Identifying Secure Domination in the Cartesian and Lexicographic Products of Graphs

Haridel A. Aquiles¹, Marie Cris A. Bulay-og², Grace M. Estrada³, Carmelita M. Loquias⁴, Enrico L. Enriquez⁵

¹,²,³,⁴,⁵Department of Computer, Information Sciences and Mathematics School of Arts and Sciences University of San Carlos, 6000 Cebu City, Philippines

Abstract
Let G be a connected simple graph. A subset S of V(G) is a dominating set of G if for every v ∈ V(G) \ S, there exists x ∈ S such that xv ∈ E(G). An identifying code of a graph G is a dominating set C ⊆ V(G) such that for every v ∈ V(G), N_G[v] ∩ C is distinct. An identifying code of a graph G is an identifying secure dominating set if for each u ∈ V(G) \ C, there exists v ∈ C such that uv ∈ E(G) and the set (C\{v}) ∪ {u} is a dominating set of G. The minimum cardinality of an identifying secure dominating set of G, denoted by γ_{SD}^S, is called the identifying secure domination number of G. In this paper, the researchers initiate the study of the concept and give some important results. In particular, the researchers show some properties of the identifying secure dominating sets in the Cartesian product and lexicographic product of two connected graphs.

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1 Introduction
Claude Berge in 1958 and Oystein Ore in 1962 [1] introduced the domination in graphs. Domination in graphs started to flourish in 1977 when Ernie Cockayne and Stephen Hedetniemi published an article "Towards a theory of domination in graphs" [2]. Accordingly, a subset S of V(G) is a dominating set of G if for every v ∈ V(G) \ S, there exists x ∈ S such that xv ∈ E(G), that is N[S] = V(G). The domination number γ(G) of G is the smallest cardinality of a dominating set of G. Some studies on domination in graphs were found in the papers [3 - 31].

One type of domination parameter is identifying code of a graph. This was studied in 1998 by Karpovsky, et al in their paper "On a new class of codes for identifying vertices in graphs"[32]. They observed that a graph is identifiable if and only if it is twin-free. An Identifying code of a graph G is a dominating set C ⊆ V(G) such that for every v ∈ V(G), N_G[v] ∩ C is distinct. The minimum cardinality of an identifying code of G, denoted by γ^{ID}(G), is called the identifying code number of G. An identifying code of cardinality γ^{ID}(G) is called a γ^{ID} - set of G. Identifying code in graphs is also found in the paper [33].

One of the prominent extension topics of dominating sets is secure dominating sets [34]. A dominating set S is a
secure dominating set of $G$ if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{u\}) \cup \{u\}$ is a dominating set of $G$. The minimum cardinality of a secure dominating set of $G$, denoted by $\gamma_s(G)$, is called the secure domination number of $G$. Secure domination in graphs is studied in the papers [35 - 43].

The identifying code and secure domination in graphs motivate the researchers to introduce a new domination parameter, the identifying secure domination. An identifying code of a graph $G$ is an identifying secure dominating set $C$ if for each $u \in V(G) \setminus C$, there exists $v \in C$ such that $uv \in E(G)$ and the set $(C \setminus \{v\}) \cup \{u\}$ is a dominating set of $G$. The minimum cardinality of a identifying secure dominating set of $G$, denoted by $\gamma_{s}^{ID}(G)$, is called the identifying secure domination number of $G$. In this paper, the researchers initiate the study of the concept and give some important results. In particular, the researchers show some realization problems of identifying secure dominating sets and give some properties of the identifying secure dominating sets in the Cartesian product and lexicographic product of two connected graphs.

For the general terminology in graph theory, readers may refer to [44].

2 Results

From the definitions, the following result is immediate.

**Remark 2.1** Let $G$ be a nontrivial connected graph of order $n$. Then

\[ 1 \leq \gamma(G) \leq \gamma_{s}^{ID}(G) \leq n - 1. \]

Let $G$ be a nontrivial connected graph and $C = \{v\}$ be a dominating set in $G$. Then, $N_G[v] \cap C = \{v\}$ and $N_G[x] \cap C = \{v\}$ for all $x \in V(G) \setminus \{v\}$. This implies that $N_G[v] \cap C$ is not distinct and hence $C$ is not an identifying secure dominating set of $G$. Thus, the following remark holds.

**Remark 2.2** If $G$ has an identifying code, then $\gamma_{s}^{ID}(G) \geq 2$.

**Theorem 2.3** Let $a$, $b$, and $n$ be positive integers such that $1 \leq a \leq b \leq n - 1$. Then, there exists a connected nontrivial graph $G$ with $|V(G)| = n$ such that $\gamma(G) = a$ and $\gamma_{s}^{ID}(G) = b$.

