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Identifying Secure Domination in the Join and Corona of Graphs

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Abstract
Let G be a connected simple graph. A subset S of $V(G)$ is a dominating set of G if for every $v \in V(G)\backslash S$, there exists $x \in S$ such that $xv \in E(G)$. An Identifying code of a graph G is a dominating set $C \subseteq V(G)$ such that for every $v \in V(G), N_G[v] \cap C$ is distinct. An identifying code of a graph G is an identifying secure dominating set if for each $u \in V(G)\backslash C$, there exists $v \in C$ such that $uv \in E(G)$ and the set $(C\backslash\{v\}) \cup \{u\}$ is a dominating set of G. The minimum cardinality of an identifying secure dominating set of G, denoted by γ_s^{ID}, is called the identifying secure domination number of G. In this paper, the researchers initiate the study of the concept and give some important results. In particular, the researchers show some properties of the identifying secure dominating sets in join and corona of two graphs.

Mathematics Subject Classification: 05C69

Keywords: domination, secure, identifying, join, corona

Introduction
The emergence of the Königsberg problem gives the first concept of solving graph theory problems. Leonhard Euler (1707-1783) [1], a Swiss mathematician proposed a solution for the said problem in 1736. He provided solution to the problem by analyzing the structure of points and the line segments that connected them. Claude Berge in 1958 [2] first introduced the idea of domination in graphs. In the succeeding years, some prominent mathematicians supported the concept and added relevant studies. In 1962, Oystein Ore provided relevant definitions to support the concept of "dominating set" and the "domination number" [3].

In addition to dominating sets and their respective domination numbers, researchers around the world explored terms specific to dominating sets to serve as future reference for more studies. The Secure domination in graphs was studied and introduced by E.J. Cockayne et.al [4, 5]. In [6] Enriquez and Canoy, introduced a variant of domination in graphs, the concept of secure convex domination in graphs. Some studies on secure domination in graphs were found in the paper [7, 8, 9, 10, 11, 12].

One type of domination parameter is the identifying code of a graph. This was studied in 1998 by M.G. Karpovsky, et.al [13] in their paper "On a new class of codes for identifying vertices in graphs". They observed that the concept of identifying codes is that a graph is identifiable if and only if it is twin-free.

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the *vertex-set* of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the *edge-set* of G. The elements of $V(G)$ are called *vertices* and the cardinality $|V(G)|$ of $V(G)$ is the *order* of G. The elements of $E(G)$ are called *edges* and the cardinality $|E(G)|$ of $E(G)$ is the *size* of G. If $|V(G)| = 1$, then G is called a trivial graph. If $E(G) = \emptyset$, then G is called an empty graph. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. The elements of $N_G(v)$ are called *neighbors of* v. The *closed neighborhood* of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \cup_{v\in X} N_G(v)$. The *closed neighborhood* of X in G is the set $N_G[X] = \cup_{v\in X} N_G[v] = N_G(X) \cup X$. When no confusion arises, $N_G[x]$ [resp. $N_G(x)$] will be denoted by $N[x]$ [resp. N(x)]. A subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G)\backslash S$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. An identifying code of a graph G is a dominating set $C \subseteq V(G)$ such that for every $v \in V(G)$, $N_G[v] \cap C$ is distinct. The minimum cardinality of an identifying code of G, denoted by $\gamma^{ID}(G)$, is called the *identifying code number* of G. An identifying code of cardinality $\gamma^{ID}(G)$ is called a $\gamma^{ID} - set$ of G.

In addition to dominating sets and their respective domination numbers, researchers around the world explored terms specific to dominating sets to serve as future reference for more studies. This drives the researcher's interest to explore and introduce a new domination parameter, the identifying secure domination in graphs. Accordingly, an identifying code of a graph G is an identifying secure dominating set C if for each $u \in V(G)\backslash C$, there exists $v \in C$ such that $uv \in E(G)$ and the set $(C\backslash \{v\}) \cup \{u\}$ is a dominating set of G. The minimum cardinality of a identifying secure dominating set of G, denoted by $\gamma_s^{ID}(G)$, is called the identifying secure domination number of G. In this paper, the researchers initiate the study of the concept and give some important results. In particular, the researchers show some properties of the identifying secure dominating sets in join and corona of two graphs.

