International Journal for Multidisciplinary Research (IJFMR)

E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

Identifying Secure Domination in the Cartesian and Lexicographic Products of Graphs

Haridel A. Aquiles[1], Marie Cris A. Bulay-og[2], Grace M. Estrada[3], Carmelita M. Loquias[4], Enrico L. Enriquez[5]

[1,2,3,4,5]Department of Computer, Information Sciences and Mathematics School of Arts and Sciences
University of San Carlos, 6000 Cebu City, Philippines

Abstract

Let G be a connected simple graph. A subset S of $V(G)$ is a dominating set of G if for every $v \in V(G)\backslash S$, there exists $x \in S$ such that $xv \in E(G)$. An identifying code of a graph G is a dominating set $C \subseteq V(G)$ such that for every $v \in V(G), N_G[v] \cap C$ is distinct. An identifying code of a graph G is an identifying secure dominating set if for each $u \in V(G)\backslash C$, there exists $v \in C$ such that $uv \in E(G)$ and the set $(C\backslash\{v\}) \cup \{u\}$ is a dominating set of G. The minimum cardinality of an identifying secure dominating set of G, denoted by γ_S^{ID}, is called the identifying secure domination number of G. In this paper, the researchers initiate the study of the concept and give some important results. In particular, the researchers show some properties of the identifying secure dominating sets in the Cartesian product and lexicographic product of two connected graphs.

Mathematics Subject Classification: 05C69

Keywords: domination, secure, identifying, Cartesian, lexicographic

1 Introduction

Claude Berge in 1958 and Oystein Ore in 1962 [1] introduced the domination in graphs. Domination in graphs started to flourish in 1977 when Ernie Cockayne and Stephen Hedetniemi published an article "Towards a theory of domination in graphs" [2]. Accordingly, a subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G)\backslash S$, there exists $x \in S$ such that $xv \in E(G)$, that is $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. Some studies on domination in graphs were found in the papers [3 - 31].

One type of domination parameter is identifying code of a graph. This was studied in 1998 by Karpovsky, et al in their paper "On a new class of codes for identifying vertices in graphs"[32]. They observed that a graph is identifiable if and only if it is twin-free. An Identifying code of a graph G is a dominating set $C \subseteq V(G)$ such that for every $v \in V(G)$, $N_G[v] \cap C$ is distinct. The minimum cardinality of an identifying code of G, denoted by $\gamma^{ID}(G)$, is called the *identifying code number* of G. An identifying code of cardinality $\gamma^{ID}(G)$ is called a $\gamma^{ID} - set$ of G. Identifying code in graphs is also found in the paper [33].

One of the prominent extension topics of dominating sets is secure dominating sets [34]. A dominating set S is a secure dominating set of G if for each $u \in V(G)\setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S\setminus\{u\}) \cup \{u\}$ is a dominating set of G. The minimum cardinality of a secure dominating set of G, denoted by $\gamma_s(G)$, is called the secure domination number of G. Secure domination in graphs is studied in the papers [35 - 43].

The identifying code and secure domination in graphs motivate the researchers to introduce a new domination parameter, the identifying secure domination in graphs. An identifying code of a graph G is an identifying secure dominating set C if for each $u \in V(G)\setminus C$, there exists $v \in C$ such that $uv \in E(G)$ and the set $(C\setminus\{v\}) \cup \{u\}$ is a dominating set of G. The minimum cardinality of a identifying secure dominating set of G, denoted by $\gamma_s^{ID}(G)$, is called the identifying secure domination number of G. In this paper, the researchers initiate the study of the concept and give some important results. In particular, the researchers show some realization problems of identifying secure dominating sets and give some properties of the identifying secure dominating sets in the Cartesian product and lexicographic product of two connected graphs.

For the general terminology in graph theory, readers may refer to [44].

2 Results

From the definitions, the following result is immediate.

Remark 2.1 *Let G be a nontrivial connected graph of order n. Then*

$$1 \leq \gamma(G) \leq \gamma_s^{ID}(G) \leq n-1.$$

Let G be a nontrivial connected graph and $C = \{v\}$ be a dominating set in G. Then, $N_G[v] \cap C = \{v\}$ and $N_G[x] \cap C = \{v\}$ for all $x \in V(G)\setminus \{v\}$. This implies that $N_G[v] \cap C$ is not distinct and hence C is not an identifying secure dominating set of G. Thus, the following remark holds.