**Proof:** Consider the following cases:

**Case 1.** Suppose that $1 = a < b < n - 1$.

Let $n \equiv 1 \pmod{5}$ and $5b = 2n + 3$. Consider the graph $G \cong F_n$, where $F_n = K_1 + P_{n-1}$ and $P_{n-1} = \{v_1, v_2, \ldots, v_{n-1}\}$. Then the set $A = V(K_1)$ is the $\gamma$-set and the set $B = \{v_{5k-4}, v_{5k-1} : k = 1, 2, \ldots, \frac{n-1}{5}\} \cup \{v_{n-1}\}$ is a $\gamma^{ID}$-set in $G$. Thus, $|V(G)| = n$, $\gamma(G) = |A| = 1 = a$, and $\gamma_{s}^{ID}(G) = |B| = \frac{n-1}{5} + \frac{n-1}{5} + 1 = \frac{2n+3}{5} = b$.

**Case 2.** Suppose that $1 < a = b < n - 1$. Let $n = 2a$. Consider the graph $G = P_a \circ K_1$ (see Figure 2.1).
The set $A = \{v_1, v_2, \ldots, v_a\}$ is a $\gamma - set$ and $\gamma_{s ID} - set$ in $G$. Thus, $|V(G)| = 2a = n$, $\gamma(G) = |A| = a$, and $\gamma_{s ID}(G) = a = b$.

Case 3. Suppose that $1 < a < b = n - 1$. Consider the graph $G \cong S_n$ where $S_n = K_1 + \bar{P}_{n-1}$ (see Figure 2.2).

Case 4. Suppose that $1 < a < b = n - 1$. Consider the graph $G$ with $C_4 = [v_1, \ldots, v_4]$, vertices $x_1, \ldots, x_{n-4}$, and edges $v_4x_1, \ldots, v_4x_{n-4}$ (see Figure 2.3).
The set $A = \{v_1, v_6\}$ is the $\gamma$-set and the set $B = \{v_1, v_2, v_3, x_1, x_2, ..., x_{n-4}\}$ is the set $\gamma^D_s$-set in $G$. Thus, $|V(G)| = n$, $\gamma(G) = |A| = 2 = a$, and

$$\gamma^D_s(G) = |B| = 3 + (n - 4) = n - 1 = b.$$ 

**Case 5.** Suppose that $1 < a < b < n - 1$. Let $b = n - 4$ and consider the graph $G$ with $C_6 = [v_1, ..., v_6]$, vertices $x_1, ..., x_{n-6}$, and edges $v_6x_1, ..., v_6x_{n-6}$ (see Figure 2.4).

![Figure 2.4: A graph $G$ with $\gamma(G) = 2$, and $\gamma^D_s(G) = n - 3$](image)

The set $A = \{v_3, v_6\}$ is the $\gamma$-set and the set $B = \{v_2, v_4, x_1, x_2, ..., x_{n-6}\}$ is the $\gamma^D_s$-set in $G$. Thus, $|V(G)| = n$, $\gamma(G) = |A| = 2 = a$, and

$$\gamma^D_s(G) = |B| = 2 + (n - 6) = n - 4 = b.$$ 

This completes the proofs. ■

The following corollary is an immediate consequence of Theorem 2.3.

**Corollary 2.4** The difference $\gamma^D_s(G) - \gamma(G)$ can be made arbitrarily large.

**Proof:** Let $n = 5k + 1$ where $k$ is a positive integer. By Theorem 3.3, there exists a connected graph $G$ such that $\gamma^D_s(G) = \frac{2n+3}{5}$ and $\gamma(G) = 1$. As a result, we have, $\gamma^D_s(G) - \gamma(G) = \frac{2n+3}{5} - 1 = \frac{2(n-1)}{5} = \frac{10k}{2} = 2k$, showing that $\gamma^D_s(G) - \gamma(G)$ can be made arbitrarily large. ■

The lexicographic product of two graphs $G$ and $H$ is the graph $G[H]$ with vertex-set $V(G[H]) = V(G) \times V(H)$ and the edge-set $E(G[H])$ satisfying the following conditions: $(x, u)(y, v) \in E(G[H])$ if and only if either $xy \in E(G)$ or $x = y$ and $uv \in E(H)$.

Note that a non-empty subset $C$ of $V(G[H]) = V(G) \times V(H)$ can be written as $C = \bigcup_{x \in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$. In the following results, we shall be using this form to denote any non-empty subset $C$ of $V(G[H])$.

