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The Petersen Graph and its Generalizations

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Abstract

The Petersen graph, arecurring mathematical object, is the subject of this paper.The Petersen graph, which bears the name of the Danish mathematician Julius Petersen, who first proposed it in 1898, is one of the most fascinating and well-known graphs in graph theory.The Petersen graph is a compact yet extremely intriguing graph with a number of distinctive characteristics and uses. Some properties and significance of Petersen graph and its generalizations are explored in this paper.

Keywords: Petersen graph, generalized Petersen graph

1. Introduction

The field of mathematics known as graph theory deals with the study of graphs, which are mathematical representations of pairwise relationships between objects. One of the most intriguing and well-known graphs in graph theory is the Petersen graph, which retains the name of the Danish mathematician Julius Petersen who initially presented it in 1898. The Petersen graph is a small yet incredibly fascinating graph with a lot of unique features and applications. Petersen conducted research on factorizations of regular graphs in the 1890s, and in 1891 he produced a significant paper [1] that is honoured in this book. In his publication, Petersen demonstrated that a l-factor exists in every 3-regular graph with a maximum of two bridges. A few years prior, Tait [2] had said that he had demonstrated the l-factorability of every 3-regular graph, but that this finding was 'not true without limitation'. Tait's assertion may not have been what he intended, but Petersen interpreted it that way in 1898, claiming that every 3-regular bridgeless graph is l-factorable. If accurate, this conclusion would have been more reliable than Petersen's theorem. By producing a 3-regular bridgeless graph that is factor-incompressible, Petersen, however, refuted it [3].

The Petersen graph is an undirected regular 3-valent graph with 10 vertices and 10 edges. The Petersen graph is "a remarkable configuration that serves as a counterexample to many optimistic predictions about what might be true for graphs in general," according to a Donald Knuth quote that was obtained from Wikipedia. The book cover of Frank Harary's Russian translation was where the author first noticed this graph [4]. Its visual expression is distinguished by a unique and beautiful structure. The graph has five vertex points on the outer cycle that are connected to five vertex points that form the central pentagon. Two vertices on the pentagon are connected to each vertex on the outer cycle. The Petersen graph is visually appealing because it looks like a pentagram. Its structure, which is both straightforward and complex, makes it a fantastic subject for research in graph theory.

The Danish mathematician Julius Petersen (1839–1910), who made contributions to the growth of graph theory, is commemorated in the graph. His revolutionary work on regular graphs is what made him most famous. It was there earlier, as usual, in the work of English mathematician Sir Alfred Kempe (1849-1922), who is renowned for developing the four-color theorem and invariant theory. Kempe defined the Petersen graph as the graph whose edges connect two vertices if two lines do not intersect at one of the ten points in the Desargues configuration of 10 lines and 10 points in projective plane.

The main properties of the Petersen graph were discussed at length by Chartrand et.al in [5] and [6]. However, the Petersen graph continues to appear throughout the literature of graph theory. In this paper, an updated review is presented including recent results relating to the Petersen graph.

2. Definition and Preliminaries

The Petersen graph (Fig:1) is a graph with 10 vertices and 15 edges. It can be described in the following ways [5].

- The Kneser graph K_G (5, 2), of pairs on 5 elements, where edges are formed by disjoint edges.
- It can also be seen as the complement of the line graph of K_5i.e the vertices of the line graph are the edges of K_5 and two edges are joined if they share a vertex.
- Another way to obtain Petersen graph is to take two disjoint copies of C_5: (v_1 , v_2 , v_3 , v_4 , v_5) and (w_1 , w_2 , w_3 , w_4, w_5). Then add a matching of 5 edges between them: (v_1, w_1); (v_2, w_3); (v_3 ,w_5); (v_4 ,w_2); (v_5 ,v_4).