Remark 2.2 *If G has an identifying code, then* $\gamma_s^{ID}(G) \geq 2$.

Theorem 2.3 *Let a, b, and n be positive integers such that* $1 \leq a \leq b \leq n-1$. *Then, there exists a connected nontrivial graph G with* $|V(G)| = n$ *such that* $\gamma(G) = a$ *and* $\gamma_s^{ID}(G) = b$.

Proof: Consider the following cases:

<u>Case1.</u> Suppose that $1 = a < b < n-1$.

Let $n \cong 1(\bmod 5)$ and $5b = 2n+3$. Consider the graph $G \cong F_n$ where $F_n = K_1 + P_{n-1}$ and $P_{n-1} = [v_1, v_2, \dots, v_{n-1}]$. Then the set $A = V(K_1)$ is the $\gamma-set$ and the set $B = \left\{v_{5k-4}, v_{5k-1} : k = 1, 2, \dots, \frac{n-1}{5}\right\} \cup \{v_{n-1}\}$ is a $\gamma^{ID}-set$ in G. Thus, $|V(G)| = n$, $\gamma(G) = |A| = 1 = a$, and $\gamma_s^{ID}(G) = |B| = \frac{n-1}{5} + \frac{n-1}{5} + 1 = \frac{2n+3}{5} = b$.

<u>Case2.</u> Suppose that $1 < a = b < n-1$. Let $n = 2a$. Consider the graph $G = P_a \circ K_1$ (see Figure 2.1).

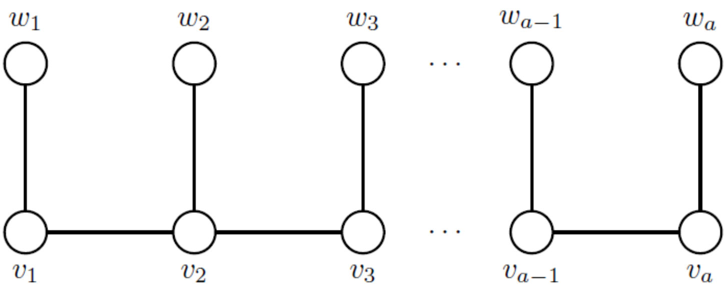

Figure 2.1: A graph G with $\gamma(G) = a = \gamma_s^{ID}(G)$

The set $A = \{v_1, v_2, \dots, v_a\}$ is a $\gamma - set$ and $\gamma_s^{ID} - set$ in G. Thus, $|V(G)| = 2a = n, \gamma(G) = |A| = a$, and $\gamma_s^{ID}(G) = a = b$.

Case3. Suppose that $1 = a < b = n - 1$. Consider the graph $G \cong S_n$ where $S_n = K_1 + \bar{P}_{n-1}$ (see Figure 2.2).

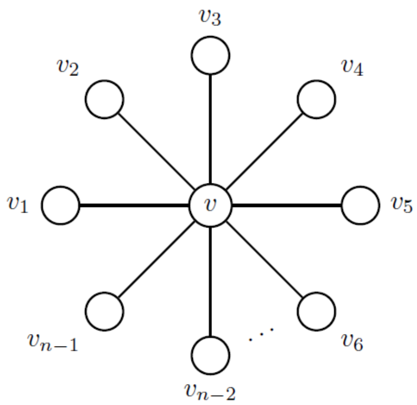

Figure 2.2: A graph G with $\gamma(G) = 1$ and $\gamma_s^{ID}(G) = n - 1$

The set $A = \{v\}$ is the $\gamma - set$ and the set $B = \{v_1, v_2, \dots, v_{n-1}\}$ is the $\gamma_s^{ID} - set$ in G. Thus, $|V(G)| = n$, $\gamma(G) = |A| = 1 = a$, and $\gamma_s^{ID}(G) = |B| = n - 1 = b$.

Case4. Suppose that $1 < a < b = n - 1$. Consider the graph G with $C_4 = [v_1, \dots, v_4]$, vertices $x_1, \dots, x_{n-4}$, and edges $v_4x_1, \dots, v_4x_{n-4}$ (see Figure 2.3).

