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Light Bending and Stability Analysis in Weyl Conformal Gravity

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Abstract
Employing a recent proposed method by Rindler-Ishak, the bending of light is calculated to second order, which reveals the exact Schwarzschild terms as well as the effects arising out of the parameters of the Mannheim-Kazanas solution of Weyl conformal gravity. Next using the approach of autonomous dynamical system, the stability of circular motion of massive and massless particles in the motion has been investigated. The main results justify why Rindler-Ishak method has to be preferred over text book methods when asymptotically non flat spacetime has been concerned. It turns out that there is no stable circular radius for light motion in the considered solution.

I. Introduction

Classical Einstein's general relativity theory (EGRT) has been nicely confirmed within the weak field regime of solar gravity and binary pulsars. Certainly it continues to remain as one of the cornerstone of modern physics. However, it must be said in all fairness that within the ambit of classical EGRT, there still exists serious challenges. For instance, observations of flat rotation curves in the galactic halo still lack a universally accepted satisfactory explanation. The most widely accepted explanation, based on EGRT, hypothesizes that almost every galaxy hosts a large amount of nonluminous matter, the so called gravitational dark matter [1], consisting of unknown particles not included in the particle standard model, forming a halo around the galaxy. This dark matter provides the needed gravitational field and the required mass to match the observed galactic flat rotation curves. The exact nature of either the dark matter or dark energy is yet far too unknown except that the former has to be attractive on the galactic scale and the later repulsive on the cosmological scale. These requirements lead us to explore alternative theories, such as Modified Newtonian Dynamics (MOND) [2.3], braneworld model [4], scalar model [5]. A prominent candidate is Weyl Conformal Gravity that keep intact the weak field successes of EGRT and potentially resolves the dark matter/dark energy problem without hypothesizing them. By itself, Weyl Conformal Gravity seems quite as elegant as other theories because it is based on the conformal invariance with an associated 15- parameter largest symmetry group. An interesting solution in this theory is the Mannheim-Kazanas (MK) metric [6] that has successfully interpreted galactic flat rotation curves without invoking the elusive dark matter.

The MK solution contains two arbitrary parameters γ and κ that are expected to play prominent roles on the galactic halo and cosmological scale respectively. The fit to galactic flat rotation $\gamma > 0$ and a numerical value of the order of inverse Hubble radius [6c]. Therefore, it is expected that $\gamma > 0$ would lead to an enhanced bending of light γR due to the non-luminous halo over the usual Schwarzschild one due to luminous galactic mass. This enhancement is consistent with the observed attractive halo gravity.

Interestingly κ cancels out of the light path equation and one might be led to believe that κ has no role in light bending. The text book methods of calculation of bending using that path equation would then lead to diminished bending $-\gamma R$, which conflicts with observation.

The purpose of the present article is to justify why Rindler-Ishak method [7] has to be preferred over text book methods when asymptotically non flat spacetimes are concerned. The method not only gives the needed enhanced bending γR due to the attractive halo gravity but also leads to a new additional effect right in the first order bending of light. Next, we proceed to investigate the stability of circular orbits of massive and massless particle via approach of dynamical systems, which also suggests that $\gamma > 0$. All the results are summarized at the end.

II. Geodesic Equation

The Weyl action is given by

$$S = \alpha_g \int d^4x(-g)^{-1/2} C_{\mu\vartheta\gamma\delta} C^{\mu\vartheta\gamma\delta} \tag{1}$$

where $C_{\mu\vartheta\gamma\delta}$ is the Weyl tensor, α_g is the dimensionless gravitational coupling constant. Variation of the action with respect to the metric gives the field equations

$$(-g)^{-1/2}\, \delta S/\delta g_{\mu\vartheta} = -2\alpha_g W^{\mu\vartheta} = -T^{\mu\vartheta}/2 \tag{2}$$

where $W^{\mu\vartheta}$ is given by

$$W^{\mu\vartheta} = \frac{g^{\mu\vartheta}(R^{\alpha}_{\alpha})^{;\beta}_{;\beta}}{2} + R^{\mu\vartheta;\beta}_{;\beta} - R^{\mu\beta;\vartheta}_{;\beta} - 2R^{\mu\beta}R^{\vartheta}_{\beta} + \frac{g^{\mu\vartheta}R_{\alpha\beta}R^{\alpha\beta}}{2} - \frac{2g^{\mu\vartheta}(R^{\alpha}_{\alpha})^{;\beta}_{;\beta}}{3} + \frac{2(R^{\alpha}_{\alpha})^{;\mu;\vartheta}}{3} + \frac{2R^{\alpha}_{\alpha}R^{\mu\vartheta}}{3} - g^{\mu\vartheta}(R^{\alpha}_{\alpha})^2/6\,. \tag{3}$$