Unless otherwise stated, all graphs in this paper are assumed to be simple and connected. For general concepts, we refer the reader to [20].

Results

Let G be a connected nontrivial graph and let $C = \{v\}$ be dominating set in G. Then $N_G[v] \cap C = \{v\}$ and $N_G[x] \cap C = \{v\}$ for all $x \in V(G)\backslash\{v\}$. This implies that $N_G[v] \cap C$ is not distinct and hence C is not an identifying code of G and therefore, C is not an identifying secure dominating set of G. The following remark holds.

Remark 2.1 *If G has an identifying secure domination set, then $\gamma_s^{ID}(G) \geq 2$.*

Definition 2.2 *The join of two graphs G and H is the graph $G + H$ with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set*

$$E(G + H) = E(G) \cup E(H) \cup \{uv: u \in V(G), v \in V(H)\}.$$

The following results shows some characteristics of an identifying secure domination in graphs.

Theorem 2.3 *Let G and H be connected noncomplete graphs of order $m \geq 4$ and $n \geq 4$ respectively with*

$\gamma(G) \neq 1$ or $\gamma(H) \neq 1$. Then $S = S_G \cup S_H \subseteq V(G+H)$ is an identifying secure dominating set of $G+H$ if S_G and S_H are identifying codes of G and H respectively, and one of the following is satisfied.

i) $N_G[v] \cap S_G \neq S_G$ *for all* $v \in V(G)$.

ii) $N_H[u] \cup S_H$ *for all* $u \in V(H)$.

Proof: Suppose that $S = S_G \cup S_H \subseteq V(G+H)$ is an identifying secure dominating set of $G+H$. Suppose that S_G is not an identifying code of G.

Since the $m \geq 3$, the order of G, there exists $v \in V(G)$ such that $N_G[v] \cap S_G = N_G[v'] \cap S_G$ for some $v' \in V(G)$ where $v \neq v'$ by Remark 2.1. This implies that

$$\begin{aligned} N_{G+H}[v] \cap S &= \big(NG[v] \cup V\,(H)\big) \cap\ (SG \cup\ SH) \\ &= \big[\big(N_G[v] \cup V(H)\big) \cap S_G\big] \cup \big[\big(N_G[v] \cup V(H)\big) \cap S_H\big] \\ &= (N_G[v] \cap S_G) \cup S_H \\ &= (N_G[v'] \cap S_G) \cup S_H \\ &= (N_G[v'] \cup SH) \cap (S_G \cup S_H) \\ &= N_{G+H}[v'] \cap S \end{aligned}$$

Thus, S is not an identifying code of $G+H$ contrary to our assumption. Hence, S_G must be an identifying code of G. Similarly, S_H must be an identifying code of H.

Next, consider that $N_G[v] \cap S_G = S_G$ for some $v \in V(G)$ and $N_H[u] \cap S_H = S_H$ for some $u \in V(H)$. Then,

$$\begin{aligned} N_{G+H}[v] \cap S &= \big(NG[v] \cup V\,(H)\big) \cap\ (SG \cup\ SH) \\ &= \big[\big(N_G[v] \cup V(H)\big) \cap S_G\big] \cup \big[\big(N_G[v] \cup V(H)\big) \cap S_H\big] \\ &= (N_G[v] \cap S_G) \cup S_H \\ &= S_G \cup S_H, since\ N_G[v] \cap S_G = S_G \\ &= S_G \cup (N_H[u] \cap S_H), since\ N_G[u] \cap S_H = S_H \\ &= \big[\big(N_H[u] \cup V(G)\big) \cap S_G\big] \cup (N_H[u] \cap S_H) \\ &= \big[\big(N_H[u] \cup V(G)\big) \cap S_G\big] \cup \big[\big(N_H[u] \cup V(G)\big) \cap S_H\big] \\ &= [N_H[u] \cup V(G)] \cap (S_G \cup S_H) \\ &= N_{G+H}[u] \cap S \end{aligned}$$

This implies that S is not an identifying code of $G+H$ contrary to our assumption. Thus, either $N_G[v][\cap S_G \neq S_G$ or $N_H[u] \cap S_H \neq S_H$. If $N_G[v] \cap S_G \neq S_G$, then statement *(i)* is satisfied. If $N_H[u] \cap S_H \neq S_H$, then statement *(ii)* is satisfied. If $N_G[v] \cap S_G \neq S_G$ and $N_H[u] \cap S_H \neq S_H$, then both statement *(i)* and *(ii)* are satisfied.