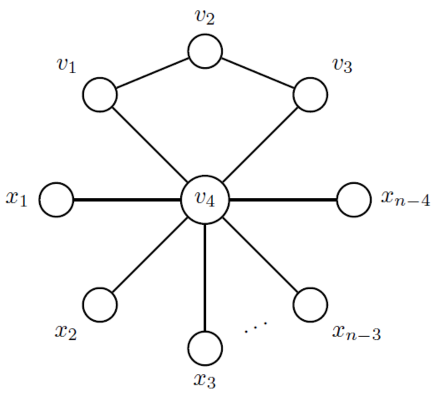

Figure 2.3: A graph G with $\gamma(G) = 2$ and $\gamma_s^{ID}(G) = n - 1$

The set $A = \{v_1, v_4\}$ is the $\gamma - set$ and the set $B = \{v_1, v_2, v_3, x_1, x_2, \dots, x_{n-4}\}$ is the set $\gamma_s^{ID} - set$ in G. Thus, $|V(G)| = n$, $\gamma(G) = |A| = 2 = a$, and

$$\gamma_s^{ID}(G) = |B| = 3 + (n-4) = n - 1 = b.$$

<u>*Case5.*</u> Suppose that $1 < a < b < n - 1$. Let $b = n - 4$ and consider the graph G with $C_6 = [v_1, \dots, v_6]$, vertices $x_1, \dots, x_{n-6}$, and edges $v_6x_1, \dots, v_6x_{n-6}$ (see Figure 2.4).

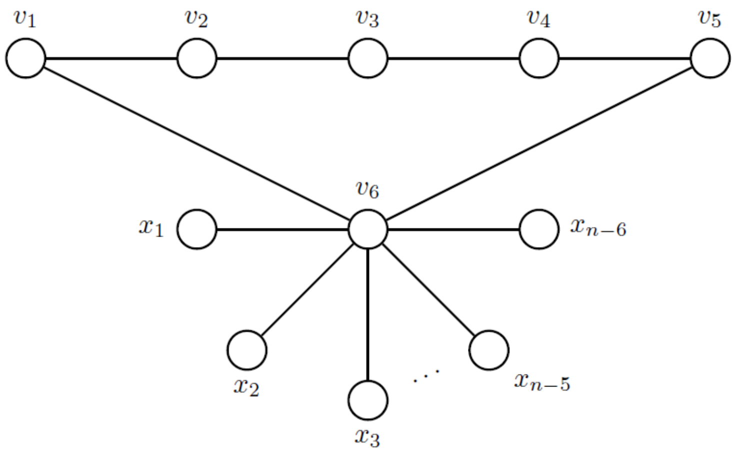

Figure 2.4: A graph G with $\gamma(G) = 2$, and $\gamma_s^{ID}(G) = n - 3$

The set $A = \{v_3, v_6\}$ is the set $\gamma - set$ and the set $B = \{v_2, v_4, x_1, x_2, \dots, x_{n-6}\}$ is the $\gamma_s^{ID} - set$ in G. Thus, $|V(G)| = n$, $\gamma(G) = |A| = 2 = a$, and

$$\gamma_s^{ID}(G) = |B| = 2 + (n-6) = n - 4 = b.$$

This completes the proofs. ■

The following corollary is an immediate consequence of Theorem 2.3.

Corollary 2.4 *The difference $\gamma_s^{ID}(G) - \gamma(G)$ can be made arbitrarily large.*

Proof: Let $n = 5k + 1$ where k is a positive integer. By Theorem 3.3, there exists a connected graph G such that $\gamma_s^{ID}(G) = \frac{2n+3}{5}$ and $\gamma(G) = 1$. As a result, we have, $\gamma_s^{ID}(G) - \gamma(G) = \frac{2n+3}{5} - 1 = \frac{2(n-1)}{5} = \frac{10k}{2} = 2k$, showing that $\gamma_s^{ID}(G) - \gamma(G)$ can be made arbitrarily large. ■

The lexicographic product of two graphs G and H is the graph $G[H]$ with vertex-set $V(G[H]) = V(G) \times V(H)$ and the edge-set $E(G[H])$ satisfying the following conditions: $(x, u)(y, v) \in E(G[H])$ if and only if either $xy \in E(G)$ or $x = y$ and $uv \in E(H)$.

Note that a non-empty subset C of $V(G[H]) = V(G) \times V(H)$ can be written as $C = \bigcup_{x \in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$. In the following results, we shall be using this form to denote any non-empty subset C of $V(G[H])$.

