International Journal for Multidisciplinary Research (IJFMR)

E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

To Study on Number of Subgroups of Finite Abelian Group $Z_m \otimes Z_n$

Mohd Haqnawaz Khan[1], Dr. Ashfaque ur Rahman[2]

[1]Research Scholar (Mathematics)
[2]Research Supervisor, Madhyanchal Professional University Bhopal India

Abstract:
In this paper, we determine the number of subgroups of group $Z_{p^m} \otimes Z_{p^n}$ which may be cyclic or non – cyclic by using simple number –theoretic formulae. We describe the subgroups of the group $Z_m \otimes Z_n$ and derive a simple formula for the total number s(m,n) of the subgroups where m, n are arbitrary positive integers. We point out that certain multiplicative properties of related counting functions for finite. Abelian groups are immediate consequences of these formula.

Keywords: Subgroups, Abelian groups ,Cyclic ,Theoretic Formula, Finite Abelian Groups,

Introduction
Consider a finite abelian group $Z_m \otimes Z_n$ of order mn. If m and n are relatively prime then $Z_m \otimes Z_n$ is cyclic , otherwise non-cyclic. In [1] ,if group $Z_m \otimes Z_n$ is cyclic then number of subgroup is equal number of divisor of mn. It is well known (Frobenius-Stickelberger,1878) that a finite abelian group is the direct sum of a finite number of cyclic groups of prime power orders.
However ,when it comes to draw the subgroup lattice of a given finite abelian group, or to find out the total number of subgroups this group has, this can be a difficult task. It is the purpose of this note to describe a new method which partially solves these two problems. The number of nontrivial subgroups of G = $Z_p \otimes Z_p$ is p+3 .If p is prime then p+3 never equal to number of divisor of p^2. In [4] (MARIUS TARNAUCANU) ,the total number of subgroups of $Z_{p^{\alpha_1}} \otimes Z_{p^{\alpha_2}}$ is $\frac{[(\alpha_2-\alpha_1+1)p^{\alpha_1+2}-(\alpha_2-\alpha_1-1)p^{\alpha_1+1}-(\alpha_2+\alpha_1+3)p+(\alpha_2+\alpha_1+1)]}{(p-1)^2}$ where $\alpha_1 \leq \alpha_2$. In this paper we derive a formula which works in both the case either group $Z_m \otimes Z_n$ cyclic or non-cyclic .

2-Preliminaries
2.1 Theorem: (The Fundamental Theorem of Arithmetic) Every positive integer greater than one can be written uniquely as a product of primes, with prime factors in the product written in order of non decreasing size.

2.2 Theorem: (The Fundamental Theorem of Finite Abelian Groups) Every finite Abelian group is a direct product of cyclic groups of prime –power order. Moreover, the factorization is unique except for rearrangement of factors.

2.3 Theorem: Prove that G $\otimes H \approx H \otimes G$ where H and G both are groups.

2.4 Theorem: If a and b are elements abelian groups G_1 and G_2 respectively and their orders are finite as well as co-prime, then $\langle a\rangle \otimes \langle b\rangle = \langle(a,b)\rangle$

Proof: We have o$\big((a,b)\big)$ = lcm $\{o(a), o(b)\} = o\,(a) o\,(b)$

Hence subgroup $\langle(a,b)\rangle$ has order o(a) o(b) which same order of $\langle a\rangle \otimes \langle b\rangle$

Now want to prove that $\langle(a,b)\rangle = \langle a\rangle \otimes \langle b\rangle$

Let x = $(a,b)^k$ be any arbitrary element of $\langle(a,b)\rangle$

$\Longrightarrow$ x = $(a^k, b^k) \Longrightarrow a^k \in \langle a\rangle$ and $b^k \in \langle b\rangle \Longrightarrow (a^k, b^k) \in \langle a\rangle \otimes \langle b\rangle$