For the converse, suppose that statement *(i)* is satisfied. Since S_G is an identifying code of G, it follows that S_G is a dominating set of G. Similarly, S_H is a dominating set of H. Clearly, $S = S_G \cup S_H$ is a dominating set of $G+H$. If $S = V(G+H)$, then S is a secure dominating set of $G+H$ (trivial).

Consider that $S \neq V(G+H)$ and let $u \in V(G+H) \backslash S$.

Case 1. If $u \in V(G) \backslash S_G$, then there exists $v \in S_G \subset S$ such that $uv \in E(G) \subset E(G+H)$. Since S_H is a dominating set of H, it follows that S_H is a dominating set of $G+H$. Thus, the set $S_H \subset S_v = (S \backslash \{v\}) \cup \{u\}$ is a dominating set of $G+H$.

Case 2. If $u \in V(H) \backslash S_H$, then there exists $v \in S_H \subset S$ such that $uv \in E(H) \subset E(G+H)$. Since S_G is a dominating set of G, it follows that S_G is a dominating set of $G+H$. Thus, the set $S_G \subset S_v = (S \backslash \{v\}) \cup \{u\}$ is a dominating set of $G+H$.

In any case, S is a secure dominating set of $G+H$ by definition.

Next, $N_G[v] \cap S_G \neq S_G$ for all $v \in V(G)$. Let $u \in V(G+H)$ and consider the following cases.

Case 1. If $u \in V(G)$, then

$$\begin{aligned} N_{G+H}[v] \cap S &= \big(N_G[v] \cup V(H)\big) \cap (S_G \cup S_H) \\ &= \big[\big(N_G[v] \cup V(H)\big) \cap S_G\big] \cup \big[\big(N_G[v] \cup V(H)\big) \cap S_H\big] \\ &= (N_G[v] \cap S_G) \cup S_H \\ &\neq (N_G[u] \cap S_G) \cup S_H, since\ S_G\ is\ an\ identifying\ code\ of\ G \\ &= \big[\big(N_G[u] \cup V(H)\big) \cap S_G\big] \cup \big[\big(N_G[u] \cup V(H)\big) \cap S_H\big] \\ &= \big(N_G[u] \cup V(H)\big) \cap (S_G \cup S_H) \\ &= N_{G+H}[u] \cap S, \end{aligned}$$

that is, $N_{G+H}[v] \cap S \neq N_{G+H}[u] \cap S$ for all $v, u \in V(G)$.

Case 2. If $u \notin V(G)$, then consider the following.

Subcase 1. $N_H[u] \cap S_H = S_H$ for some $u \in V(H)$. Then

$$\begin{aligned} N_{G+H}\,[v] \cap S &= \big(N_G[v] \cup V(H)\big) \cap (S_G \cup S_H) \\ &= \big[\big(N_G[v] \cup V(H)\big) \cap S_G\big] \cup \big[\big(N_G[v] \cup V(H)\big) \cap S_H\big] \\ &= (N_G[v] \cap S_G) \cup S_H \\ &\neq S_G \cup S_H, since\ N_G[v] \cap S_G \neq S_G \\ &= S_G \cup (N_H[u] \cap S_H) \\ &= (S_G \cup N_H[u]) \cap (S_G \cup S_H) \\ &\subseteq (V(G) \cup N_H[u]) \cap (S_G \cup S_H) \\ &= N_{G+H}[u] \cap S \end{aligned}$$

that is, $N_{G+H}[v] \cap S \neq N_{G+H}[u] \cap S$ for all $v \in V(G)$ and $u \in V(H)$.