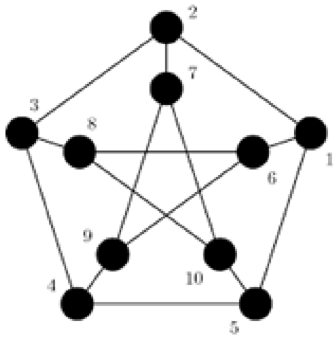

Fig 1: The Petersen graph

The Petersen graph is a very interesting small graph, which provides a counterexample to many graph-theoretic statements [5].

- It is the smallest bridgeless 3-regular graph, which has no 3-coloring of the edges so that adjacent edges get different colors (the smallest "snark").
- It is the smallest 3-regular graph of girth 5.
- It is the largest 3-regular graph of diameter 2.

- It has 2000 spanning trees, the most of any 3-regular graph on 10 vertices.
- It has eigenvalues including multiplicities (3, 1, 1, 1, 1, 1, -2, -2, -2, -2).

Fig:2 gives different drawings of the Petersen graph. The Petersen graph is ***strongly regular***. This means that not only does each vertex have the same degree 3, but each pair of vertices (u, v) in Ehas the same number of shared neighbours 0, and each pair of vertices (u, v) not in E has the same number of shared neighbours 1.

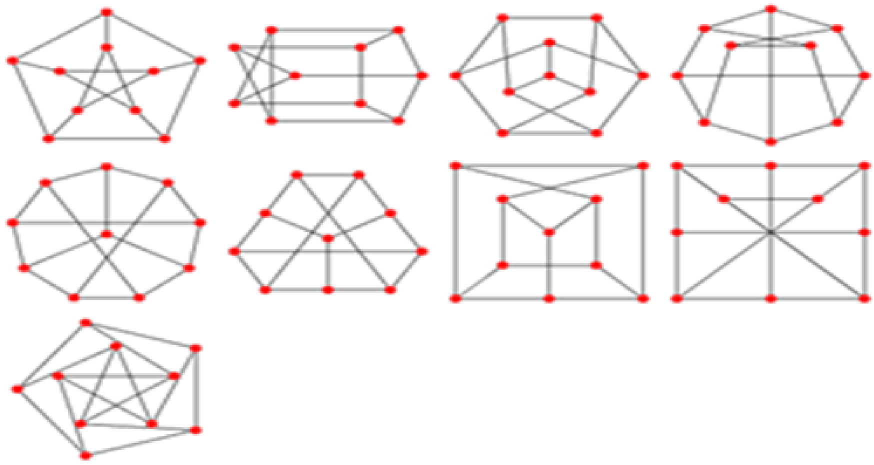

Fig:2 -Different drawings of Petersen graph

3. Characterizations and Properties

We have the following characterizations and properties of Petersen graph as detailed in [5] and [6].

Theorem 1:[5] The Petersen graph is the only 5-cage.

Theorem 2:[5] There is no decomposition of the edge set of K_{10} into 3 copies of the Petersen graph.

Theorem 3: [5] Apart from the complete graph K4, the Petersen graph is the only 3-regular graph in which any two nonadjacent vertices are mutually adjacent to just one other vertex.

A graph G to have the property $P_{1,2}$ if, for each sequence v, v_l, v_2, . . . , $v_n$of n + 1 vertices, there is another vertex of G, adjacent to v but not to v_l, v_2, . . . , v_n . If G is a given graph, a graph F is a G-frame if it is a graph of smallest order with the property that, for each vertex z, of G and each vertex w of F, there is an embedding of G into F as an induced subgraph with v appearing at W.

Theorem 4: [5] The Petersen graph is the smallest graph with the property that, given any three distinct vertices v, v_l, $v_2$there is a fourth vertex adjacent to v but not to v, or v2. In fact, the the Petersen graph is the only graph with fewer than twelve vertices having property $P_{1,2}$.

