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Fixed Point Theorems for Generalized Contractions in Complete Metric Space

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Abstract:

In this paper, we resent fixed point results for generalization on spaces with two metrics. Thefocus in on continuation results for such type of mappings.

Keywords: - Metric Space, Complete Metric Space, Self Mapping, Fixed Point, Commute Mapping 2000 AMS Classification:- 47H10, 54H25

Introduction:-

The study of common fixed point of mapping contractive type condition has been a very active field of research activity during the last three decades. The most general of the common fixed point pertain to two or three mapping of a metric space (X, d) and use either a Banach type contractive condition or other contractive condition. Many, Hardy [1], Rajput [2], Yadav [3], Sengupta[4] and so many author work in this field and prove more interesting result. Throughout this section (X, d') denotes a complete metric space and d be an another metric on X. if $x_0 \in X$ and r > 0 denote by $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$ and by clos. $B(x_0, r)^{d'}$ the d'- closer of $B(x_0, r)$

Fixed point results for Banach Generalized contractions:-

Theorem: 1:-Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, r > 0 and T be the mapping from, clos. $B(x_0, r)^{d'}$ into X, satisfying the following conditions;

$$d(Tx,Ty) \le \alpha. d(x,y) \tag{1.1}$$

Where non negative α , such that, $0 \leq \alpha < 1$ In addition assume the following three properties hold:

$$d(x_0, Tx_0) < (1-\alpha)r$$
 (1.2)

If $d \geq d'$

then T is uniformaly continuous from $(B(x_0, r), d)$ into (X, d') (1.3)

if
$$d \neq d'$$
 then T is continuous from $(clos.B(x_0,r)^{d'},d')$ into (X,d') (1.4)

then T has fixed point, that is there exists $x \in clos. B(x_0, r)^{d'}$ with Tx = x.

Proof:

Let $x_1 = Tx_0$ then from (1.2), we have $d(x_0, x_1) = d(x_0, Tx_0) < (1 - \alpha) r \le r$ So that, $x_1 \in B(x_0, r)$

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Next let $x_2 = Tx_1$ then we note that, $d(x_1, x_2) = d(Tx_0, Tx_1)$ *From* (1.1) $d(Tx_0, Tx_1) \leq \alpha d(x_0, x_1)$ $d(Tx_0, Tx_1) \leq \alpha (1-\alpha) r$ Now $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2)$ $d(x_0, x_2) \leq (1 - \alpha) r + \alpha (1 - \alpha) r$ $d(x_0, x_2) \leq (1 - \alpha) r (1 + \alpha)$ $d(x_0, x_2) < (1 - \alpha) r (1 + \alpha + \alpha^2 + \alpha^3 + \cdots \dots)$ $d(x_0, x_2) \ < \ (1-\alpha) \, r \ (1-\alpha \,)^{-1}$ $d(x_0, x_2) < r$ So that, $x_2 \in B(x_0, r)$ Proceeding inductively we obtain $d(x_{n+1}, x_n) \leq \alpha^n d(x_0, x_1)$ $d(x_0, x_{n+1}) < (1-\alpha)^n r (1-\alpha)^{-1}$ It follows $d(x_0, x_{n+1}) < r$ and $x_{n+1} \in B(x_0, r)$ In this way we construct a sequence $\{x_n\}$ of elements of X, such that $\{x_n\}$ is a Cauchy sequence with respect to, d, which converges to x. We claim that $\{x_n\}$ is a Cauchy sequence with respect to d'. If $d \ge d'$ then this is trivial. Next we suppose that, $d \ge d'$ Let $\varepsilon > 0$ be given. Now from (1.3) that there exists $\delta > 0$ such that, $d'(Tx,Ty) < \varepsilon$ whenever $x, y \in B(x_0,r)$ and $d(x,y) < \delta$ (1.5)From the above the sequence $\{x_n\}$ is a Cauchy sequence with respect to d, so we know that there exists N with $d(x_n, x_m) < \delta$ for all $n, m \ge N$ (1.6)Now from (1.5) and (1.6) implies $d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \varepsilon$ whenever $n, m \ge N$ Which proves that $\{x_n\}$ is a Cauchy sequence with respect to d'. Now since (X, d') is complete there exists $x \in clos.B(x_0, r)^{d'}$ with $d'(x_n, x) \to 0 \text{ and } n \to \infty$. We claim that, x = Tx(1.7)First consider the case, when $d \neq d'$. $d'(x,Tx) \leq d(x,x_n) + d(x_n,Tx) = d(x,x_n) + d(Tx_{n-1},Tx)$ Let $n \to \infty$ and using (1.4), we obtain $d'(x,Tx) \leq d(x,x) + d(Tx,Tx)$ d'(x,Tx) = 0And thus (8.7) is true, Next we suppose that d = d' then $d'(x,Tx) \leq d(x,x_n) + d(Tx_{n-1},Tx)$ *From* (1.1),



 $d'(x,Tx) \leq d(x,x_n) + \alpha d(x_{n-1},Tx)$

 $As \to \infty$, $Tx_n = x = Tx$ and above inequality can be written as, $(1 - \alpha)d(x, Tx) \le 0$

So that, d(x,Tx) = 0 and (1.7) holds.

This the proof of the theorem.

