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Number of Pythagorean Triples that Contain a Given Number

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Abstract

Background: The study of the generation of Pythagorean triples is a topic that has been examined extensively in the past. One of the prominent formulae, which Euclid introduced in a parametric form, can produce all primitive Pythagorean triples by utilising two integers.

Problem Statement: To find a Pythagorean triple that includes a given number necessitates a distinct representation of the given number. Determining the number of potential Pythagorean triples becomes increasingly arduous since it mandates the expression of all factors of the given number in a specific format, which can be a laborious undertaking.

Aim: The aim of this article is to create a set of constraints that generates the number of Pythagorean triples that contain a given number.

Result: We have two constraint-based sets for when the given number is the hypotenuse and when it is a side other than the hypotenuse. It also will generate all Pythagorean triples with a parametric formula. It is apparent that this method is much simpler and more efficient to calculate the number of possible Pythagorean triples of a given number than Euclid's formula or other contemporary methods.

Keywords: Euclid’s formula, Pythagorean triples, Number of Pythagorean triples, Parametric formula

1. Introduction

Pythagorean Triple

A Pythagorean triple is a set of three integers {a, b, c} such that they follow the Pythagorean theorem i.e., $a^2 + b^2 = c^2$.

Geometrically the three sides of a right-angled triangle, if integer, will constitute a Pythagorean triple. We can differentiate between the elements of the triple by categorizing them as either being the hypotenuse i.e., **c** or as being a side other than the hypotenuse i.e., **a**, **b**.

Some questions

Some questions that come to mind about Pythagorean triples are:

1. Generate a Pythagorean triple
2. Generate a Pythagorean triple for a given number
3. Generate all Pythagorean triples that contain a given number.
4. Generate the number of Pythagorean triples that contain a given number.

Euclid's formula

Euclid's formula generates Pythagorean triples using a parametric form using two arbitrary integers. [3]

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Consider two integers **m** and **n** such that $m > n > 0$.
The Pythagorean triples {a, b, c} can be represented as
$a = m^2 - n^2;\ b = 2mn;\ c = m^2 + n^2$
(1a)
Euclid's formula provides a solution for the first two of the four problems mentioned in the previous sub-section.
Let $m = 7, n = 3$ then
$a = 7^2 - 3^2 = 40\ ;\ b = 2 \times 7 \times 3 = 42\ ;\ c = 7^2 + 3^2 = 58$
Therefore, $\{a, b, c\} = \{40, 42, 58\}$.

To generate a triple with a specific number, one must find values of **m** and **n** such that the given number can be represented as an element of the triple.
Let the given number be 11 and because $11 = 6^2 - 5^2$
We can find a Pythagorean triple that contains 11 from Equation (1a) by putting
$m = 6; n = 5$.
We get $\{a, b, c\} = \{11, 60, 61\}$.

As for our third question, this formula cannot generate all the Pythagorean triples, but addition of a variable **k** is a possible solution.
The Pythagorean triples {a, b, c} can be represented as
$a = k \cdot (m^2 - n^2);\ b = k \cdot (2mn);\ c = k \cdot (m^2 + n^2)$
(1b)
To generate all the Pythagorean triples for a given number, one would have to represent the given number and its factors as each of **a**, **b,** and **c**.
The process of finding the number of Pythagorean triples for a given number (4^{th} question) is even more tedious. One would have to count the number of times one can represent the given number and its factors as any of **a**, **b,** or **c** from Equation (1a).

Aims and Objectives
Aim: To create a set of constraints that generate the number of possible Pythagorean triples that contain a given number.

Objectives:
- The result should be fundamentally connected to Euclid's formula.
- Must obtain separate sets of constraints for when the given number is the hypotenuse and when it is not.

2. Results
Given number is not the hypotenuse
For a given number **T**
Consider a set

$$set(A) = \{x : x \in Z^+, x < T, x \equiv T(mod2), x = \frac{T^2}{p}, p \equiv x(mod2)\}$$

(2a)

Then the number of Pythagorean triples that contain **T**, such that **T** is not the hypotenuse, is equal to the number of elements of set(A) i.e., n(A).

Given number is the hypotenuse

For a given number **T**

Consider a set

$$set(B) = \{y : y \in Z^+, y < \frac{T(\sqrt{2}-1)}{\sqrt{2}}, y = \frac{2T}{q^2+1}, q \in Z^+\}$$

(2b)

Then the number of Pythagorean triples that contain **T**, such that **T** is the hypotenuse, is equal to the number of elements of set(B) i.e., n(B).