$\Longrightarrow$ x $\in \langle a\rangle \otimes \langle b\rangle$

Hence $\langle(a,b)\rangle \subseteq \langle a\rangle \otimes \langle b\rangle$

Let x = (a^k, b^l) be any arbitrary element of $\langle a\rangle \otimes \langle b\rangle$ and without loss of generality assume that k $\leq l$

Case1:

If k = 1 then x = (a^k, b^k) then x $\in \langle(a,b)\rangle$

Case 2: If k < 1 then x = $(a^k, b^l) = (a^k, b^k)(e, b^{l-k})$

We know that o(a) and o(b) are co-prime ,then there exists integers α and β such that $1 = \alpha.o(a) + \beta.o(b)$

Then (e,b) = $\left(e, b^{\alpha.o(a)+\beta.o(b)}\right) = \left(e, \left(b^{o(a)}\right)^{\alpha}\left(b^{o(b)}\right)^{\beta}\right) = \left(e, \left(b^{o(a)}\right)^{\alpha}\right) = \left(e, b^{o(a)}\right)^{\alpha}$

Here (a,b) $\in \langle(a,b)\rangle \Rightarrow (a,b)^{o(a)} \in \langle(a,b)\rangle \Rightarrow \left(e, b^{o(a)}\right) \in \langle(a,b)\rangle$

$\Rightarrow (e,b) \in \langle(a,b)\rangle \Rightarrow (e, b^{l-k}) \in \langle(a,b)\rangle$.............*

$\Rightarrow$ Also $(a^k, b^k) \in \langle(a,b)\rangle$ ....**

From $(*)$ *and* $(**)$, we get $(a^k, b^l) \in \langle(a,b)\rangle \Rightarrow$ x $\in \langle(a,b)\rangle$

On basis of case 1 and case 2 , we conclude that $\langle a\rangle \otimes \langle b\rangle \subseteq \langle(a,b)\rangle$

Therefore $\langle a\rangle \otimes \langle b\rangle = \langle(a,b)\rangle$

2.5 Theorem: If order of abelian group G_1 and G_2 are finite as well as co-prime ,then every subgroup of $G_1$and G_2 are finite as well as co-prime ,then every subgroup of $G_1 \otimes G_2$can be written as be written as external product of subgroup of G_1 and subgroup of G_2 .

Proof: Assume H is any subgroup of $G_1 \otimes G_2$, we have to prove that H can be written as external product of G_1 and subgroup G_2.

Firstly, we prove that H must have a subgroup which can be written as external product of subgroup of G_1 and subgroup of G_2.

Here $\{(e_1, e_2)\}$ is a subgroup of H and $\{(e_1, e_2)\} \approx \{e_1\} \otimes \{e_2\}$

Therefore ,our first claim is proved.

Assume that $H_1 \otimes H_2$ is the largest subgroup of H which can be written as external product of subgroup of G_1 and subgroup of G_2. There are two cases arises here.

$(i)\quad H_1 \otimes H_2 = H$

$(ii)\ H_1 \otimes H_2 \subseteq H$

Case 1: If $H_1 \otimes H_2$ = H ,then nothing to prove.

Case 2: If $H_1 \otimes H_2 \subseteq$ H, then there exists (a,b) $\in$ H such that $(a,b) \notin H_1 \otimes H_2$

It is given that G_1 is an abelian group; Therefore H_1 and $\langle a\rangle$ are also abelian subgroup of G_1 ; Hence $H_1\langle a\rangle = \langle a\rangle H_1 \Rightarrow H_1\langle a\rangle$ is a subgroup of G_1.Similarly we can prove that $H_2\langle b\rangle$ is a subgroup of G_2 .Then $H_1\langle a\rangle \otimes H_2\langle b\rangle$ is also subgroup of $G_1 \otimes G_2$