Theorem:- 2

Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, r > 0 and T be the mapping from, clos. $B(x_0, r)^{d'}$ into X, satisfying the following conditions;

 $d(Tx,Ty) \le \alpha.d(x,y) + \beta[d(x,Tx) + d(y,Ty)] + \gamma[d(x,Ty) + d(y,Tx)]$ (2.1) Where non negative α , β , γ , such that, $0 \le \alpha + \beta + \gamma < 1$

In addition assume the following three properties hold:

$$d(x_0, Tx_0) < \left(1 - \frac{\alpha + \beta}{1 - \beta - \gamma}\right) r$$
(2.2)

If $d \geq d'$

then T is uniformaly continuous from $(B(x_0, r), d)$ into (X, d') (2.3)

if $d \neq d'$ then T is continuous from $(clos.B(x_0,r)^{d'},d')$ into (X,d') (2.4)

then T has fixed point, that is there exists $x \in clos.B(x_0,r)^{d'}$ with Tx = x.

Proof:

Let $x_1 = Tx_0$ then from (2.2), we have

$$d(x_0, x_1) = d(x_0, Tx_0) < \left(1 - \frac{\alpha + \beta}{1 - \beta - \gamma}\right) r \le r$$

So that, $x_1 \in B(x_0, r)$ Next let $x_2 = Tx_1$ then we note that, $d(x_1, x_2) = d(Tx_0, Tx_1)$

From (2.1)

$$\begin{aligned} &d(Tx_0, Tx_1) \leq \alpha \, d(x_0, x_1) + \beta [d(x_0, x_1) + d(x_1, x_2)] + \gamma \, d(x_0, x_2) \\ &d(Tx_0, Tx_1) \leq \left(\frac{\alpha + \beta}{1 - \beta - \gamma}\right) \left(1 - \frac{\alpha + \beta}{1 - \beta - \gamma}\right) r \end{aligned}$$

Now

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ d(x_0, x_2) &\leq \left(1 - \frac{\alpha + \beta}{1 - \beta - \gamma}\right) r + \left(\frac{\alpha + \beta}{1 - \beta - \gamma}\right) \left(1 - \frac{\alpha + \beta}{1 - \beta - \gamma}\right) r \\ d(x_0, x_2) &\leq \left(1 - \frac{\alpha + \beta}{1 - \beta - \gamma}\right) r \left(1 + \frac{\alpha + \beta}{1 - \beta - \gamma}\right) \\ d(x_0, x_2) &< \left(1 - \frac{\alpha + \beta}{1 - \beta - \gamma}\right) r \left(1 + \left[\frac{\alpha + \beta}{1 - \beta - \gamma}\right] + \left[\frac{\alpha + \beta}{1 - \beta - \gamma}\right]^2 + \left[\frac{\alpha + \beta}{1 - \beta - \gamma}\right]^3 + \end{aligned}$$

....

$$d(x_0, x_2) < \left(1 - \left[\frac{\alpha + \beta}{1 - \beta - \gamma}\right]\right) r \left(1 - \left[\frac{\alpha + \beta}{1 - \beta - \gamma}\right]\right)^{-1}$$

$$d(x_0, x_2) < r$$

So that, $x_2 \in B(x_0, r)$

Proceeding inductively we obtain

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$$d(x_{n+1}, x_n) \leq \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]^n d(x_0, x_1)$$

$$d(x_0, x_{n+1}) < \left(1 - \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right)^n r \left(1 - \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right)^{-1}$$

It follows $d(x_0, x_{n+1}) < r$ and $x_{n+1} \in B(x_0, r)$ In this way we construct a sequence $\{x_n\}$ of elements of X, such that $\{x_n\}$ is a Cauchy sequence with respect to, d, which converges to x. We claim that $\{x_n\}$ is a Cauchy sequence with respect to d'. If $d \ge d'$ then this is trivial. Next we suppose that, $d \ge \neq d'$ Let $\varepsilon > 0$ be given. Now from (1.3) that there exists $\delta > 0$ such that, $d'(Tx,Ty) < \varepsilon$ whenever $x, y \in B(x_0, r)$ and $d(x, y) < \delta$ (2.5)From the above the sequence $\{x_n\}$ is a Cauchy sequence with respect to d, so we know that there exists N with $d(x_n, x_m) < \delta$ for all $n, m \geq N$ (2.6) Now from (2.5) and (2.6) implies $d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \varepsilon$ whenever $n, m \ge N$ Which proves that $\{x_n\}$ is a Cauchy sequence with respect to d'. Now since (X, d') is complete there exists $x \in clos.B(x_0, r)^{d'}$ with $d'(x_n, x) \to 0 \text{ and } n \to \infty$. We claim that, x = Tx(2.7)First consider the case, when $d \neq d'$. $d'(x,Tx) \leq d(x,x_n) + d(x_n,Tx) = d(x,x_n) + d(Tx_{n-1},Tx)$ Let $n \to \infty$ and using (2.4), we obtain $d'(x,Tx) \leq d(x,x) + d(Tx,Tx)$ d'(x,Tx) = 0And thus (2.7) is true, Next we suppose that d = d' then $d'(x,Tx) \leq d(x,x_n) + d(Tx_{n-1},Tx)$ *From* (2.1), $d'(x,Tx) \leq d(x,x_n) + \alpha d(x_{n-1},Tx) + \beta [d(x_{n-1},Tx_{n-1}) + d(x,Tx)] +$ $\gamma[d(x_{n-1}, Tx) + d(x, Tx_{n-1})]$ $As \rightarrow \infty$, $Tx_n = x = Tx$ and above inequality can be written as, $\left(1 - \left[\frac{\alpha + \beta}{1 - \beta - \gamma}\right]\right) d(x, Tx) \leq 0$ So that, d(x,Tx) = 0 and (2.7) holds. This complete proof of the theorem.

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