Theorem 2.5 *Let $G = P_m$ and $H = P_n$ with $m \geq 3$ and $n = 2k + 3$ for some positive integer k. If $C = \bigcup_{x \in V(G)}(\{x\} \times T_x)$ where T_x is an identifying secure domination set of H and $N_H[u] \cap T_x \neq T_x$ for each $x \in V(G)$ and for some $u \in V(H)$, then C is an identifying secure domination set of $G[H]$.*

Proof: Suppose that $C = \bigcup_{x \in V(G)}(\{x\} \times T_x)$ where T_x is an identifying secure dominating set of $H = P_n = [u_1, u_2, \dots, u_n]$ for each $x \in V(G)$ and $N_H[u] \cap T_x \neq T_x$ for some $u \in V(H)$. If $n = 2k + 3$ for some positive integer k, then

Case1. Consider $T_x = \left\{u_{2k-1}: k = 1, 2, 3, \dots, \frac{n+1}{2}\right\}$ for all $x \in V(G)$.

$$C = \bigcup_{x \in V(G)} (\{x\} \times T_x)$$
$$= \bigcup_{x \in V(G)} \left(\{x\} \times \left\{u_{2k-1}: k = 1, 2, 3, \dots, \frac{n+1}{2}\right\}\right)$$
$$= \bigcup_{x \in V(G)} \left\{(x, u_{2k-1}): k = 1, 2, 3, \dots, \frac{n+1}{2}\right\}$$

Let $(v, u) \in V(G[H]) \backslash C$. Then $(v, u) = (v, u_{2k})$ for any $k \in \left\{1, 2, 3, \dots, \frac{n-1}{2}\right\}$ and $v \in V(G)$. Further, there exists $(v, u_{2k-1}) \in C$ for some $k \in \left\{1, 2, 3, \dots, \frac{n+1}{2}\right\}$ such that $(v, u)(v, u_{2k-1}) \in E(G[H])$ and $(C \backslash \{(v, u_{2k-1})\}) \cup \{(v, u)\}$ is a dominating set of $G[H]$. Hence, C is a secure dominating set of $G[H]$ be definition.

Let $G = [v_1, v_2, v_3, \dots, v_m]$ and $(v, u), (v', u') \in V(G[H]), (v, u) \neq (v', u')$. Then $v, v' \in V(G) = \{v_1, v_2, v_3, \dots, v_m\}$ and $u, u' \in V(H) = \{u_1, u_2, u_3, \dots, u_n\}$.

The $N_{G[H]}[(v, u)]$ in $G[H]$ can be expressed as one of the following.

$$N_{G[H]}[(v_1, u_j)] = \left\{\{(v_1, u_j)\} \cup (\{v_2\} \times T_{v_2})\right\} \text{ for } j = 1, 3, 5, \dots, n$$
$$\text{or } \left\{\{(v_1, u_{j-1})\} \cup \{(v_1, u_{j+1})\} \cup (\{v_2\} \times T_{v_2})\right\} \text{ for } j = 2, 4, 6, \dots, n-1$$
$$N_{G[H]}[(v_i, u_j)] = \left\{\{(v_i, u_j)\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})\right\}$$
$$\text{for } i \neq 1 \text{ or } i \neq m \text{ and } j = 2k-1, k = 1, 2, \dots, \frac{n+1}{2}$$
$$\text{or } \left\{\{(v_i, u_{j-1})\} \cup \{(v_i, u_{j+1})\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})\right\}$$
$$\text{for } i \neq 1, \text{ or } i \neq m \text{ and } j = 2k, k = 1, 2, \dots, \frac{n-1}{2}$$
$$N_{G[H]}[(v_m, u_j)] = \left\{\{(v_m, u_j)\} \cup (\{v_{m-1}\} \times T_{v_{m-1}})\right\} \text{ for } j = 2k-1, k = 1, 2, \dots, \frac{n+1}{2}$$
$$\text{or } \left\{\{(v_m, u_{j-1})\} \cup \{(v_m, u_{j+1})\} \cup (\{v_{m-1}\} \times T_{v_{m-1}})\right\}$$
$$\text{for } j = 2k \text{ and } k = 1, 2, \dots, \frac{n-1}{2}$$