**Theorem 2.5** Let $G = P_m$ and $H = P_n$ with $m \geq 3$ and $n = 2k + 3$ for some positive integer $k$. If $C = \bigcup_{x \in V(G)}(\{x\} \times T_x)$ where $T_x$ is an identifying secure domination set of $H$ and $N_H[u] \cap T_x \neq T_x$ for each $x \in V(G)$ and for some $u \in V(H)$, then $C$ is an identifying secure domination set of $G[H]$.

**Proof:** Suppose that $C = \bigcup_{x \in V(G)}(\{x\} \times T_x)$ where $T_x$ is an identifying secure dominating set of $H = P_n = [u_1, u_2, ..., u_n]$ for each $x \in V(G)$ and $N_H[u] \cap T_x \neq T_x$ for some $u \in V(H)$. If $n = 2k + 3$ for some positive
integer \( k \), then

**Case I.** Consider \( T_x = \{ u_{2k-1}: k = 1, 2, 3, \ldots, \frac{n+1}{2} \} \) for all \( x \in V(G) \).

\[
C = \bigcup_{x \in V(G)} \{ x \} \times T_x \\
= \bigcup_{x \in V(G)} \{ x \} \times \{ u_{2k-1}: k = 1, 2, 3, \ldots, \frac{n+1}{2} \} \\
= \bigcup_{x \in V(G)} \{ (x, u_{2k-1}): k = 1, 2, 3, \ldots, \frac{n+1}{2} \}
\]

Let \((v, u) \in V(G[H]) \setminus C\). Then \((v, u) = (v, u_{2k})\) for any \( k \in \{ 1, 2, 3, \ldots, \frac{n-1}{2} \} \) and \( u \in V(G) \). Further, there exists \((v, u_{2k-1}) \in C\) for some \( k \in \{ 1, 2, 3, \ldots, \frac{n+1}{2} \} \) such that \((v, u)(v, u_{2k-1}) \in E(G[H])\) and \((C \setminus \{ (v, u_{2k-1}) \}) \cup \{ (v, u) \} \) is a dominating set of \( G[H] \). Hence, \( C \) is a secure dominating set of \( G[H] \) be definition.

Let \( G = [v_1, v_2, v_3, \ldots, v_m] \) and \((v, u), (v', u') \in V(G[H]), (v, u) \neq (v', u')\). Then \( v, v' \in V(G) = \{ v_1, v_2, v_3, \ldots, v_m \} \) and \( u, u' \in V(H) = \{ u_1, u_2, u_3, \ldots, u_n \} \).

The \( N_{G[H]}[(v, u)] \) in \( G[H] \) can be expressed as one of the following.

\[
N_{G[H]}[(v_1, u_j)] = \left\{ \left( (v_1, u_j) \right) \cup \left( \{ v_2 \} \times T_{v_2} \right) \right\} \text{ for } j = 1, 3, 5, \ldots, n \\
\text{or } \left\{ \left( (v_1, u_{j-1}) \right) \cup \left( (v_1, u_{j+1}) \right) \cup \left( \{ v_2 \} \times T_{v_2} \right) \right\} \text{ for } j = 2, 4, 6, \ldots, n - 1
\]

\[
N_{G[H]}[(v_i, u_j)] = \left\{ \left( (v_i, u_j) \right) \cup \left( \{ v_i \} \times T_{v_{i-1}} \right) \cup \left( \{ v_{i+1} \} \times T_{v_{i+1}} \right) \right\} \\
\text{for } i \neq 1 \text{ or } i \neq m \text{ and } j = 2k - 1, k = 1, 2, \ldots, \frac{n+1}{2} \\
\text{or } \left\{ \left( (v_i, u_{j-1}) \right) \cup \left( (v_i, u_{j+1}) \right) \cup \left( \{ v_{i-1} \} \times T_{v_{i-1}} \right) \cup \left( \{ v_{i+1} \} \times T_{v_{i+1}} \right) \right\} \\
\text{for } i \neq 1 \text{ or } i \neq m \text{ and } j = 2k, k = 1, 2, \ldots, \frac{n-1}{2}
\]

\[
N_{G[H]}[(v_m, u_j)] = \left\{ \left( (v_m, u_j) \right) \cup \left( \{ v_{m-1} \} \times T_{v_{m-1}} \right) \right\} \text{ for } j = 2k - 1, k = 1, 2, \ldots, \frac{n+1}{2} \\
\text{or } \left\{ \left( (v_m, u_{j-1}) \right) \cup \left( (v_m, u_{j+1}) \right) \cup \left( \{ v_{m-1} \} \times T_{v_{m-1}} \right) \right\} \\
\text{for } j = 2k \text{ and } k = 1, 2, \ldots, \frac{n-1}{2}
\]