Subcase 2. $N_H[u] \cap S_H \neq S_H$ for all $u \in V(H)$. Then,

$$\begin{aligned} N_{G+H}[v] \cap S &= \big(N_G[v] \cup V(H)\big) \cap (S_G \cup S_H) \\ &= \big[\big(N_G[v] \cup V(H)\big) \cap S_G\big] \cup \big[\big(N_G[v] \cup V(H)\big) \cap S_H\big] \\ &= (N_G[v] \cap S_G) \cup S_H \\ &\neq S_G \cup S_H, since\ N_G[v] \cap S_G \neq S_G \\ &\neq S_G \cup (N_H[u] \cap S_H), since\ N_G[u] \cap S_H \neq S_H \\ &= (S_G \cup N_H[u]) \cap (S_G \cup S_H) \\ &\subseteq (V(G) \cup N_H[u]) \cap (S_G \cup S_H) \\ &= N_{G+H}[u] \cap S, \end{aligned}$$

that is, $N_{G+H}[v] \cap S \neq N_{G+H}[u] \cap S$ for all $v \in V(G)$ and $u \in V(H)$.

In any case, S is an identifying code of $G+H$. Since S is a secure dominating set of $G+H$, it follows that S is an identifying secure dominating set of $G+H$. Similarly, if statement *(ii)* is satisfied, then S is an identifying secure dominating set of $G+H$. The proof is completed. ■

Remark 2.4 $\gamma_s^{ID}(P_m + P_n) = \gamma^{ID}(P_m) + \gamma^{ID}(P_n)$ *for some positive integers* $m \geq 5$ *and* $n \geq 5$.

The following Corollary is an immediate consequence of Theorem 2.3.

Corollary 2.5 *Let* $G = P_m$ *and* $H = P_n$. *Then for all positive integers* r *and* s,

$$\gamma_s^{ID}(G+H) = \begin{cases} \frac{n+m+2}{2}, & if\ n = 2r+3\ and\ m = 2s+3 \\ \frac{n+m+3}{2}, & if\ n = 2r+3\ and\ m = 2s+6 \\ \frac{n+m+4}{2}, & if\ n = 2r+6\ and\ m = 2s+6 \end{cases}$$

Proof: Suppose that $G = P_m$ and $H = P_n$ with S_G and S_H are identifying codes of G and H respectively, and $N_G[v] \cap S_G \neq S_G$ for all $v \in V(G)$. Then $S = S_G \cup S_H$ is an identifying secure dominating set of $G + H$ by Theorem 3.6(i).

Case 1. If $n = 2r + 3$ and $m = 2s + 3$ for all positive integers r and s, then let $G = [v_1, v_2, \dots, v_m]$ and $H = [u_1, u_2, \dots, u_n]$. The set

$$S_G = \left\{v_{2i-1} : i = 1,2, \dots, \frac{m+1}{2}\right\}$$

is a minimum identifying code of G and the set

$$S_H = \left\{u_{2i-1} : i = 1,2, \dots, \frac{n+1}{2}\right\}$$

is a minimum identifying code of H. Thus,

$$\begin{aligned} |S| &= |S_G \cup S_H| \\ &= |S_G| + |S_H| \\ &= \gamma^{ID}(G) + \gamma^{ID}(H) = \frac{m+1}{2} + \frac{n+1}{2} = \frac{n+m+2}{2} \end{aligned}$$

Thus, $\gamma^{ID}(G+H) = \frac{n+m+2}{2}$, by Remark 2.4

Case 2. If $n = 2r + 3$ and $m = 2s + 6$ for all positive integers r and s, then let $G = [v_1, v_2, \dots, v_m]$ and $H = [u_1, u_2, \dots, u_n]$. The set

$$S_G = \left\{v_{2i-1} : i = 1,2, \dots, \frac{m+1}{2}\right\}$$

is a minimum identifying code of G and the set

$$S_H = \left\{u_{2i-1} : i = 1,2, \dots, \frac{n-2}{2}\right\} \cup \{u_{n-2}, u_{n-1}\}$$

is a minimum identifying code of H. Thus,

$$\begin{aligned} |S| &= |S_G \cup S_H| \\ &= |S_G| + |S_H| \\ &= \gamma^{ID}(G) + \gamma^{ID}(H) = \left(\frac{m-2}{2} + 2\right) + \left(\frac{n-1}{2} + 2\right) = \frac{n+m+4}{2} \end{aligned}$$

Thus, $\gamma_s^{ID}(G+H)=\frac{n+m+4}{2}$, by Remark 2.4. ■

Let G and H be graphs of order m and n, respectively. The *corona* of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and m copies of H, and then joining the ith vertex of G to every vertex of the ith copy of H. The join of vertex v of G and a copy of H^v of H in the corona of G and H is denoted by $v + H^v$.