Subcase 1. Consider $(v, u) = (v_1, u_j)$ for some $j \in \{1, 3, 5, \dots, n\}$ and $q \in \{2, 4, 6, \dots, n-1\}$. Then

$$N_{G[H]}[(v, u)] \cap C = N_{G[H]}[(v_1, u_j)] \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)$$
$$= \left\{\{(v_1, u_j)\} \cup (\{v_2\} \times T_{v_2})\right\} \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)$$
$$= \left\{\{(v_1, u_j)\} \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)\right\} \cup \left[(\{v_2\} \times T_{v_2}) \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)\right]$$

$$= \{(v_1, u_j)\} \cup (\{v_2\} \times T_{v_2})$$
$$\neq \{(v_1, u_{q-1})\} \cup \{(v_1, u_{q+1})\} \cup (\{v_2\} \times T_{v_2})$$
$$= \{\{(v_1, u_{q-1})\} \cup \{(v_1, u_{q+1})\} \cup (\{v_2\} \times T_{v_2})\} \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x) \right)$$
$$= N_{G[H]}[(v_1, u_q)] \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x) \right)$$
$$= N_{G[H]}[(v, u')] \cap C \text{ with } (v, u') = (v_1, u_q)$$

This implies that $N_{G[H]}[(v,u)] \cap C \neq N_{G[H]}[(v,u')] \cap C$ for $(v,u) \neq (v,u')$.

Subcase 2. If $(v,u) = (v_m, u_j)$ for some $j \in \{1,3,5,\dots,n\}$ and $q \in \{2,4,6,\dots,n-1\}$. Then

$$N_{G[H]}[(v,u)] \cap C = N_{G[H]}[(v_m, u_j)] \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x) \right)$$
$$= \{\{(v_m, u_j)\} \cup (\{v_{m-1}\} \times T_{v_{m-1}})\} \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x) \right)$$
$$\neq \{(v_m, u_{q-1})\} \cup \{(v_m, u_{q+1})\} \cup (\{v_{m-1}\} \times T_{v_{m-1}}) \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x) \right)$$
$$= N_{G[H]}[(v_m, u_q)] \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x) \right), q = 2,4,6,\dots,n-1$$
$$= N_{G[H]}[(v, u')] \cap C \text{ with } (v, u') = (v_m, u_q)$$

This implies that $N_{G[H]}[(v,u)] \cap C \neq N_{G[H]}[(v,u')] \cap C$ for $(v,u) \neq (v,u')$.

Subcase 3. If $(v,u) = (v_i, u_j)$ with $i \neq 1$ or $i \neq m$, for some $j \in \{1,3,5,\dots,n\}$ and $q = \{2,4,6,\dots,n-1\}$. Then

$$N_{G[H]}[(v,u)] \cap C = N_{G[H]}[(v_i, u_j)] \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x) \right)$$
$$= \{\{(v_i, u_j)\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})\} \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x) \right)$$
$$= \left[\{(v_i, u_j)\} \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x) \right) \right] \cup \left[(\{v_{i-1}\} \times T_{v_{i-1}}) \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x) \right) \right]$$
$$\cup \left[(\{v_{i+1}\} \times T_{v_{i+1}}) \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x) \right) \right]$$
$$= \{(v_i, u_j)\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})$$
$$\neq \{\{(v_i, u_{q-1})\} \cup \{(v_i, u_{q+1})\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})\}$$

$$= \left[\{(v_i, u_{q-1})\} \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)\right] \cup \left[\{(v_i, u_{q+1})\} \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)\right]$$

$$\cup \left[(\{v_{i-1}\} \times T_{v_{i-1}}) \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)\right] \cup \left[(\{v_{i+1}\} \times T_{v_{i+1}}) \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)\right]$$

$$= \left\{\{(v_i, u_{q-1})\} \cup \{(v_i, u_{q+1})\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})\right\}$$

$$\cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)$$

$$= N_{G[H]}[(v_i, u_q)] \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)$$

$$= N_{G[H]}[(v, u')] \cap C \text{ with } (v, u') = (v_1, u_q) \text{ for some } q \in \{2, 4, 6, \dots, n-1\}$$

This implies that $N_{G[H]}[(v,u)] \cap C \neq N_{G[H]}[(v,u')] \cap C$ for $(v,u) \neq (v,u')$.

Thus, $N_{G[H]}[(v,u)] \cap C$ is distinct. Hence, C is an identifying code of $G[H]$. Since C is also a secure dominating set of $G[H]$, it follows that C is an identifying secure dominating set of $G[H]$.