**Subcase I.** Consider \((v, u) = (v_j, u_j)\) for some \( j \in \{ 1, 3, 5, \ldots, n \} \) and \( q \in \{ 2, 4, 6, \ldots, n - 1 \} \). Then

\[
N_{G[H]}[(v, u)] \cap C = N_{G[H]}[(v_1, u_j)] \cap \left( \bigcup_{x \in V(G)} \{ x \} \times T_x \right) \\
= \left\{ (v_1, u_j) \cup (v_2) \times T_{v_2} \right\} \cap \left( \bigcup_{x \in V(G)} \{ x \} \times T_x \right) \\
= \left\{ (v_1, u_j) \cap \left( \bigcup_{x \in V(G)} \{ x \} \times T_x \right) \right\} \cup \left( (v_2) \times T_{v_2} \right) \cap \left( \bigcup_{x \in V(G)} \{ x \} \times T_x \right)
\]
= \{(v_1, u_j)\} \cup \{v_2 \times T_{v_2}\}

\neq \{(v_1, u_{q-1})\} \cup \{(v_1, u_{q+1})\} \cup \{v_2 \times T_{v_2}\}

= \left[\{(v_1, u_{q-1})\} \cup \{(v_1, u_{q+1})\} \cup \{v_2 \times T_{v_2}\}\right] \cap \bigcup_{x \in \mathcal{V}(G)} \{(x) \times T_x\}

= N_{G[H]}[(v_1, u_q)] \cap \bigcup_{x \in \mathcal{V}(G)} \{(x) \times T_x\}

= N_{G[H]}[(v, u')] \cap \mathcal{C} with (v, u') = (v_1, u_q)

This implies that \(N_{G[H]}[(v, u)] \cap \mathcal{C} \neq N_{G[H]}[(v, u')] \cap \mathcal{C}\) for \((v, u) \neq (v, u')\).

Subcase 2. If \((v, u) = (v_m, u_j)\) for some \(j \in \{1, 3, 5, ..., n\}\) and \(q \in \{2, 4, 6, ..., n-1\}\). Then

\[ N_{G[H]}[(v, u)] \cap \mathcal{C} = N_{G[H]}[(v_m, u_j)] \cap \bigcup_{x \in \mathcal{V}(G)} \{(x) \times T_x\} \]

\[ = \left[\{(v_m, u_j)\} \cup \{v_{m-1} \times T_{v_{m-1}}\}\right] \cap \bigcup_{x \in \mathcal{V}(G)} \{(x) \times T_x\} \]

\[ \neq \{(v_m, u_{q-1})\} \cup \{(v_m, u_{q+1})\} \cup \{v_{m-1} \times T_{v_{m-1}}\} \cap \bigcup_{x \in \mathcal{V}(G)} \{(x) \times T_x\} \]

\[ = N_{G[H]}[(v_m, u_q)] \cap \bigcup_{x \in \mathcal{V}(G)} \{(x) \times T_x\}, q = 2, 4, 6, ..., n-1 \]

This implies that \(N_{G[H]}[(v, u)] \cap \mathcal{C} \neq N_{G[H]}[(v, u')] \cap \mathcal{C}\) for \((v, u) \neq (v, u')\).

Subcase 3. If \((v, u) = (v_i, u_j)\) with \(i \neq 1\) or \(i \neq m\), for some \(j \in \{1, 3, 5, ..., n\}\) and \(q = \{2, 4, 6, ..., n-1\}\). Then

\[ N_{G[H]}[(v, u)] \cap \mathcal{C} = N_{G[H]}[(v_i, u_j)] \cap \bigcup_{x \in \mathcal{V}(G)} \{(x) \times T_x\} \]

\[ = \left[\{(v_i, u_j)\} \cup \{v_{i-1} \times T_{v_{i-1}}\} \cup \{v_{i+1} \times T_{v_{i+1}}\}\right] \cap \bigcup_{x \in \mathcal{V}(G)} \{(x) \times T_x\} \]

\[ = \left[\{(v_i, u_j)\} \cap \bigcup_{x \in \mathcal{V}(G)} \{(x) \times T_x\}\right] \cup \{v_{i-1} \times T_{v_{i-1}}\} \cap \bigcup_{x \in \mathcal{V}(G)} \{(x) \times T_x\} \]

\[ \cup \{\{v_{i+1} \times T_{v_{i+1}}\} \cap \bigcup_{x \in \mathcal{V}(G)} \{(x) \times T_x\}\} \]