Remark 2.6 *For any connected graph G, $V(G)$ is a minimum dominating set in $G \circ H$.*

The following lemmas are needed to show some of the properties of the identifying secure dominating set in the corona of two graphs.

Lemma 2.7 *Let G be a connected graph and H be a connected non-complete graph of order $n \geq 4$. If $S = \bigcup_{v\in V(G)} S_v$, where S_v is an identifying code of H^v for all $v \in V(G)$ and $N_{H^v}[x] \cap S_v \neq S_v$ for all $x \in V(H^v)$, then S is an identifying secure domination of $G \circ H$.*

Proof: Suppose that $S = \bigcup_{v\in V(G)} S_v$ where S_v is an identifying code of H^v for all $v \in V(G)$ and $N_{H^v[x]} \cap S_v \neq S_v$ for all $x \in V(H^v)$. Since S_v is an identifying code of H^v for all $v \in V(G)$, it follows that S_v is a dominating set of H^v for all $v \in V(G)$. Hence, $S = \bigcup_{v\in V(G)} S_v$ is a dominating set of $G \circ H$. Now, let $u \in V(G \circ H)\backslash S$.

Case 1. If $u \in V(G)$, then $(S\backslash\{x\}) \cup \{u\}$ is a dominating set of $G \circ H$ for all $x \in S$, Remark 2.6. Since S_u is a dominating set of H^u for all $u \in V(G)$, there exists $x \in S_u \subset S$ such that $ux \in E(u + H^u) \subset E(G \circ H)$. Thus, for every $u \in V(G \circ H)\backslash S$ there exists $x \in S$ such that $ux \in E(G \circ H)$ and $(S\backslash\{x\}) \cup \{u\}$ is a dominating set of $G \circ H$ for all $u \in V(G)$.

Case 2. If $u \notin V(G)$, then $u \in V(H^v)\backslash S_v$. Since S_v is a dominating set of H^v for all $u \in V(H^v)\backslash S_v \subset V(G \circ H)\backslash S$, there exists $x \in S_v \subset S$ such that $ux \in E(H^v) \subset E(G \circ H)$. Further, S_v is an identifying code of H^v, implies that $(S_v\backslash\{x\}) \cup \{u\} \subset (S\backslash\{x\}) \cup \{u\}$ is a dominating set of $H^v \subset G \circ H$ for all $v \in V(G)$. Thus, for every $u \in V(G \circ H)\backslash S$ there exists $x \in S$ such that $ux \in E(G \circ H)$ and $(S\backslash\{x\}) \cup \{u\}$ is a dominating set of $G \circ H$ for all $v \in V(G)$.

Thus, in any case, S is a secure dominating set of $G \circ H$. Now, let $y \in V(G \circ H)$. Then, $N_{G\circ H}[y] \cap (\bigcup_{v\in V(G)} S_v)$.

Case 1. If $y \in V(G)$, then $N_{G\circ H}[y] \cap S = [\{y\} \cup V(H^y)] \cap S_y = S_y$. Since for all $x \in V(H^y)$, $N_{H^y}[x] \cap S_y \neq S_y$ where S_y is an identifying code of H^y, it follows that $N_{G\circ H} \cap S = [\{y\} \cup V(H^y)] \cap S_y = S_y$ is distinct.

Case 2. If $y \notin V(G)$, then $y \in V(H^{v'})$ for some $v' \in V(G)$. Thus, $N_{G\circ H}[y] \cap S = N_{H^{v'}}[y] \cap S_{v'}$ is distinct since $S_{v'}$ is an identifying code of $H^{v'}$.

In any case, the $N_{G\circ H}[y] \cap S$ is distinct. Thus, S is an identifying code of $G \circ H$. Since S is a secure dominating set of $G \circ H$, it follows that S is an identifying secure dominating set of $G \circ H$. ■

Lemma 2.8 *Let G be a connected graph and H be a connected non-complete graph of order $n \geq 4$. If $S = S_G \cup (\bigcup_{v\in V(G)} S_v)$, where $S_G \subset V(G)$ $(S_G \neq \emptyset)$ and S_v is an identifying code of $H^v$for all $v \in V(G)$, and $N_{H^v}[x] \cap S_v \neq S_v$ for all $x \in V(H^v)$, then S is an identifying secure dominating of $G \circ H$.*