Number of Pythagorean Triples

With the same sets as Equation (2a) and Equation (2b), the number of possible Pythagorean triples that contain a given number is simply the sum of the number of elements of set(A) and set(B) i.e., $n(A) + n(B)$.

Let us look at an example.

Let the given number be $T = 75$.

Then from Equation (2a),

$set(A) = \{1, 3, 5, 9, 15, 25, 45\}$

And from Equation (2b),

$set(B) = \{3, 15\}$

The number of Pythagorean triples that contain 75 as a side other than the hypotenuse is equal to $n(A) = 7$.

The number of Pythagorean triples that contain 75 as the hypotenuse is equal to $n(B) = 2$.

The total number of possible Pythagorean triples that contain 75 is equal to

$n(A) + n(B) = 9$.

Generating a Pythagorean triple

The formula to generate the Pythagorean triples for a given number T

Using variable $x \in A$.

$$a = T\ ;\ b = \frac{T^2 - x^2}{2x}\ ;\ c = \frac{T^2 + x^2}{2x}$$

(3a)

E.g., $T = 75, x = 9$

$$a = 75\ ;\ b = \frac{75^2 - 9^2}{2 \times 9} = 308;\ c = \frac{75^2 + 9^2}{2 \times 9} = 317$$

We get $\{a, b, c\} = \{75, 308, 317\}$

Using variable $y \in B$ is

$$a = \sqrt{2yT - y^2}\ ;\ b = T - y\ ;\ c = T$$

(3b)
E.g., $T = 75, y = 3$
$a = \sqrt{2 \times 3 \times 75 - 3^2} = 21 \; ; \; b = 75 - 3 = 72 \; ; \; c = 75$
We get $\{a, b, c\} = \{21, 72, 75\}$

3. Discussion

Derivation of Equation (3a)
Consider a Pythagorean triple a, b, c.
Let us assume that $c = b + x$.
By applying the Pythagorean theorem, we get
$a^2 + b^2 = (b + x)^2$
$a^2 + b^2 = b^2 + 2bx + x^2$
$a^2 = 2bx + x^2$
Therefore, we have
$$b = \frac{a^2 - x^2}{2x}$$
Because $c = b + x$
$$c = \frac{a^2 + x^2}{2x}$$
Thus, we have
$$a = a \; ; \; b = \frac{a^2 - x^2}{2x} \; ; \; c = \frac{a^2 + x^2}{2x}$$
(4a)
If the given number **T** is a side other than the hypotenuse, then we can assume $a = T$. By making this substitution in Equation (4a), we get Equation (3a).

Derivation of Equation (3b)
From Equation (4a) for convenience, we will switch the variable from **x** to **y**.
In doing so, we get
$$c = \frac{a^2 + y^2}{2y}$$
$a^2 = 2yc - y^2$
Therefore, we have
$a = \sqrt{2yC - y^2}$
And we had assumed that $b = c - y$.
Thus, we have
$a = \sqrt{2yC - y^2} \; ; \; b = c - y \; ; \; c = c$
(4b)
If the given number **T** is the hypotenuse, then we can assume $c = T$. By making this substitution in Equation (4b), we get Equation (3b).

Constraints on x

Let us analyse Equation (3a).

We have

$$b = \frac{T^2 - x^2}{2x}$$

Because all the variables are assumed to belong to the set of positive integers, we have $b > 0$, which implies $T^2 - x^2 > 0$.

Because **T** and **x** are positive integers as well, we can say

$$x < T$$

(5a)

Also, we have

$$\frac{T^2 - x^2}{2x} \in Z^+$$

We can manipulate this expression by displacing the denominator to the RHS. This can be done by moving either the 2 or the **x**.

Let us first move the **x** to the RHS. In doing so, we get

$$\frac{T^2 - x^2}{2} \in Z^+$$

This implies that both $T^2$and x^2must simultaneously be either odd or even.

$$x^2 \equiv T^2 (mod 2)$$

$$x \equiv T (mod 2)$$

(5b)

Next, we will move the 2 to the RHS. In doing so, we get

$$\frac{T^2 - x^2}{x} = \frac{T^2}{x} - x \in Z^+$$

Because x is a positive integer, $\frac{T^2}{x}$ must also be a positive integer. Hence, we can conclude,

$$x = \frac{T^2}{p} \; ; \; p \in Z^+$$

(5c)

If we substitute the value of $T^2$from Equation (5c) into Equation (3a) we get

$$b = \frac{p - x}{2}$$

In similar line of logic to the derivation of Equation (5c), we can infer

$$x \equiv p (mod 2)$$

(5d)

By combining all the equations derived in this sub-section as constraints for a set, we will reach the conclusion of Equation (2a).