\[ = \{(v_i, u_j)\} \cup \{v_{i-1} \times T_{v_{i-1}}\} \cup \{v_{i+1} \times T_{v_{i+1}}\} \neq \{(v_i, u_{q-1})\} \cup \{(v_i, u_{q+1})\} \cup \{v_{i-1} \times T_{v_{i-1}}\} \cup \{v_{i+1} \times T_{v_{i+1}}\} \]
\[ u = \bigcup \{ x \times T_x \} \]

\[ \bigcup \{ (v_{i-1}, x) \times T_{v_{i-1}} \} \cap \bigcup \{ (v_{i+1}, x) \times T_{v_{i+1}} \} \cup \bigcup \{ (v_{i-1}, x) \times T_{v_{i-1}} \} \cup \bigcup \{ (v_{i+1}, x) \times T_{v_{i+1}} \} \]

\[ = \bigcup \{ \{ (v_i, u_{q-1}) \} \cup \{ (v_i, u_{q+1}) \} \cup \{ (v_{i-1}, x) \times T_{v_{i-1}} \} \cup \{ (v_{i+1}, x) \times T_{v_{i+1}} \} \cap \bigcup \{ x \times T_x \} \]

\[ = N_{G[H]}[(v_i, u_q)] \cap \bigcup \{ x \times T_x \} \]

This implies that \( N_{G[H]}[(v, u')] \cap C = N_{G[H]}[(v, u)] \cap C \) for some \( q \in \{2, 4, 6, ..., n - 1\} \). Thus, \( N_{G[H]}[(v, u)] \cap C \) is distinct. Hence, \( C \) is an identifying code of \( G[H] \). Since \( C \) is also a secure dominating set of \( G[H] \), it follows that \( C \) is an identifying secure dominating set of \( G[H] \).

**Case 2.** Consider \( T_x = \{ u_{4k-2}, u_{4k-1}, u_{4k}: k = 1, 2, 3, ..., \frac{n-1}{4} \} \) for all \( x \in V(G) \).

\[ C = \bigcup \{ \{ x \times T_x \} \} \]

\[ = \bigcup \{ \{ x \times \{ u_{4k-2}, u_{4k-1}, u_{4k}: k = 1, 2, 3, ..., \frac{n-1}{4} \} \} \}

\[ = \bigcup \{ \{ (x, u_{4k-2}), (x, u_{4k-1}), (x, u_{4k}): k = 1, 2, 3, ..., \frac{n-1}{4} \} \}

Let \((v, u) \in V(G[H]) \setminus C\). Then \((v, u) = (v, u_{4k-3})\) for any \( k \in \{1, 2, 3, ..., \frac{n+3}{4} \} \) and \( v \in V(G) \). Further, there exists \((v, u_{4k-2}) \in C\) for some \( k \in \{1, 2, 3, ..., \frac{n+3}{4} \} \) such that \((v, u)(v, u_{4k-2}) \in E(G[H])\) and \( (C \setminus \{(v, u_{4k-2})\}) \cap \{(v, u)\} \) is a dominating set of \( G[H] \). Hence, \( C \) is a secure dominating set of \( G[H] \) by definition.

Let \( G = \{ v_1, v_2, v_3, ..., v_m \} \) and \((v, u), (v', u') \in V(G[H]), (v, u) \neq (v', u')\). Then \( v, v' \in V(G) = \{ v_1, v_2, v_3, ..., v_m \} \) and \( u, u' \in V(H) = \{ u_1, u_2, u_3, ..., u_m \} \).

The \( N_{G[H]}[(v, u)] \) in \( G[H] \) can be expressed as one of the following.

1. If \((v, u) = (v_1, u_j),\) then \( N_{G[H]}[(v_1, u_j)] = \{(v_1, u_j), (v_1, u_{j+1})\} \cup \{(v_2) \times T_{v_2}\} \) for \( j = 2, 6, 10, ..., n - 3 \)

or \( \{(v_1, u_{j-1}), (v_1, u_j), (v_1, u_{j+1})\} \cup \{(v_2) \times T_{v_2}\} \) for \( j = 3, 7, 11, ..., n - 2 \)

or \( \{(v_1, u_{j-1}), (v_1, u_j)\} \cup \{(v_2) \times T_{v_2}\} \) for \( j = 4, 8, 12, ..., n - 1 \)

or \( \{(v_1, u_{j-1}), (v_1, u_{j+1})\} \cup \{(v_2) \times T_{v_2}\} \)
for } j = 5, 9, 13, \ldots, n - 4, (n \neq 5)
\text{ or } \{v_1, u_n\} \cup \{(v_2) \times T_{v_2}\} \text{ for } j = n