Now we have to show that $H_1 \otimes H_2 \subseteq H_1\langle a\rangle \otimes H_2\langle b\rangle \subseteq H$
Let $(x, y) \in H_1 \otimes H_2 \Rightarrow$ x $\in H_1$ and y $\in H_2 \Rightarrow$ x $\in H_1\langle a\rangle$ *and* $y \in H_2\langle b\rangle$
$\Rightarrow (x, y) \in H_1 \langle a\rangle \otimes H_2\langle b\rangle$
Hence $H_1 \otimes H_2 \subseteq H_1 \langle a\rangle \otimes H_2\langle b\rangle$ ……………………(i)
It is given in this case that $(a, b) \notin H_1\otimes H_2$
But a $\in \langle a\rangle \Rightarrow$ a $\in H_1\langle a\rangle$
Similarly , b $\in H_2\langle b\rangle$
Hence $(a, b) \in H_1\langle a\rangle \otimes H_2\langle b\rangle$ ……………………(ii)
From (*) and (**) ,we get $H_1 \otimes H_2 \subseteq H_1\langle a\rangle \otimes H_2\langle b\rangle$ ………………(iii)
Let $(x, , y) \in H_1\langle a\rangle \otimes H_2\langle b\rangle$
$\Rightarrow x \in H_1\langle a\rangle$ *and* $y \in H_2\langle b\rangle$
$\Rightarrow$ x = h_1a^k and y = h_2b^l where $h_1 \in H_1, h_2 \in H_2$ and k, $l \in Z$
(x, y) = (h_1a^k, h_2b^l) = (h_1, h_2) (a^k, b^l) ………………(iv)
It is also given that order of groups G_1 and G_2 are finite as well as co-prime ,hence o(a) and o(b) are also co-prime and finite
By use of theorem 2.4 , we get $\langle a\rangle \otimes \langle b\rangle \subseteq H$
Hence $(a^k, b^l) \in H$ …………….(v)
Also $(h_1, h_2) \in H_1 \otimes H_2 \subseteq H \ldots\ldots\ldots.. (vi)$
By use of (iv) ,(v) and (vi) with use of concept that H is subgroup, we get
$(x, y) = (h_1a^K, h_2b^l) = (h_1, h_2)(a^k, b^l) \in H$
Hence $H_1\langle a\rangle \otimes H_2\langle b\rangle \subseteq H$ ………………..(vii)
From (iii) and (vii) ,we get $H_1 \otimes H_2 \subseteq H_1\langle a\rangle \otimes H_2\langle b\rangle \subseteq H$ Above the result is a contradiction with fact that $H_1 \otimes H_2$ is largest subgroup of H which can be external product of two subgroups of G_1 and G_2. Hence H is itself external product of two subgroups of G_1 and G_2.
But H is any arbitrary subgroup of $G_1 \otimes G_1$,hence every subgroup of $G_1 \otimes G_2$ can be written as be written as external product of subgroup of G_1 and subgroup of G_2.

3- New Number-theoretic Formula for number of subgroup of $Z_{p^{\alpha_1}} \otimes Z_{p^{\alpha_2}}$

3.1 Theorem: Prove that the total number of subgroups of $Z_{p^{\alpha_1}} \otimes Z_{p^{\alpha_2}}$ are $\sum_{q|(p^{\alpha_1},p^{\alpha_2})} \tau\left(\frac{p^{\alpha_1}}{q}\right)\tau\left(\frac{p^{\alpha_2}}{q}\right)\boldsymbol{\Phi}(\boldsymbol{q})$.