We can immediately confirm that the Schwarzschild $R^{\mu\vartheta} = 0$ solution is indeed an exterior solution to the theory so that the success of solar system tests are already embedded in to Weyl gravity.

An interesting solution of the field equation is MK metric given by [6] (vacuum speed of light $c_0 = 1$, unless restored):

$$d\tau^2 = B(r)dt^2 - \frac{1}{B(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2),$$

$$B(r) = 1 - \frac{2M}{r} + \gamma r - kr^2 \tag{4}$$

where k and γ are constants. The numerical value of $k \approx 10^{-56} cm^{-2}$ and $\gamma \approx 3.06 \times 10^{-30} cm^{-1}$ as determined from the fit to galactic flat rotation curve data [6c].

Using $= \frac{1}{r}$, we get the following path equation for a test particle m_0 on the equatorial plane $\theta = \pi/2$ as follows:

$$\frac{d^2u}{d\varphi^2} = -u + 3Mu^2 - \frac{\gamma}{2} + \frac{M}{h^2} + \frac{1}{2h^2u^2}\left(\gamma - \frac{2k}{u}\right); \tag{5}$$

where $h = \frac{U_0}{m_0}$, the angular momentum per unit test mass. Due to conformal invariance of the theory, geodesics for massive particles would in general depend on the conformal factor $\Omega^2(x)$, but here a fixed conformal frame has been considered not the other conformal variants of the metric. For photon, $m_0 = 0$ implies that $h \to \infty$ and one ends up with the conformally invariant equation but without k making its appearance:

$$\frac{d^2u}{d\varphi^2} = -u + 3Mu^2 - \frac{\gamma}{2} \tag{6}$$

In the Schwarzschild-de Sitter (SdS) metric, such a cancellation has been noted for long [8]. The cosmological constant ∧ does not appear in the light path differential equation and hence it is believed that

$\wedge$ does not influence light bending [9-13]. Here we find that the cancellation of k occurs despite the presence of γ in the metric. Exactly, as in the SdS case, one would now expect that the bending of light would be the same, to any order, with or without k. However, Rindler and Ishak [7] have shown that this need not be the case. They argued that "the differential equation and its integral are only half of the story. The other half is the metric itself, which determines the actual observations that can be made on the r, φ orbit equation. When that is taken in to account a quite different picture emerges: the cosmological constant $\wedge$ does contribute to the observed bending of light". This argument also finds support in the fact that the effect due to k must appear via the consideration of the full metric in the calculation of physically observable effects, such as the bending of light rays.

III. Bending of light rays

Although the MK metric is different from the SdS metric, it will be shown that the influence of k still appears in the bending provided the calculations are done using the Rindler-Ishak method. Thus the light path equation in zeroth order is

$$\frac{d^2u_0}{d\varphi^2} + u_0 = 0 \tag{7}$$

where $u_0 = \overline{u_0} + \frac{\gamma}{2}$. The solution of Eq.(7) is a straight line $u_0 = \frac{cos\varphi}{R}$ parallel to x-axis, where R is the distance of closest approach to the origin (just perpendicular distance). Following the method of small perturbations [14], we derive the solution up to second order in M^2 as

$$u = \frac{1}{r} = \frac{cos\varphi}{R} + \frac{3M}{2R^2}\left(1 - \frac{1}{3}cos2\varphi\right) + \frac{3M^2}{16R^3}(20\varphi sin\varphi + cos3\varphi + 22cos\varphi) \tag{8}$$

Assuming that $u \to 0$ for $\varphi \to \frac{\pi}{2} - \varphi$, the solution can be rewritten as,

$$\frac{1}{r} = \frac{sin\varphi}{R} + \frac{3M}{2R^2}\left(1 + \frac{1}{3}cos2\varphi\right) + \frac{3M^2}{16R^3}[10(\pi - 2\varphi)cos\varphi - sin3\varphi + 22sin\varphi] \tag{9}$$