Theorem 5: [5] The Petersen graph is the only T-frame for the tree T of Fig. 3

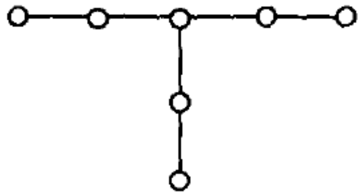

Fig3: T graph

Theorem 6: [5] If G is a 3-connected 3-regular graph, then G has a cycle passing through any ten vertices if and only if Gis not of 'Petersen form'-that is, G cannot be contracted to the Petersen graph with each of the ten vertices mapped to a distinct vertex of the Petersen graph.

Theorem 7: [5] Except for the Petersen graph, every 2-connected graph with minimum degree at least 3 contains a cycle whose length is congruent to 1 modulo 3.

Theorem 8: [5] If G is a 3-edge-connected graph with at most ten edge-cuts of size 3, then either G has a spanning closed trail, or G is contractible to the Petersen graph.

Theorem 9: [6] If G a bridgeless graph with at most 13 edge-cuts of size 3, then either G has a 3-colorable double cycle cover, or G is contractible to the Petersen graph.

Theorem 10: [6] Every 2-connected k-regular graph of order at most 3k is Hamiltonian.

Theorem 11: [6] The only 2-connected k-regular non-Hamiltonian graph of order 3k + 1 is the Petersen graph.

Theorem 12: [6] Let G be a 2-connected k-regular graph of order 2k + 3 or 2k + 4 (k 3 2). Then G is Hamiltonian if and only if G is not the Petersen graph.

Theorem 13:[6] If G has no triangles and has order 10, then G has at least five independent sets of 4 vertices; there are exactly five such sets if G is one of three graphs, one of which is the Petersen graph.

4. Generalized Petersen Graph

The Petersen Graph is a famous graph, not only because it appears on the cover of lots of books, but also due to its usefulness in several graph problems as example or counterexample. In order to generalize this concept, H. S. M. Coxeter, in [8], constructed a family of graphs that years later would be named as Generalized Petersen Graph and formalized by M. E. Watkins [7]. The idea was the following: draw a n-cycle (n-gon) surrounding a star n-gon, and joining their corresponding vertices through spokes. Prime labelling of generalized Petersen graph is studied in [9]. Coverings of Generalized Petersen Graphs are discussed in [10]. Various aspects of generalized Petersen graph wad studied by [14],[15] and [16].

Definition: [7] The Generalized Petersen graphs P(n; k) with $n \geq 3$ and $1 \leq k \leq [n/2]$are defined to be a graph with V (P(n; k)) = $\{v_i, u_i : 1 \leq i \leq 6\}$ and E(P(n; k)) = $\{ v_iv_{i+1}, v_iu_i, u_iu_{i+k}: 1 \leq i \leq n$, subscripts modulo n}.

The three subsets in which is divided E(G(n, k)) would define the n-gon (also called outer rim), the star n-gon (the inner rims) and the spokes respectively. Due to its construction, one can easily see that all Generalized Petersen Graphs are cubic graphs.
Theorem 14: [9] A Generalized Petersen Graph G(n, k) is bipartite if and only if n is even and k is odd.

Theorem 15: [10]P(n; k) is not a prime graph for odd n.

Theorem 16:[10] P(n; k) is not a prime graph if both n and k are even numbers.

Theorem 17: [10] If generalized Petersen graph P(n; k) is prime then n must be even and k must be odd.