Proof: Without loss of generality, assume that $\alpha_1 \leq \alpha_2$
Let S = $\sum_{q|(p^{\alpha_1},p^{\alpha_2})} \tau\left(\frac{p^{\alpha_1}}{q}\right)\tau\left(\frac{p^{\alpha_2}}{q}\right)\boldsymbol{\Phi}(q)$ where $\alpha_1 \leq \alpha_2$

$$S = \sum_{q|p^{\alpha_1}} \tau\left(\frac{p^{\alpha_1}}{q}\right)\tau\left(\frac{p^{\alpha_2}}{q}\right)\Phi(q)$$

$$S = \tau(p^{\alpha_1})\tau(p^{\alpha_2})\Phi(1) + \tau(p^{\alpha_1-1})\tau(p^{\alpha_2-1})\Phi(p) + \tau(p^{\alpha_1-2})\tau(p^{\alpha_2-2})\Phi(p^2) + \ldots + \tau(p^{\alpha_1-\alpha_1})\tau(p^{\alpha_2-\alpha_1})\Phi(p^{\alpha_1})$$

$$S = (\alpha_1 + 1)(\alpha_2 + 1) + (\alpha_1)(\alpha_2)(p - 1) + (\alpha_1 - 1)(\alpha_2 - 1)p(p - 1) + (\alpha_2 - 2)(\alpha_2 - 2)p^2(p - 1) + \cdots \ldots + (\alpha_1 - \alpha_1 + 1)(\alpha_2 - \alpha_1 + 1)p^{\alpha_1-1}(p - 1) \ldots\ldots.\text{(i)}$$

Multiply (i) by p , we obtain

$$S_p = (\alpha_1 + 1)(\alpha_2 + 1)p + (\alpha_1)(\alpha_2)(p - 1)p + (\alpha_1 - 1)(\alpha_2 - 1)p^2(p - 1) + (\alpha_1 - 2)(\alpha_2 - 2)p^3(p - 1) + \cdots \ldots + (\alpha_1 - \alpha_1 + 1)(\alpha_2 - \alpha_1 + 1)p^{\alpha_1}(p - 1) \ldots. \quad (ii)$$

Then (ii) and (i) we obtain

S(p-1) = $(\alpha_1+1)(\alpha_2+1)(p-1)-(\alpha_1)(\alpha_2)(p-1)+(\alpha_1+\alpha_2-1+2(1))p(p-1)+\left(\alpha_1+\alpha_2-1+2(2)\right)p^2(p-1)+\left(\alpha_1+\alpha_1-1+2(3)\right)p^3(p-1)+\cdots+\left(\alpha_1+\alpha_2-1+2\,(\alpha_1-1)\right)p^{\alpha_1-1}(p-1)+(\alpha_2-\alpha_1+1)p^{\alpha_1}(p-1)$

S = $(\alpha_1+1)(\alpha_2+1)-(\alpha_1)(\alpha_2)+\left(\alpha_1+\alpha_2+1-2(1)\right)p+\left(\alpha_1+\alpha_2+1-2(2)\right)p^2+(\alpha_1+\alpha_2+1-2(3))p^3+\cdots+\left(\alpha_1+\alpha_2+1-2(\alpha_1-1)\right)p^{\alpha_1-1}+(\alpha_2-\alpha_1+1)p^{\alpha_1}$

S = $(\alpha_1+1)(\alpha_2+1)-(\alpha_1)(\alpha_2)+\left(\alpha_1+\alpha_2+1-2(1)\right)p+\left(\alpha_1+\alpha_2+1-2(2)\right)p^2+(\alpha_1+\alpha_2+1-2(3))\,p^3+\ldots+\left(\alpha_1+\alpha_2+1-2(\alpha_1-1)\right)p^{\alpha_1-1}+(\alpha_2-\alpha_1+1)p^{\alpha_1}$

S = $(\alpha_1+\alpha_2+1)(1+p+p^2+\cdots+p^{\alpha_1-1})-2(p+2p^2+3p^3+\cdots+(\alpha_1-1)p^{\alpha_1-1})+(\alpha_2-\alpha_1+1)p^{\alpha_1}$

S = $(\alpha_1+\alpha_2+1)\frac{p^{\alpha_1-1}}{p-1}-2\left(\frac{\alpha_1 p^{\alpha_1}}{p-1}-\frac{p(p^{\alpha_1-1})}{(p-1)^2}\right)+(\alpha_2-\alpha_1+1)p^{\alpha_1}$

