxtimes G')$, $\langle V(K_2 \boxtimes G') - S\rangle$ has more than one component with no isolated vertices. Suppose $S' = \{(a_1, u_i), (a_1, y), (b_1, w_i), (b_1, y)\}$,$|S'| < |S|$ is a geodetic set where $\langle V(K_2 \boxtimes G') - S'\rangle$ is connected. Hence clearly S is the minimum split restrained geodetic set. There fore $g_{sr}(K_2 \boxtimes G') = 6$.
Case 2. For odd cycle
Let $S = \{(a_1, y), (b_1, y), (a_1, u_i), (a_1, u_j), (b_1, w_i), (b_1, w_j)\}$,where $d\{(a_1, y), (a_1, u_i)\} = 2$, $d\{(a_1, u_i), (a_1, u_j)\} = \left\lfloor \frac{n}{2} \right\rfloor$ similarly $d\{(b_1, y), (b_1, w_i)\} = 2$, $d\{(b_1, w_i), (b_1, w_j)\} = \left\lfloor \frac{n}{2} \right\rfloor$. Clearly $I[S] = V(K_2 \boxtimes G')$, since $\langle V(K_2 \boxtimes G') - S\rangle$ has more than one component with no isolated vertices, S is the minimum split restrained geodetic set of $K_2 \boxtimes G'$. Thus $g_{sr}(K_2 \boxtimes G') = 6$.

5. Lexicographic Product

The composition $G = G_1[G_2]$ has $V = V_1 \times V_2$ as its vertex set, and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever [u_1 adjacent to v_1] or [$u_1 = v_1$ and u_2 adjacent to v_2].

Theorem 5.1 For any cycle C_n of order n >6, $g_{sr}(K_2[C_n]) = n + 2$.
Proof: Let $G = K_2[C_n]$ be the graph formed from two copies of G_1 and G_2 of C_n. Let $U = \{u_1, u_2, \dots, u_n\} \in V(G_1)$, $W = \{w_1, w_2, \dots, w_n\} \in V(G_2)$ and $V = U \cup W$.
Consider $S = H_1 \cup H_2$ be the minimum split restrained geodetic set of G, such that $H_1 = \{u_i, u_j, w_i, w_j\}$ be the set of vertices where $\{(u_i, u_j), (w_i, w_j)\} \notin E(K_2[C_n])$ and $d(u_i, u_j) = d(w_i, w_j) = \operatorname{diam}(K_2[C_n])$, $I[H_1] = V(K_2[C_n])$ and $H_2 = \{u_2, u_4, \dots, u_{n-1}, u_n, w_2, w_4, \dots, w_{n-2}, w_{n-1}, w_n \subseteq V(G) - H_1$, $|H_2 = n - 2|$. Hence $|S| = |H_1 \cup H_2| = 4 + n - 2 = n + 2$. There fore $g_{sr}(K_2[C_n]) = n + 2$.

Theorem 5.2 G' be the graph obtained by adding an end edge (x, y) to cycle $C_n = G$ of order n > 5, with $x \in G$ and $y \notin G$. Then $g_{sr}(K_2[G']) = n + 3$.
Proof: Let $K_2[G']$ be the graph formed fromtwo copies G_1' and G_2' of G'. Let $U = \{u_1, u_2, \dots, u_{n+1}\} \in V(G_1')$, $W = \{w_1, w_2, \dots, w_{n+1}\} \in V(G_2')$ such that $V(K_2[G']) = U \cup W$. Consider $S = H_1 \cup H_2$, where $H_1 = \{u_i, u_j, w_i, w_j\}$ such that $(u_i, u_j), (w_i, w_j) \notin E(K_2[G'])$ and these vertices are not formed by the end-vertex of G', $I[H_1] = V(K_2[G'])$ and $H_2 = \{u_1, u_2, \dots, u_{n+1}\} \subseteq V(K_2[G']) - H_1$, $|H_2| = n - 1$. Such that the induced sub graph $\langle V(K_2[G']) - S\rangle$ is disconnected with no isolated vertices. Thus S is the minimum

split restrained geodetic set of $K_2[G']$. Hence $|S| = |H_1 \cup H_2| = 4 + n - 1 = n + 3$. Therefore $g_{sr}(K_2[G']) = n + 3$.

6. References

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