Constraints on y

Let us analyse Equation (3b)

We have

$$a = \sqrt{2yT - y^2}$$

Because $a \in Z^+$, we can say $\sqrt{2yT - y^2} \in Z^+$.
By squaring both the sides we get
$2yT - y^2 = r^2$, for some positive integer **r**.
This can be re-written as $y(2T - y) = r^2$.
Because the RHS is a perfect square, $2T - y$, must be a multiple of **y** and another perfect square.
$2T - y = yq^2$
Therefore, we can say

$$y = \frac{2T}{q^2 + 1}\ ;\ q \in Z^+$$

(6a)
The next point we must consider is that **a** and **b** are interchangeable variables. Without a specific relation between them, we will generate overlapping pairs, meaning non-unique pairs will be counted twice.
To avoid this, we can simply assume a relation between them.
Let $b > a$. From Equation (3b), we can say
$T - y > \sqrt{2yT - y^2}$
By squaring both sides and bringing all the terms together, we get
$2y^2 - 4yT + T^2 > 0$
By solving this inequality, we get
$y > T + \frac{T}{\sqrt{2}}$, or $y < T - \frac{T}{\sqrt{2}}$
Because the prior violates the assumption $b = c - y$, we shall proceed with the latter.
Thus, we have

$$y < \frac{T\left(\sqrt{2} - 1\right)}{\sqrt{2}}$$

(6b)
By combining the equations derived in this sub-section as constraints for a set, we will reach the conclusion of Equation (2b).

4. Conclusions

Verification

I shall now demonstrate the constraint-based solution (step-by-step).
Let us take an example. Let $T = 200$.

To calculate the number of Pythagorean triples that contain 200 as a side other than the hypotenuse, we must find the elements of $set(A)$.
$A = x : x \in Z^+$
By applying constraint from Equation (5a)
$A = \{1, 2, 3, 4, 5, \dots\dots\dots.., 199\}$
By applying the constraint from Equation (5b)
$A = \{2, 4, 6, 8, \dots\dots\dots\dots., 198\}$
By applying the constraint from Equation (5c)
$A = \{2, 4, 8, 10, 16, 20, 32, 40, 50, 64, 80, 100, 160\}$
By applying the constraint from Equation (5d)

$A = \{2, 4, 8, 10, 16, 20, 32, 40, 50, 80, 100, 160\}$
The number of Pythagorean triples that contain 200 as a side other than the hypotenuse is
$n(A) = 12$.

To calculate the number of Pythagorean triples that contain 200 as the hypotenuse, we need to find the elements of $set(B)$.
$B = y{:}\, y \in Z^+$
By applying the constraint from Equation (6b)
$B = \{1, 2, 3, 4, \dots \dots \dots .., 58\}$
By applying the constraint from Equation (6a)
$B = \{8, 40\}$
The number of Pythagorean triples that contain 200 as the hypotenuse is
$n(B) = 2$.

The total number of possible Pythagorean triples that contain 200 is
$n(A) + n(B) = 14$.

Conclusion

The principal aim of this paper was to create a set of constraints that generate the number of possible Pythagorean triples that contain a given number. It is apparent that we have done so from Equation (2a) and Equation (2b).

In view of the objectives, we can say that the results line well with the first. The result of Equation (3a) can be viewed as a reinterpretation of improved Euclid's formula. It can be derived from Equation (1b) by making the following logically sound changes in the variables,
$m \to T\ ;\ n \to x\ ;\ k \to \frac{1}{2x}$.

The second objective can also be deemed satisfied. This is because, we have obtained two different results each specifying if the given number is the hypotenuse or a side other than the hypotenuse.

References

1. R. A. Beauregard, E. R. Suryanarayan, Parametric representation of primitive Pythagorean triples, Proofs Without Words: More Exercises in Visual Thinking II (2000) 120–120.
2. D. Romik, The dynamics of Pythagorean triples, Trans. Amer. Math. Soc (2008).
3. D. E. Joyce, Euclid's Elements, in: Book X, Vol. Proposition 29, Clark University, 1997.
4. E. Maor, The Pythagorean Theorem, appendix b Edition, Princeton University Press, 2007.