Proof: Suppose that $S = V(G) \cup \left(\bigcup_{v\in V(G)} S_v\right)$, where S_v is an identifying code of H^v for all $v \in V(G)$ and $N_{H^v}[x] \cap S_v \neq S_v$ for all $x \in V(H^v)$. Note that $\bigcup_{v\in V(G)} S_v$ is an identifying secure domination of $G \circ H$ by Lemma 2.7. Since $\bigcup_{v\in V(G)} S_v \subset V(G) \cup (\bigcup_{v\in V(G)} S_v)$ it follows that $V(G) \cup (\bigcup_{v\in V(G)} S_v)$ is an identifying secure domination of $G \circ H$. Hence, S is an identifying secure domination of $G \circ H$.■

The following result shows some properties of the identifying secure dominating set in the corona of two connected graphs.

Theorem 2.10 *Let G be a connected graph and H be a connected non-complete graph of order $n \geq 4$. Then $S \subseteq V(G \circ H)$ is an identifying secure domination of $G \circ H$ if S_v is an identifying code of H^v, $N_{H^v}[x] \cap S_v \neq S_v$ for all $v \in V(G)$ and of the following is satisfied.*

(i) $S = \bigcup_{v \in V(G)} S_v$.

(ii) $S = S_G \cup \left(\bigcup_{v \in V(G)} S_v\right)$ *where* $S_G \neq \emptyset \subset V(G)$.

(iii) $S = V(G) \cup (\bigcup_{v \in V(G)} S_v)$.

Proof: Suppose that statement *(i)* is satisfied. Then $S = \bigcup_{v \in V(G)} S_v$ and $N_{H^v}[x] \cap S_v \neq S_v$ for all $x \in V(H^v)$, where S_v is an identifying code of H^v for all $v \in V(G)$. By Lemma 2.7, S is an identifying secure dominating set of $G \circ H$.

Suppose that statement *(ii)* is satisfied. The $S = S_G \cup \left(\bigcup_{v \in V(G)} S_v\right)$ where $S_G \subset V(G)$ $(S_G \neq \emptyset)$, S_v is an identifying code of H^v for all $v \in V(G)$, and $N_{H^v}[x] \cap S_v \neq S_v$ for all $x \in V(H^v)$. By Lemma 2.8, S is an identifying secure dominating set of $G \circ H$.

Suppose that statement *(iii)* is satisfied. Then, $S = V(G) \cup (\bigcup_{v \in V(G)} S_v)$ where S_v is an identifying code of H^v for all $v \in V(G)$, and and $N_{H^v}[x] \cap S_v \neq S_v$ for all $x \in V(H^v)$. By Lemma 2.9, S is an identifying secure dominating set of $G \circ H$.

The proof is complete. ■

The next result is a direct consequence of Theorem 2.10.

Corollary 2.11 *Let G be a connected graph and H be a graph. Then*

$$\gamma_s^{ID}(G \circ H) = \begin{cases} |V(G)|, & \text{if } |V(H)| = 1 \\ |V(G)| \cdot |V(H)|, & \text{if } H \text{ is an empty graph} \\ |V(G)| \cdot \gamma^{ID}(H), & \text{if otherwise.} \end{cases}$$

Proof: Let $S = \bigcup_{v \in v(G)} S_v$ and $N_{H^v}[x] \cap S_v \neq S_v$ for all $v \in V(G)$ and S_v is an identifying code of H^v for all $v \in V(G)$. Then by Theorem 2.10*(iii)*, S is an identifying secure dominating set of $G \circ H$.

Case 1. If $|V(H)| = 1$, then

$$\begin{aligned} \gamma_s^{ID}(G \circ H) \leq |S| &= \bigcup_{v \in V(G)} S_v \\ &= \sum_{v \in V(G)} |S_v| \\ &= |V(G)| \cdot |S_v| \text{ for all } S_v \text{ where } S_v \text{ is an identifying code of } H^v \text{ for all } v \in V(G) \\ &\leq |V(G)| \cdot 1 \text{ since } |V(H)| = 1 \\ &= |V(G)|. \end{aligned}$$

Thus, $\gamma_s^{ID}(G \circ H) \leq |V(G)|$. Since $|V(G)|$ is a minimum dominating set of $G \circ H$ by Remark 2.6, it follows that $\gamma_s^{ID}(G \circ H) \geq |V(G)|$. Hence, $\gamma_s^{ID}(G \circ H) = |V(G)|$.

