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Fixed Point Theorems for Generalized Contractions in Complete Metric Space

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Abstract:
In this paper , we resent fixed point results for generalization on spaces with two metrics. The focus in on continuation results for such type of mappings.

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Introduction:-
The study of common fixed point of mapping contractive type condition has been a very active field of research activity during the last three decades. The most general of the common fixed point pertain to two or three mapping of a metric space (X, d) and use either a Banach type contractive condition or other contractive condition. Many, Hardy [1], Rajput [2] , Yadav [3] , Sengupta[4] and so many author work in this field and prove more interesting result. Throughout this section $(\mathrm{X}, \mathrm{d}')$ denotes a complete metric space and d be an another metric on X. if $\mathrm{x}_0 \in \mathrm{X}$ and $\mathrm{r} > 0$ denote by $\mathrm{B}(\mathrm{x}_0, \mathrm{r}) = \{\mathrm{x} \in \mathrm{X} : \mathrm{d}(\mathrm{x}_0, \mathrm{x}) < \mathrm{r}\}$ and by $\mathrm{clos.B}(x_0, r)^{d'}$ *the d'- closer of* $B(x_0, r)$.

Fixed point results for Banach Generalized contractions:-

Theorem: 1 :- Let (X, d') *be a complete metric space,* d *another metric on* X, $x_0 \in X$, $r > 0$ *and* T *be the mapping from,* $clos.B(x_0, r)^{d'}$ *into* X *, satisfying the following conditions;*

$$d(Tx, Ty) \leq \alpha . d(x, y) \qquad (1.1)$$

Where non negative α *, such that,* $0 \leq \alpha < 1$
In addition assume the following three properties hold:

$$d(x_0, Tx_0) < (1 - \alpha) r \qquad (1.2)$$

If $d \ngeq d'$
then T *is uniformaly continuous from* $(B(x_0, r), d)$ *into* (X, d') *(1.3)*
if $d \neq d'$ *then* T *is continuous from* $(clos.B(x_0, r)^{d'}, d')$ *into* (X, d') *(1.4)*
then T *has fixed point , that is there exists* $x \in clos.B(x_0, r)^{d'}$ *with* $Tx = x$.

Proof:
Let $x_1 = Tx_0$ *then from (1.2), we have*

$$d(x_0, x_1) = d(x_0, Tx_0) < (1 - \alpha) r \leq r$$

So that, $x_1 \in B(x_0, r)$

Next let $x_2 = Tx_1$ *then we note that,*

$$d(x_1, x_2) = d(Tx_0, Tx_1)$$

From (1.1)

$$d(Tx_0, Tx_1) \leq \alpha\, d(x_0, x_1)$$
$$d(Tx_0, Tx_1) \leq \alpha\,(1-\alpha)\, r$$

Now

$$d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2)$$
$$d(x_0, x_2) \leq (1-\alpha)\, r + \alpha\,(1-\alpha)\, r$$
$$d(x_0, x_2) \leq (1-\alpha)\, r\ (1+\alpha)$$
$$d(x_0, x_2) < (1-\alpha)\, r\ (1+\alpha + \alpha^2 + \alpha^3 + \cdots \ldots\ldots..)$$
$$d(x_0, x_2) < (1-\alpha)\, r\ (1-\alpha\,)^{-1}$$
$$d(x_0, x_2) < r$$

So that, $x_2 \in B(x_0, r)$

Proceeding inductively we obtain

$$d(\,x_{n+1},\ x_n) \leq \alpha^n\, d(x_0, x_1)$$
$$d(x_0, x_{n+1}) < (1-\alpha)^n\, r\ (1-\alpha\,)^{-1}$$

It follows $d(x_0, x_{n+1}) < r$ *and* $x_{n+1} \in B(x_0, r)$

In this way we construct a sequence $\{x_n\}$ *of elements of X, such that* $\{x_n\}$ *is a Cauchy sequence with respect to,* d *, which converges to* x.

We claim that $\{x_n\}$ *is a Cauchy sequence with respect to d'.*

If $d \geq d'$ *then this is trivial.*

Next we suppose that, $d \ngeq d'$

Let $\varepsilon > 0$ *be given. Now from (1.3) that there exists* $\delta > 0$ *such that,*

$$d'(Tx, Ty) < \varepsilon \text{ whenever } x, y \in B(x_0\,, r) \text{ and } d(x, y) < \delta \qquad (1.5)$$

From the above the sequence $\{x_n\}$ *is a Cauchy sequence with respect to d, so we know that there exists N with*

$$d(x_n, x_m) < \delta \text{ for all } n, m \geq N \qquad (1.6)$$

Now from (1.5) and (1.6) implies

$$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \varepsilon \text{ whenever } n, m \geq N$$

Which proves that $\{x_n\}$ *is a Cauchy sequence with respect to d'.*

Now since (X, d') *is complete there exists* $x \in clos.B(x_0, r)^{d'}$ *with* $d'(x_n, x) \to 0$ *and* $n \to \infty$.

We claim that, $x = Tx$ *(1.7)*

First consider the case, when $d \neq d'$.

$$d'(x, Tx) \leq d(x, x_n) + d(x_n, Tx) = d(x, x_n) + d(Tx_{n-1}, Tx)$$

Let $n \to \infty$ *and using (1.4), we obtain*

$$d'(x, Tx) \leq d(x, x) + d(Tx, Tx)$$
$$d'(x, Tx) = 0$$

And thus (8.7) is true,

Next we suppose that $d = d'$ *then*

$$d'(x, Tx) \leq d(x, x_n) + d(Tx_{n-1}, Tx)$$

From (1.1),

$$d'(x,Tx) \leq d(x,x_n) + \alpha d(x_{n-1},Tx)$$

As $\to \infty$, $Tx_n = x = Tx$ *and above inequality can be written as,*

$$(1-\alpha)d(x,Tx) \leq 0$$

So that, $, d(x,Tx) = 0$ *and (1.7) holds.*

This the proof of the theorem.