(ii) If } (v, u) = (v_m, u_j), \text{ then }
N_{G[H]}[(v_m, u_j)] = \{(v_m, u_{j+1})\} \cup \{(v_{m-1}) \times T_{v_{m-1}}\} \text{ for } j = 2, 6, 10, \ldots, n - 3
\text{ or } \{(v_m, u_{j-1}), (v_m, u_{j+1})\} \cup \{(v_{m-1}) \times T_{v_{m-1}}\} \text{ for } j = 3, 7, 11, \ldots, n - 2
\text{ or } \{(v_m, u_{j-1})\} \cup \{(v_{m-1}) \times T_{v_{m-1}}\} \text{ for } j = 4, 8, 12, \ldots, n - 1
\text{ or } \{(v_m, u_{j-1}), (v_m, u_{j+1})\} \cup \{(v_{m-1}) \times T_{v_{m-1}}\} \text{ for } j = 5, 9, 13, \ldots, n - 4, (n \neq 5)
\text{ or } \{(v_m, u_{j-1})\} \cup \{(v_{m-1}) \times T_{v_{m-1}}\} \text{ for } j = 1
\text{ or } \{(v_m, u_{n-1})\} \cup \{(v_{m-1}) \times T_{v_{m-1}}\} \text{ for } j = n

(iii) If } (v, u) = (v_i, u_j) \text{ with } i \neq 1 \text{ or } i \neq m, \text{ then }
N_{G[H]}[(v_i, u_j)] = \{(v_i, u_{j+1})\} \cup \{(v_{i-1}) \times T_{v_{i-1}}\} \cup \{(v_{i+1}) \times T_{v_{i+1}}\} \text{ for } j = 2, 6, 10, \ldots, n - 3
\text{ or } \{(v_i, u_{j-1}), (v_i, u_{j+1})\} \cup \{(v_{i-1}) \times T_{v_{i-1}}\} \cup \{(v_{i+1}) \times T_{v_{i+1}}\} \text{ for } j = 3, 7, 11, \ldots, n - 2
\text{ or } \{(v_i, u_{j-1}), (v_i, u_{j+1})\} \cup \{(v_{i-1}) \times T_{v_{i-1}}\} \cup \{(v_{i+1}) \times T_{v_{i+1}}\} \text{ for } j = 4, 8, 12, \ldots, n - 1
\text{ or } \{(v_i, u_{j-1}), (v_i, u_{j+1})\} \cup \{(v_{i-1}) \times T_{v_{i-1}}\} \cup \{(v_{i+1}) \times T_{v_{i+1}}\} \text{ for } j = 5, 9, 13, \ldots, n - 4, (n \neq 5)
\text{ or } \{(v_i, u_{j-1}), (v_i, u_{j+1})\} \cup \{(v_{i-1}) \times T_{v_{i-1}}\} \cup \{(v_{i+1}) \times T_{v_{i+1}}\} \text{ for } j = 1
\text{ or } \{(v_i, u_{n-1})\} \cup \{(v_{i-1}) \times T_{v_{i-1}}\} \cup \{(v_{i+1}) \times T_{v_{i+1}}\} \text{ for } j = n

Subcase 1. Consider } (v, u) = (v_1, u_j) \text{ for some } j \in \{2, 6, 10, \ldots, n - 3\} \text{ and } q \in \{5, 9, \ldots, n - 4\}.
Then

\begin{align*}
N_{G[H]}[(v, u)] \cap C &= N_{G[H]}[(v_1, u_j)] \cap \left( \bigcup_{x \in V(G)} ((x) \times T_x) \right) \\
&= \{(v_1, u_j), (v_1, u_{j+1})\} \cup \{(v_2) \times T_{v_2}\} \cap \left( \bigcup_{x \in V(G)} ((x) \times T_x) \right) \\
&= \left[ \{(v_1, u_j)\} \cap \left( \bigcup_{x \in V(G)} ((x) \times T_x) \right) \right] \\
&\cup \left[ \{(v_1, u_{j+1})\} \cap \left( \bigcup_{x \in V(G)} ((x) \times T_x) \right) \right] \\
&\cup \left[ \{(v_2) \times T_{v_2}\} \cap \left( \bigcup_{x \in V(G)} ((x) \times T_x) \right) \right]
\end{align*}
\[
= \{(v_1, u_j)\} \cup \{(v_1, u_{j+1})\} \cup \{(v_2) \times T_{v_2}\}
\]

\[
\neq \{(v_1, u_{q-1})\} \cup \{(v_1, u_{q+1})\} \cup \{(v_2) \times T_{v_2}\}
\]

\[
= \left\{\left(\{(v_1, u_{q-1})\} \cup \{(v_1, u_{q+1})\} \cup \{(v_2) \times T_{v_2}\}\right) \cap \bigcup_{x \in V(G)} \{(x) \times T_x\}\right\}
\]

\[
= N_{G[H]}(v_1, u_q) \cap \bigcup_{x \in V(G)} \{(x) \times T_x\}
\]

This implies that \(N_{G[H]}(v, u) \cap C \neq N_{G[H]}(v, u') \cap C\) with \((v, u') = (v_1, u_q)\).