Case 2. If H is an empty graph, then H is a graph where there are no edges between its vertices. Thus,

$$\gamma_s^{ID}(G \circ H) \leq |S| = \bigcup_{v \in V(G)} S_v$$

$$= \sum_{v \in V(G)} |S_v|$$
$$= |V(G)| \cdot |S_v| \; for\ all\ S_v\ where\ S_v\ is\ an\ identifying\ code\ of\ H^v\ for\ all\ v \in V(G)$$
$$\leq |V(G)| \cdot |V(H)| \; since\ S_v = |V(H)|\ as\ H\ is\ an\ empty\ graph$$
$$= |V(G)| \cdot |V(H)|.$$

Thus, $\gamma_s^{Id}(G \circ H) \leq |V(G)| \cdot |V(H)|$.

Now, let S^o be a $\gamma^{ID} - set$ of $G \circ H$. Then $S^o = \bigcup_{v^o \in V(G)} S_{v^o}$ for some identifying code S_{v^o} of H^{v^o} where $v^o \in V(G)$. Thus,

$$\gamma_s^{ID}(G \circ H) = |S^o|$$
$$= |\bigcup_{v^o \in V(G)} S_{v^o}|$$
$$= \sum_{v^o \in V(G)} |S_{v^o}|$$
$$= |V(G)| \cdot |S_{v^o}|$$
$$\geq |V(G)| \cdot \gamma^{ID}(H)$$
$$\gamma_s^{ID}(G \circ H) \geq |V(G)| \cdot |V(H)|, since\ \gamma^{ID}(H) = V(H)\ as\ H\ is\ an\ empty\ graph.$$

Hence, $\gamma_s^{ID}(G \circ H) \leq |V(G)| \cdot |V(H)|$ and $\gamma_s^{ID}(G \circ H) \geq |V(G)| \cdot |V(H)|$ implies that $\gamma_s^{ID}(G \circ H) = |V(G)| \cdot |V(H)|$.

Case 3. If otherwise,

$$\gamma_s^{ID}(G \circ H) \leq |S| = \bigcup_{v \in V(G)} S_v$$
$$= \sum_{v \in V(G)} |S_v|$$
$$= |V(G)| \cdot |S_v| \; for\ all\ S_v\ where\ S_v\ is\ an\ identifying\ code\ of\ H^v\ for\ all\ v \in V(G)$$
$$\leq |V(G)| \cdot \gamma^{ID}(H)$$

Now, let S^o be a $\gamma_s^{ID} - set$ of $G \circ H$. Then $S^o = \bigcup_{v^o \in V(G)} S_{v^o}$ for some identifying code S_{v^o} of H^{v^o} where $v_o \in V(G)$. Thus,

$$\gamma_s^{ID}(G \circ H) = |S^o|$$
$$= |\bigcup_{v^o \in V(G)} S_{v^o}|$$
$$= \sum_{v^o \in V(G)} |S_{v^o}|$$
$$= |V(G)| \cdot |S_{v^o}|$$
$$\geq |V(G)| \cdot \gamma^{ID}(H)$$
$$\gamma_s^{ID}(G \circ H) \geq |V(G)| \cdot \gamma^{ID}(H)$$

Hence, $\gamma_s^{ID}(G \circ H) \leq |V(G)| \cdot \gamma^{ID}(H)$ and $\gamma_s^{ID}(G \circ H) \geq |V(G)| \cdot \gamma^{ID}(H)$ implies that $\gamma_s^{ID}(G \circ H) = |V(G) \cdot \gamma^{ID}(H)$. ■

Conclusion and Recommendations

In this paper, we introduced a new parameter of domination in graphs – the identifying secure domination in graphs. Some properties of identifying secure domination in the join two graphs were given with the corresponding identifying secure domination number. Further, the identifying secure dominating set of the corona of two graphs were characterized and its corresponding identifying secure domination number was computed. This study will pave the way to new researches such as bounds and other binary operations of two connected graphs. Other parameters involving the identifying secure domination in graphs may also be explored. Finally, the characterization of a identifying secure domination in graphs of the lexicographic product, Cartesian product, and their bounds are promising extension of this study.

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