S = $\frac{(\alpha_1+\alpha_2+1)(p^{\alpha_1}-1)(p-1)-2\left(\alpha_1 p^{\alpha_1}(p-1)-p(p^{\alpha_1}-1)\right)+(\alpha_2-\alpha_1+1)p^{\alpha_1}(p-1)^2}{(p-1)^2}$

S= $\frac{[(\alpha_2-\alpha_1+1)p^{\alpha_1+2}-(\alpha_2-\alpha_1-1)p^{\alpha_1+1}-(\alpha_2+\alpha_1+3)p+(\alpha_2+\alpha_1+1)]}{(p-1)^2}$

Which is same as in [4] (MARIUS TARNAUCEANU) ,the total number of subgroups o+f $Z_{p^{\alpha_1}}\otimes Z_{p^{\alpha_2}}$ are

$\frac{[(\alpha_2-\alpha_1+1)p^{\alpha_1+2}-(\alpha_2-\alpha_1-1)p^{\alpha_1+1}-(\alpha_2+\alpha_1+3)p+(\alpha_2+\alpha_1+1)]}{(p-1)^2}$ where $\alpha_1\le\alpha_2$

Hence, total number of subgroups of of $Z_{p^{\alpha_1}}\otimes Z_{p^{\alpha_2}}$ are $\sum_{q|(p^{\alpha_1},p^{\alpha_2})}\tau\left(\frac{p^{\alpha_1}}{q}\right)\tau\left(\frac{p^{\alpha_2}}{q}\right)\boldsymbol{\Phi}(q)$

4- New Number-theoretic Formula for number of subgroup of $Z_m\otimes Z_n$

4.1 Theorem: If order of abelian groups G_1 and G_2 are finite as well as co-prime, then number of subgroups of $G_1\otimes G_2$ is product of number of subgroups of G_1 with number of subgroups of G_2.

Proof: Total number of subgroups of $G_1\otimes G_2$ which can be written as be written as external product of subgroup of G_1 and subgroup of G_2 is product of number of subgroups of G_1 with number of subgroups of G_2.

By use of theorem 2.5 ,every subgroup of $G_1\otimes G_2$ can be written as be written as external product of subgroup of G_1 and subgroup of G_2. Hence there is no subgroup of $G_1\otimes G_2$ which cannot be written as external product of subgroup of G_1 and subgroup of G_2.

Hence total number of subgroups of $G_1\otimes G_2$ is product of number of subgroups of G_1 with number of subgroups of G_2.

4.2 Corollary : If p and q are different primes, then number of subgroup of group $Z_{p^\alpha}\otimes Z_{q^\beta}$ are $\tau(p^\alpha q^\beta)$

Proof: Number of subgroup of Z_{p^α} is $\tau(p^\alpha)$ and Number of subgroup of Z_{q^β} is $\tau(q^\beta)$

Here order of abelian groups Z_{p^α} and Z_{q^β} are finite as well as co-prime ,then number of subgroups of $G_1\otimes G_2\ is\ \tau(p^\alpha)\tau(q^\beta)=\tau(p^\alpha q^\beta)$

4.3 Theorem: If finite abelian group G_i for i = 1,2,….,k and $(|G_i|,|G_j|)=1\ \forall\, i\neq j$, then number of subgroups of $G_1\otimes G_2\otimes G_3\otimes\ldots\otimes G_k\ is\ \prod_{i=1}^{k}(Number\ of\ subgroup\ of\ G_i)$

Proof: We use the principal of Mathematical Induction to prove this result. Take k =2, then by use of theorem 4.1 , we have

Number of subgroups of $G_1 \otimes G_2$ is $\prod_{i=1}^{2}(Number\ of\ subgroup\ of\ G_i)$

Let us assume that the given result is true for k-1, we have

Number of subgroups of $G_1 \otimes G_2 \otimes G_3 \otimes \ldots . \otimes G_{k-1}$ is $\prod_{i=1}^{k-1}(Number\ of\ subgroup\ of G_i)$

We have to prove that the given result is true for k

Say H = $G_1 \otimes\ G_2 \otimes G_3 \otimes \ldots . . \otimes G_{k-1}$ and K = G_k

It is given that order of each G_i for i = 1,2, …..k-1 is finite , therefore order H is finite.