Similarly, if \(N_{G[H]}(v_1, u_j) = \{(v_1, u_{j-1}), (v_1, u_j), (v_1, u_{j+1})\}\) for \(j = 3, 7, 11, ..., n - 2\), or \(N_{G[H]}(v_1, u_j) = \{(v_1, u_{j-1}), (v_1, u_j)\} \cup \{(v_2) \times T_{v_2}\}\) for \(j = 4, 8, 12, ..., n - 1\), or \(N_{G[H]}(v_1, u_j) = \{(v_1, u_{j-1}), (v_1, u_{j+1})\} \cup \{(v_2) \times T_{v_2}\}\) for \(j = 5, 9, 13, ..., n - 4\), \((n \neq 5)\), then \(N_{G[H]}(v, u) \cap C \neq N_{G[H]}(v, u') \cap C\) for \((v, u) \neq (v, u')\).

**Subcase 2.** Consider \((v, u) = (v_m, u_j)\) for some \(j \in \{2, 6, 10, ..., n - 3\}\) and \(q \in \{5, 9, ..., n - 4\}\).

Then

\[
N_{G[H]}(v, u) \cap C = N_{G[H]}(v_m, u_j) \cap \left\{\bigcup_{x \in V(G)} \{(x) \times T_x\}\right\}
\]

\[
= \{(v_m, u_j), (v_m, u_{j+1})\} \cup \{(v_{m-1}) \times T_{v_{m-1}}\} \cap \left\{\bigcup_{x \in V(G)} \{(x) \times T_x\}\right\}
\]

\[
= \left\{(v_m, u_j) \cap \left\{\bigcup_{x \in V(G)} \{(x) \times T_x\}\right\}\right\} \cup \{(v_m, u_{j+1}) \cap \left\{\bigcup_{x \in V(G)} \{(x) \times T_x\}\right\}\right\}
\]

\[
= \left\{(v_m, u_{j-1}) \cup \{(v_m, u_{j+1})\} \cup \{(v_{m-1}) \times T_{v_{m-1}}\}\right\} \cap \left\{\bigcup_{x \in V(G)} \{(x) \times T_x\}\right\}
\]

This implies that \(N_{G[H]}(v, u) \cap C \neq N_{G[H]}(v, u') \cap C\) with \((v, u') = (v_m, u_q)\).

Similarly, if \(N_{G[H]}(v_m, u_j) = \{(v_m, u_{j-1}), (v_m, u_j), (v_m, u_{j+1})\}\) for \(j = 3, 7, 11, ..., n - 2\), or \(N_{G[H]}(v_m, u_j) = \{(v_m, u_{j-1}), (v_m, u_j)\} \cup \{(v_{m-1}) \times T_{v_{m-1}}\}\) for \(j = 4, 8, 12, ..., n - 1\) or...
\[N_{G[H]}([v_m, u_j]) = \{(v_m, u_{j-1}), (v_m, u_{j+1})\} \cup \{(v_{m-1}) \times T_{v_{m-1}}\} \text{ for } j = 5, 9, 13, \ldots, n - 4, (n \neq 5), \text{ then } \]

\[N_{G[H]}([v, u]) \cap C \neq N_{G[H]}([v, u']) \cap C \text{ for } (v, u) \neq (v, u').\]

**Subcase 3.** Consider \((v, u) = (v_i, u_j)\) for some \(j \in \{2, 6, 10, \ldots, n - 3\}\) and \(q \in \{5, 9, \ldots, n - 4\}\). Then

\[N_{G[H]}([v, u]) \cap C = N_{G[H]}([v, u]) \cap \left( \bigcup_{x \in V(G)} ([x] \times T_x) \right) \]

\[= \left( \{(v_i, u_j), (v_i, u_{j+1})\} \cup \{(v_{i-1}) \times T_{v_{i-1}}\} \cup \{(v_{i+1}) \times T_{v_{i+1}}\} \right) \cap \left( \bigcup_{x \in V(G)} ([x] \times T_x) \right) \]