The method of Rindler and Ishak [7] is based on the invariant formula for the cosine of the angle ψ between two coordinate directions d and δ such that

$$cos\psi = \frac{g_{ij}d^i\delta^j}{(g_{ij}d^id^j)^{1/2}(g_{ij}\delta^i\delta^j)^{1/2}} \tag{10}$$

Differentiating Eq.(9) with respect to φ, and denoting $\frac{dr}{d\varphi} = A(r, \varphi)$, we get

$$A(r,\varphi) = \frac{r^2}{16R^3}[2(16MRsin\varphi - 3M^2 - 8R^2)cos\varphi + 3M^2\{3cos3\varphi + 10(\pi - 2\varphi)sin\varphi\}]$$

(11)

Eq.(10) then yields

$$cos\psi = \frac{|A|}{[A^2 + B(r)r^2]^{1/2}} \tag{12}$$

or in a more convenient form

$$tan\psi = \frac{B^{1/2}r}{|A|} \tag{13}$$

when $\varphi = 0$, we get from Eq.(12)

$$r = \frac{16R^3(3M^2 - 16R^2)}{(32MR + 30\pi M^2)^2} \tag{15}$$

The one sided bending angle is given by $\epsilon = \psi - \varphi$and let us calculate $\epsilon = \psi = \psi_0$ when $\varphi = 0$. Putting the value from Eqs.(14),(15) in Eq. (13), we get

$$tan\psi \cong \frac{M(15\pi M + 16R)}{8R^2}\left(1 + \frac{3M^2}{16R^2}\right)\left[1 - \frac{M^2(15\pi M + 16R)}{8R^3} + \frac{4\gamma R^3}{15\pi M^2 + !6MR} - \frac{32kR^6}{3M^2(15\pi M + 16R)^2}\right] \tag{16}$$

Expanding in powers of M in second order for a small angle ψ_0 (or, $tan\psi_0 \cong \psi_0$), we obtain the following expression:

$$\psi_0 \cong \frac{2M}{R}\left(1+\frac{15\pi M}{16R}-\frac{kR^4}{24M^2}+\frac{\gamma R^2}{4M}+\frac{3\gamma M}{64}\right) \quad (17)$$

The roles of γ and k are quite evident in the above. It is found that the contribution is exactly same to the bending due to k as in Ref.[7] as well as the exact first and second order Schwarzschild terms in $\frac{M}{R}$ derived by Bodenner and Will [14]. The result shows that the effect of k does influence the bending although the trajectory equation (6) does not contain k. For $k = 0$ it is found that the total Schwarzschild bending is enhanced by a halo contribution γR. The result is quite consistent with the attractive halo gravity.
Next, an entirely new effect has been noticed: The last term in Eq.(17) contains a coupling between γ and M giving rise to a dimensionless factor $\frac{3\gamma M}{64}$ that adds a constant to unity. This leads to a Weyl gravity modification to the observed first order bending itself. To get an idea of the magnitude involved, let α arcsec be the total first order bending in the solar gravity. Then

$$\alpha = \frac{4M}{R}\left(\frac{1+\gamma_0}{2}\right)\left(1+\frac{3\gamma M}{64}\right) \rightarrow \gamma = \frac{64}{3M}\left(\frac{\alpha R}{4M}\frac{2}{1+\gamma_0}-1\right) \quad (18)$$

where γ_0 is the first post-Newtonian parameter. The prediction from EGRT gives $\alpha = \frac{4M}{R} = 1.7504$ arcsec. Putting this in the above, we get

$$\gamma = \frac{64}{3M}\left(\frac{2}{1+\gamma_0}-1\right) \quad (19)$$

Assuming $\gamma \approx 10^{-30}cm^{-1}$ and the solar mass to be $M \cong 3\times10^5 cm$, we get a value $\gamma_0 \approx 1 - 1.5\times10^{-27}$. Currently estimated value is $\gamma_0 = 2\times(0.99992 \pm 0.00014) - 1$ [15], which is close to 1 up to an accuracy of 10^{-4}. Note that the second post Newtonian correction demands an accuracy of the order of 10^{-6} but its measurement is already beset with some technical difficulties though not unsurmountable (See Refs [14-16]). Naturally, the accuracy of 10^{-27} demanded by the matching of γ from the rotation curve data with that from solar gravity is technologically unattainable even in far future.