It is also given that each G_i for i = 1,2,…k-1 is abelian group, therefore H is also abelian group.

Here $(|G_1|, |G_k|) = 1$, $(|G_2|, |G_k|) = 1$,….. $(|G_{k-1}|, |G_k|) = 1 \Rightarrow (|G_1||G_k| \ldots |G_{k-1}|, |G_k|) = 1$
$\Rightarrow (|G_1 \otimes G_2 \otimes G_3 \otimes \ldots \otimes\ G_{k-1}|, |G_k|) = 1 \Rightarrow (|H|, |K|) = 1$

Hence number of subgroups of H $\otimes K$ is (number of subgroup of H) $\times$ $(number\ of\ subgroup$ with k).

$=[\prod_{i=1}^{k-1}(Number\ of\ subgroup\ of\ G_i)] \times Number\ of\ subgroup\ of\ G_k$ = $\prod_{i=1}^{k}(Number\ of\ subgroup\ of\ G_i)$.

4.4 Corollary:- Number of subgroups of $Z_m \otimes\ Z_n\ are\ \sum_{q|(m,n)} \tau\left(\frac{m}{q}\right) \tau\left(\frac{n}{q}\right) \Phi(q)$. **Proof:** By use of the Fundamental Theorem of Arithmetic m and n can be written as m = $p_1^{\alpha_1} p_2^{\alpha_2}\ p_3^{\alpha_3} \ldots\ \ldots p_r^{\alpha_r}$ and n = $p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} \ldots \ldots\ p_r^{\beta_r}$ throughout the paper. Here it is not necessary that all α_i and β_i are not zero.

Hence, by use of Fundamental Theorem of finite Abelian Groups ,we have

$Z_m \approx Z_{{p_1}^{\alpha_1}} \otimes Z_{{p_2}^{\alpha_2}} \otimes Z_{{p_3}^{\alpha_3}} \otimes \ldots \otimes Z_{{p_r}^{\alpha_r}}$ and $Z_n \approx\ Z_{{p_1}^{\beta_1}} \otimes Z_{{p_2}^{\beta_2}} \otimes Z_{{p_3}^{\beta_3}} \otimes \ldots \otimes Z_{{p_r}^{\beta_r}}$ Then, we can write

$Z_m \otimes Z_n \approx Z_{{p_1}^{\alpha_1}} \otimes Z_{{p_2}^{\alpha_2}} \otimes \ldots \ldots \otimes Z_{{p_r}^{\alpha_r}} \otimes Z_{{p_1}^{\beta_1}} \otimes Z_{{p_2}^{\beta_2}} \otimes \ldots \otimes Z_{{p_r}^{\beta_r}}$

Now apply theorem 2.3 ,we get

$Z_m \otimes Z_n \approx\ Z_{{p_1}^{\alpha_1}} \otimes\ Z_{{p_1}^{\beta_1}} \otimes Z_{{p_2}^{\alpha_2}} \otimes Z_{{p_2}^{\beta_2}} \otimes \ldots \ldots \otimes Z_{{p_r}^{\alpha_r}} \otimes Z_{{p_r}^{\beta_r}}$.