\[= \left( \{(v_i, u_j), (v_i, u_{j+1})\} \cup \{(v_{i-1}) \times T_{v_{i-1}}\} \cup \{(v_{i+1}) \times T_{v_{i+1}}\} \right) \cap \left( \bigcup_{x \in V(G)} ([x] \times T_x) \right) \]

\[= \{(v_i, u_{q-1})\} \cup \{(v_i, u_{q+1})\} \cup \{(v_{i-1}) \times T_{v_{i-1}}\} \cup \{(v_{i+1}) \times T_{v_{i+1}}\} \]

\[= N_{G[H]}([v_i, u_q]) \cap \left( \bigcup_{x \in V(G)} ([x] \times T_x) \right) \]

This implies that \(N_{G[H]}([v, u]) \cap C \neq N_{G[H]}([v, u']) \cap C \text{ for } (v, u) \neq (v, u').\)

Similarly, if \(N_{G[H]}([v, u]) = \{(v_i, u_{j-1}), (v_i, u_j), (v_i, u_{j+1})\} \cup \{(v_{i-1}) \times T_{v_{i-1}}\} \cup \{(v_{i+1}) \times T_{v_{i+1}}\} \) for \(j = 3, 7, 11, \ldots, n - 2\), or \(N_{G[H]}([v, u]) = \{(v_i, u_{j-1}), (v_i, u_j)\} \cup \{(v_{i-1}) \times T_{v_{i-1}}\} \cup \{(v_{i+1}) \times T_{v_{i+1}}\} \) for \(j = 4, 8, 12, \ldots, n - 1\), or \(N_{G[H]}([v, u]) = \{(v_i, u_{j-1}), (v_i, u_{j+1})\} \cup \{(v_{i-1}) \times T_{v_{i-1}}\} \cup \{(v_{i+1}) \times T_{v_{i+1}}\} \) for \(j = 5, 9, 13, \ldots, n - 4, (n \neq 5), \text{ then } N_{G[H]}([v, u]) \cap C \neq N_{G[H]}([v, u']) \cap C \text{ for } (v, u) \neq (v, u').\)

Thus, \(N_{G[H]}([v, u]) \cap C\) is distinct. Hence, \(C\) is an identifying code of \(G[H]\). Since \(C\) is also a secure dominating set of \(G[H]\), it follows that \(C\) is an identifying secure domination set of \(G[H]\).

The following result is an immediate consequence of **Theorem 2.5**.

**Corollary 2.6** Let \(G = P_m\) and \(H = P_n\) with \(m \geq 3\) and \(n = 2k + 3\) for some positive integer \(k\). Then
\begin{equation*}
\gamma_s^{ID}(G[H]) \leq m \cdot \gamma_s^{ID}(H).
\end{equation*}

\textbf{Proof}: Given that } G = P_m \text{ and } H = P_n \text{ with } m \geq 3 \text{ and } n = 2k + 3 \text{ for some positive integer } k. \text{ Suppose that } C = \bigcup_{x \in V(G)} \{(x) \times T_x\} \text{ where } T_x \text{ is an identifying secure dominating set of } H \text{ and } N_H[u] \cap T_x \neq T_x \text{ for each } x \in V(G) \text{ and for some } u \in V(H). \text{ Then } C \text{ is an identifying secure dominating set of } G[H] \text{ by Theorem 2.5. Thus,}

\begin{align*}
\gamma_s^{ID}(G[H]) & \leq |C| \\
& = \left| \bigcup_{x \in V(G)} \{(x) \times T_x\}, T_x \in V(H) \text{ for all } x \in V(G) \right| \\
& = |V(G) \times T_x|, T_x \in V(H) \text{ for all } x \in V(G) \\
& = |V(G)| \cdot |T_x|, T_x \in V(H) \text{ for all } x \in V(G) \\
& = m \cdot \gamma_s^{ID}(H).
\end{align*}

Hence, \( \gamma_s^{ID}(G[H]) \leq m \cdot \gamma_s^{ID}(H). \) \( \blacksquare \)

\section{Conclusion}

This study showed that the identifying secure domination number of a graph exists, and the characterization of this domination parameter resulting from the lexicographic product of two graphs was presented. This study will result in new research such as bounds and other binary operations of two graphs. Other parameters involving the identifying secure domination in graphs may also be explored. Finally, the characterization of an identifying secure domination in graphs and its bounds is a promising extension of this study.

\textbf{References}

30. J.N.C. Serrano and E.L. Enriquez, Fair Doubly Connected Domination in the Corona and the Cartesian Product of Two Graphs, International Journal of Mathematics Trends and Technology,