Theorem:- 2

Let (X,d') *be a complete metric space,* d *another metric on* X, $x_0 \in X$, $r > 0$ *and* T *be the mapping from,* $clos.B(x_0,r)^{d'}$ *into* X, *satisfying the following conditions;*

$$d(Tx,Ty) \leq \alpha.d(x,y) + \beta[d(x,Tx) + d(y,Ty)] + \gamma[d(x,Ty) + d(y,Tx)] \quad (2.1)$$

Where non negative α, β, γ, *such that,* $0 \leq \alpha + \beta + \gamma < 1$

In addition assume the following three properties hold:

$$d(x_0,Tx_0) < \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r \quad (2.2)$$

If $d \ngeq d'$

then T *is uniformaly continuous from* $(B(x_0,r),d)$ *into* (X,d') (2.3)

if $d \neq d'$ *then* T *is continuous from* $\left(clos.B(x_0,r)^{d'},d'\right)$ *into* (X,d') (2.4)

then T *has fixed point, that is there exists* $x \in clos.B(x_0,r)^{d'}$ *with* $Tx = x$.

Proof:

Let $x_1 = Tx_0$ *then from (2.2), we have*

$$d(x_0,x_1) = d(x_0,Tx_0) < \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r \leq r$$

So that, $x_1 \in B(x_0,r)$

Next let $x_2 = Tx_1$ *then we note that,*

$$d(x_1,x_2) = d(Tx_0,Tx_1)$$

From (2.1)

$$d(Tx_0,Tx_1) \leq \alpha\, d(x_0,x_1) + \beta[d(x_0,x_1) + d(x_1,x_2)] + \gamma\, d(x_0,x_2)$$

$$d(Tx_0,Tx_1) \leq \left(\frac{\alpha+\beta}{1-\beta-\gamma}\right)\left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r$$

Now

$$d(x_0,x_2) \leq d(x_0,x_1) + d(x_1,x_2)$$

$$d(x_0,x_2) \leq \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r + \left(\frac{\alpha+\beta}{1-\beta-\gamma}\right)\left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r$$

$$d(x_0,x_2) \leq \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r \left(1 + \frac{\alpha+\beta}{1-\beta-\gamma}\right)$$

$$d(x_0,x_2) < \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r \left(1 + \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right] + \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]^2 + \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]^3 + \cdots\cdots\cdots\right)$$

$$d(x_0,x_2) < \left(1 - \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right) r \left(1 - \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right)^{-1}$$

$$d(x_0,x_2) < r$$

So that, $x_2 \in B(x_0,r)$

Proceeding inductively we obtain

$$d(x_{n+1}, x_n) \leq \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]^n d(x_0, x_1)$$

$$d(x_0, x_{n+1}) < \left(1-\left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right)^n r \left(1-\left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right)^{-1}$$

It follows $d(x_0, x_{n+1}) < r$ *and* $x_{n+1} \in B(x_0, r)$

In this way we construct a sequence $\{x_n\}$ *of elements of* X*, such that* $\{x_n\}$ *is a Cauchy sequence with respect to,* d *, which converges to* x.

We claim that $\{x_n\}$ *is a Cauchy sequence with respect to d'.*

If $d \geq d'$ *then this is trivial.*

Next we suppose that, $d >\neq d'$

Let $\varepsilon > 0$ *be given. Now from (1.3) that there exists* $\delta > 0$ *such that,*

$$d'(Tx, Ty) < \varepsilon \text{ whenever } x, y \in B(x_0, r) \text{ and } d(x, y) < \delta \tag{2.5}$$

From the above the sequence $\{x_n\}$ *is a Cauchy sequence with respect to* d*, so we know that there exists* N *with*

$$d(x_n, x_m) < \delta \text{ for all } n, m \geq N \tag{2.6}$$

Now from (2.5) and (2.6) implies

$$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \varepsilon \text{ whenever } n, m \geq N$$

Which proves that $\{x_n\}$ *is a Cauchy sequence with respect to d'.*

Now since (X, d') *is complete there exists* $x \in clos.B(x_0, r)^{d'}$ *with* $d'(x_n, x) \to 0$ *and* $n \to \infty$.

We claim that, $x = Tx$ *(2.7)*

First consider the case, when $d \neq d'$.

$$d'(x, Tx) \leq d(x, x_n) + d(x_n, Tx) = d(x, x_n) + d(Tx_{n-1}, Tx)$$

Let $n \to \infty$ *and using (2.4), we obtain*

$$d'(x, Tx) \leq d(x, x) + d(Tx, Tx)$$

$$d'(x, Tx) = 0$$

And thus (2.7) is true,

Next we suppose that $d = d'$ *then*

$$d'(x, Tx) \leq d(x, x_n) + d(Tx_{n-1}, Tx)$$

From (2.1),

$$d'(x, Tx) \leq d(x, x_n) + \alpha d(x_{n-1}, Tx) + \beta[d(x_{n-1}, Tx_{n-1}) + d(x, Tx)] + \gamma[d(x_{n-1}, Tx) + d(x, Tx_{n-1})]$$

As $\to \infty$, $Tx_n = x = Tx$ *and above inequality can be written as,*

$$\left(1-\left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right) d(x, Tx) \leq 0$$

So that, $, d(x, Tx) = 0$ *and (2.7) holds.*

This complete proof of the theorem.

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