IV. Stability of Circular Orbits via Hamiltonian Approach

Analysis of dynamical system involves converting this second order equation in to two first order equations. For this purpose, we set the notation.

$$u = x, y = \dot{x} = \frac{dx}{d\varphi} \quad (20)$$

To reduce Eq.(5) into a pair of first order autonomous system in the (x, y) phase plane

$$\dot{x} = X(x,y) = y \quad (21)$$

$$\dot{y} = Y(x,y) = a + bx + cx^2 + dx^{-2} + ex^{-3} \quad (22)$$

where

$$a = \frac{M}{h^2} - \frac{\gamma}{2}, b = -1, c = 3M, d = \frac{\gamma}{2h^2}, e = -\frac{k}{h^2} \quad (23)$$

(a) Massive particle motion

The equilibrium points are given by $\dot{x} = 0$ and $\dot{y} = 0$. The equation $\dot{x} = 0$ gives $r = R = constant$, while $\dot{y} = 0$ gives

$$h^2 = -\frac{2MR^2 + R^4(\gamma - 2kR)}{R(2+\gamma R) - 6M} \quad (24)$$

The autonomous system (21), (22) can be phrased as a Hamiltonian system as follow

$$\frac{\partial H}{\partial x} = -Y(x,y) = -(a + bx + cx^2 + dx^{-2} + ex^{-3}) \quad (25)$$

$$\frac{\partial H}{\partial y} = -X(x,y) = y \tag{26}$$

The necessary and sufficient condition for the system (25), (26) to be Hamiltonian system, namely, $\frac{\partial X}{\partial x'} + \frac{\partial Y}{\partial y'} = 0$, is fulfilled for all x and y. Moreover, $\frac{dH}{d\varphi} = 0$ and therefore $H(x',y') = constant$ (independent of φ). Intergrating Eqs.(25) , (26) , we get

$$H(x,y) = -\left(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 - dx^{-1} - \frac{e}{2}x^{-2}\right) + u(y) \tag{27}$$

$$H(x,y) = \frac{1}{2}y^2 + v(x) \tag{28}$$

Where $u(y)$ and $v(x)$ are arbitrary functions subject to the consistency of Eqs.(27) and (28). These two equations will match only if $u(y) = \frac{1}{2}y^2 + C$ (29)

$$v(x) = -\left(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 - dx^{-1} - \frac{e}{2}x^{-2}\right) + E \tag{30}$$

Where C, E are arbitrary constants. The family of Hamiltonian path on the phase plane are given by

$$H(x,y) = \frac{1}{2}y^2 - \left(ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 - dx^{-1} - \frac{e}{2}x^{-2}\right) + G \tag{31}$$

Where G is a parameter. It follows that

$$\frac{\partial^2 H}{\partial x^2} = -(b + 2cx - 2dx^{-3} - 3ex^{-4}) \tag{32}$$

$$\frac{\partial^2 H}{\partial y^2} = 1 \tag{33}$$

$$\frac{\partial^2 H}{\partial x \partial y} = 0 \tag{34}$$

As before, the equilibrium points occur when $X = 0$ and $Y = 0$, which give the values $r = R = constant$ and h^2 as in Eq.(24). The quantity determining ability is [17]

$$\frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2} - \left(\frac{\partial^2 H}{\partial x \partial y}\right)^2 \tag{35}$$

Putting the value of h^2, we get at the equilibrium points the following expression

$$q_0 = 1 - \frac{6M}{R} + \frac{R(3kR-\gamma)[R(2+\gamma R)-6M]}{R^2(2kR-\gamma)-2M} \tag{36}$$

We can also take $k = 0$, then negative values of γ would lead to stable radius for some values of R, while unstable radius for some other values of R. There is no physical reason why such behaviour should occur. Therefore we conclude that negative values of γ should be ruled out.