Assume $G_1 = Z_{{p_1}^{\alpha_1}} \otimes\ Z_{{p_1}^{\beta_1}}$, $G_2 = Z_{{p_2}^{\alpha_2}} \otimes Z_{{p_2}^{\beta_2}}$,….. … $G_r =\ Z_{{p_r}^{\alpha_r}} \otimes Z_{{p_r}^{\beta_r}}$

Here each G_i is abelian finite group and $(|G_i|, |G_i|)$ =1 $\forall\ i\ \neq j$

By use of theorem , number of subgroups of $Z_m \otimes Z_n$ is $\prod_{i=1}^{k}(Number\ of\ subgroup\ of\ G_i)$

$= \prod_{i=1}^{k}\left(Number\ of\ subgroup\ of\ \ Z_{{p_i}^{\alpha_i}}\ \otimes\ Z_{{p_i}^{\beta_i}}.\right)$

$= \prod_{i=1}^{k}\left(\sum_{q_i|p_i^{\alpha_i}, p_i^{\beta_i}} \tau\left(\frac{p_i^{\alpha_i}}{q_i}\right) \tau\left(\frac{p_i^{\beta_i}}{q_i}\right) \Phi(q_i)\right) =$

$$\sum_{q_1 q_2 \ldots q_k \left|\left(p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots . . p_r^{\alpha_r}, p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} \ldots p_r^{\beta_r}\right)\right.} \tau\left(\frac{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots . . . p_r^{\alpha_r}}{q_1 q_2 \ldots q_k}\right) \tau\left(\frac{, p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} \ldots\ p_r^{\beta_r}}{q_1 q_2 \ldots q_k}\right) \Phi(q_1 q_2 \ldots q_k)$$

$= \sum_{q|(m,n)} \tau\left(\frac{m}{q}\right) \tau\left(\frac{n}{q}\right) \boldsymbol{\Phi}(q)$ [Here $q_1 q_2 \ldots . . q_k$ = q]

We get the desired result.

4.5 Corollary :- If $p_1\ and\ p_2$ are different primes, then number of subgroup of group $Z_{{p_1}^{\alpha_1}} \otimes Z_{{p_2}^{\alpha_2}} \otimes\ Z_{{p_2}^{\beta_1}} \otimes\ Z_{{p_2}^{\beta_2}}$ are

$$\sum_{q\left|\left(p_1^{\alpha_1}p_2^{\beta_1},p_1^{\alpha_2}p_2^{\beta_2}\right)\right.} \tau\left(\frac{p_1^{\alpha_1}p_2^{\beta_1}}{q}\right)\tau\left(\frac{p_1^{\alpha_2}p_2^{\beta_2}}{q}\right)\Phi(q)$$

References

1. Butler M.L.Subgroup Lattices and Symmetric Functions. Mem. AMS ,vol .112,nr .539 ,1994.
2. Delsarte S. Fonctions de Mobius sur les groups abeliens finis. Annals of Math .49,(1948) 600-609
3. Djubjuk P.E. On the number of subgroups of a finite abelian group. Izv. Akad . Nauk SSSR Ser.Mat.12 ,(1948) 351-378.
4. A. Gallian ,Contemporary Abstract Algebra ,Narosa, 1999
5. V.Murali and B.B .Makamba ,On an equivalence of fuzzy subgroups I, Fuzzy sets and system. 123 (2001) ,259-264
6. CALUGAREANU ,GR.G, The total number of subgroups of a finite abelian group., Sci.Math .Jpn.(1) 60 (2004) ,157-167
7. MARIUS TARNAUCEANU, An arithmetic method of counting the subgroups of a finite abelian group, Bull .Math .Soc .Sci .Math .Roumanie Tome 53(101) No. 4, 2010 ,373-386
8. Yeh Y, On prime power abelian groups. Bull. AMS ,54 ,(1948) 323-327.
9. Suzuki M, On the lattice of subgroups of finite groups. Trans .Amer. Math .Soc. 70,(1951) 345-371.
10. Schmidt R. Subgroup lattices of groups.de Gruyter Expositions in Mathematics ,14. Walter de Gruyter,1994.
11. Remark R, Uber Untergruppen direkter Produkte von drei Faktoren .J. Reine Angew .Math 166, (1931) , 65-100.
12. Fuchs L, Infinite Abelian Groups, vol. 1 and 2 .Academic Press, New York , London 1970, 1973.