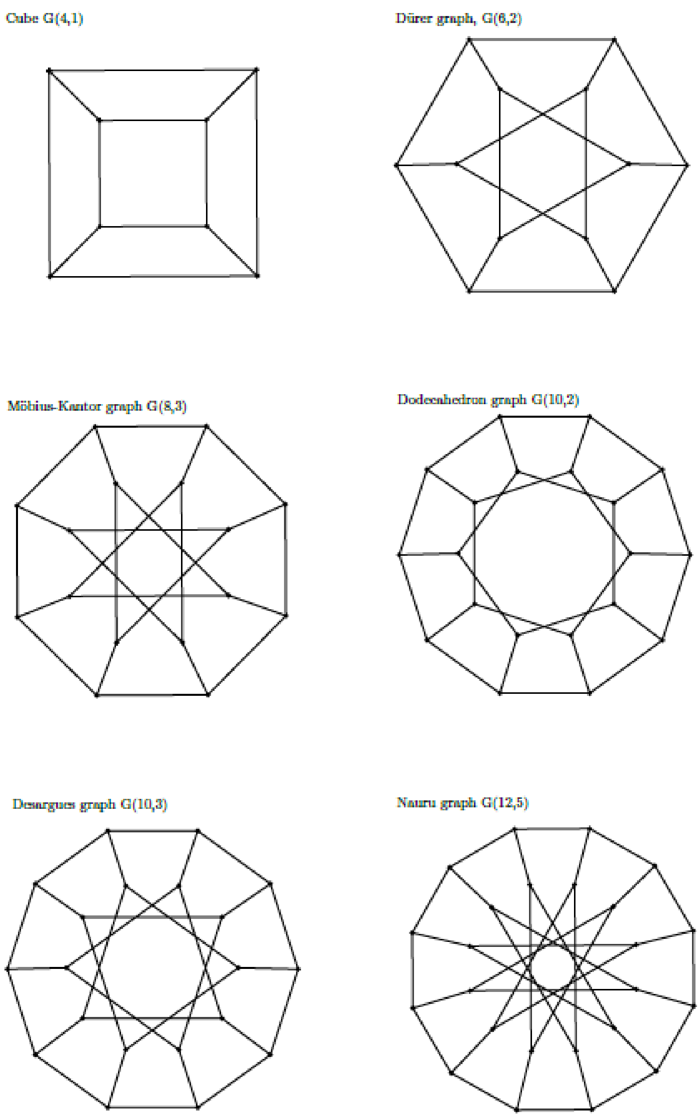

Fig:4 -Some Generalized Petersen Graphs

The Petersen Graph has the following distinctive qualities as detailed in [13]. The Petersen graph exhibits strong rotational symmetry and is symmetric. It has a unique trait among graphs in that it may be rotated by 72 degrees around its centre and still retain its original appearance. The Petersen graph has a chromatic number of 3, which indicates that only three different colours may be used to colour it without causing any adjacent vertices to share a colour. The Petersen graph is non-planar, which means that it cannot be represented on a plane without any edge crossings, despite the fact that it appears to be complicated. T. P. Petersen demonstrated this property in 1898, just a few months after introducing the graph. Since the Petersen graph is Hamiltonian, each vertex is visited exactly once by a Hamiltonian cycle, which is present in this graph. One of the oldest problems in graph theory is locating a Hamiltonian cycle in the Petersen graph. The game chromatic number of generalized Petersen graph is studied in [11]. Gera et.al discussed the spectrum of generalized Petersen graph in [12].

5. Conclusion

The extraordinary mathematical structure known as the Petersen graph serves as a showcase for the elegance and complexity of graph theory. For mathematicians and researchers from numerous fields, its special characteristics, such as its symmetry, chromatic number, planarity, and Hamiltonian cycle, make it an object of intrigue and study. The Petersen graph has practical uses in chemistry, coding theory, network design, and game theory in addition to its theoretical significance, highlighting the broad application of mathematical ideas in practical settings. Mathematicians are still fascinated by this mysterious graph and it continues to open up new directions for research in graph theory and related areas.

6. Conflict of Interest

The authordeclares that there is no conflict of interest regarding the publication of this paper.

7. Authors' Biography

Seema Varghese received the M.Sc. and Ph.d. degrees in Mathematics from Cochin University of Science and Technology in 2000 and 2011, respectively. Since 2004, She has been working as a faculty in various government colleges in Kerala. Presently, she is serving as Associate Professor in Mathematics at Government Engineering College, Thrissur affiliated to APJ Abdul Kalam Technological University, Kerala, India

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