(b) Massless particle motion

Light motion occurs in circular orbits defined by $R(2+\gamma R) - 6M = 0$, hence we must have $h^2 \to \infty$ so that $d = e = 0$ but $\gamma \neq 0$. The equilibrium points are given by $\dot{x} = 0, \dot{y} = 0$, which for light yield $\left(\frac{-b+\sqrt{b^2-4ac}}{2a}, 0\right)$ and $\left(\frac{-b-\sqrt{b^2-4ac}}{2a}, 0\right)$. To locate these points on the real phase plane (x,y), we must have $\alpha^2 \equiv b^2 - 4ac = 1 + 6\gamma M \geq 0$. For $\alpha^2 = 0 \to \gamma = -\frac{1}{6M}$, so the equilibrium points reduce to one single point given by P: $(\frac{1}{6M}, 0)$. For $\alpha^2 > 0$, or $\gamma > -\frac{1}{6M}$, there are two distinct equilibrium points $Q_{\mp}$: $\left(\frac{1+\alpha}{6M}, 0\right)$ where $\alpha = \mp\sqrt{1+6\gamma M}$. Thus $Q_{\mp}$ correspond to two γ-dependent light radii $R_{\mp} = \frac{6M}{1\pm\sqrt{1+6\gamma M}}$, which expand as follows

$$R_{+} = \frac{-1+\sqrt{1+6M\gamma}}{\gamma} \cong 3M + O(\gamma) \tag{37}$$

$$R_- = \frac{-1+\sqrt{1+6M\gamma}}{\gamma} \cong -3M + \frac{2}{\gamma} + O(\gamma) \quad (38)$$

We have from Eq,(36)

$$q_{0\pm} = 1 - \frac{6M}{R_\pm} \quad (39)$$

Which yields $q_{0+} = -\sqrt{1+6M\gamma} < 0$ leading to unstable radii $R = R_+$ whatever be the value of γ. Further $q_{0-} = -\sqrt{1+6M\gamma} > 0$ showing that $R = R_-$ are stable radii. The basic constraint (reality condition) however is that $\gamma > -\frac{1}{6M}$ and in order to be compatible with the stability criterion for massive particle motion, we can conclude that $\gamma > 0$.

V. Summary

The new result obtained in the article are the following:

(i) Since the constant k cancelled out of the light path equation, one could think that the constant would not affect the bending of light rays. Using the new method proposed by Rindler and Ishak we have shown that this is not the case. The bending beyond Schwarzschild terms comes as a combination of the two constants γ and k of the MK solution. The implication of this result is that one can have increased or decreased bending on the sign of the combination $-\frac{kR^4}{24M^2} + \frac{\gamma R^2}{4M}$. This flexibility can be tuned to the actual physical observations about attractive and repulsive gravity.

(ii) There is a new first order effect $\frac{3\gamma M}{64}$ that adds a constant to unity in the bracket in Eq. (17). However, measurement of this demand a precision level of the order of 10^{-27}, which is clearly out of question even in the far future.

(iii) A very interesting result pertains to the well discussed [18,19] special case $B(r) = 1 + \gamma r$, obtained by setting $M = 0, k = 0$ in the solution (4). This case is expected to describe purely the galactic halo gravity. Looking at Eq.(17), we see that it gives a total light deflection $2\psi_0 = \gamma R$, which would mean a deflection toward the source provided $\gamma > 0$.Recalling that the fit to rotation curve data does give $\gamma > 0$. We immediately see that the special solution consistently explains the observed attractive halo gravity. We wish to emphasize that this consistent result has come about because of the use of Rindler-Ishak method. We had used the standard text book methods for the calculation of bending [19], we would end up with a deflection away from the source $2\psi_0 = -\gamma R$, which would be in direct conflict with attractive halo gravity. The fact that one can obtain the physically correct deflection in the halo justifies that the Rindler-Ishak method has to be preferred over standard methods when asymptotically non-flat spacetimes are concerned. When the spacetime is flat all methods lead to the same result.

(iv) The dynamical system approach is elegant and the requirement of stability of massive particle circular motion revealed the negative values of γ should be ruled out. With regard to stability of light circular motion , we found that of the two light radii R_- is stable provided that $\gamma > -\frac{1}{6M}$. This reality condition does not exclude the possibility of $\gamma > 0$. Results from (iii) suggest that we should take $\gamma > 0$, which then rules out the radii $R = R_-$ because they become negative. The other radii $R = R_+$ has already been shown to be unstable. We conclude that there cannot be any stable circular light orbit in the MK solution, just like in the Schwazschild case.

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