<u>Case2</u>. Consider $T_x = \left\{u_{4k-2}, u_{4k-1}, u_{4k} : k = 1, 2, 3, \dots, \frac{n-1}{4}\right\}$ for all $x \in V(G)$.

$$C = \bigcup_{x \in V(G)} (\{x\} \times T_x)$$

$$= \bigcup_{x \in V(G)} \left(\{x\} \times \left\{u_{4k-2}, u_{4k-1}, u_{4k} : k = 1, 2, 3, \dots, \frac{n-1}{4}\right\}\right)$$

$$= \bigcup_{x \in V(G)} \left\{(x, u_{4k-2}), (x, u_{4k-1}), (x, u_{4k}) : k = 1, 2, 3, \dots, \frac{n-1}{4}\right\}$$

Let $(v,u) \in V(G[H]) \setminus C$. Then $(v,u) = (v, u_{4k-3})$ for any $k \in \left\{1, 2, 3, \dots, \frac{n+3}{4}\right\}$ and $v \in V(G)$. Further, there exists $(v, u_{4k-2}) \in C$ for some $k \in \left\{1, 2, 3, \dots, \frac{n+3}{4}\right\}$ such that $(v,u)(v,u_{4k-2}) \in E(G[H])$ and $(C \setminus \{(v, u_{4k-2})\}) \cup \{(v,u)\}$ is a dominating set of $G[H]$. Hence, C is a secure dominating set of $G[H]$ by definition.

Let $G = [v_1, v_2, v_3, \dots, v_m]$ and $(v,u), (v',u') \in V(G[H]), (v,u) \neq (v',u')$. Then $v, v' \in V(G) = \{v_1, v_2, v_3, \dots, v_m\}$ and $u, u' \in V(H) = \{u_1, u_2, u_3, \dots, u_n\}$.

The $N_{G[H]}[(v,u)]$ in $G[H]$ can be expressed as one of the following.

(i) If $(v,u) = (v_1, u_j)$, then

$$N_{G[H]}[(v_1, u_j)] = \{(v_1, u_j), (v_1, u_{j+1})\} \cup (\{v_2\} \times T_{v_2})$$
$$\text{for } j = 2, 6, 10, \dots, n-3$$
$$or\ \{(v_1, u_{j-1}), (v_1, u_j), (v_1, u_{j+1})\} \cup (\{v_2\} \times T_{v_2})$$
$$\text{for } j = 3, 7, 11, \dots, n-2$$
$$\text{or } \{(v_1, u_{j-1}), (v_1, u_j)\} \cup (\{v_2\} \times T_{v_2}) \text{ for } j = 4, 8, 12, \dots, n-1$$
$$\text{or } \{(v_1, u_{j-1}), (v_1, u_{j+1})\} \cup (\{v_2\} \times T_{v_2})$$

$$\text{for } j = 5, 9, 13, \dots, n-4, (n \neq 5)$$
$$\text{or } \{(v_1, u_2)\} \cup \left(\{v_2\} \times T_{v_2}\right) \text{ for } j = 1$$
$$\text{or } \{(v_1, u_{n-1})\} \cup \left(\{v_2\} \times T_{v_2}\right) \text{ for } j = n$$

(ii) If $(v, u) = (v_m, u_j)$, then

$$N_{G[H]}[(v_m, u_j)] = \{(v_m, u_{j+1})\} \cup (\{v_{m-1}\} \times T_{v_{m-1}}) \text{ for } j = 2, 6, 10, \dots, n-3$$
$$\text{or } \{(v_m, u_{j-1}), (v_m, u_{j+1})\} \cup (\{v_{m-1}\} \times T_{v_{m-1}})$$
$$\text{for } j = 3, 7, 11, \dots, n-2$$
$$\text{or } \{(v_m, u_{j-1})\} \cup (\{v_{m-1}\} \times T_{v_{m-1}}) \text{ for } j = 4, 8, 12, \dots, n-1$$
$$\text{or } \{(v_m, u_{j-1}), (v_m, u_{j+1})\} \cup (\{v_{m-1}\} \times T_{v_{m-1}})$$
$$\text{for } j = 5, 9, 13, \dots, n-4, (n \neq 5)$$
$$\text{or } \{(v_m, u_2)\} \cup (\{v_{m-1}\} \times T_{v_{m-1}}) \text{ for } j = 1$$
$$\text{or } \{(v_m, u_{n-1})\} \cup (\{v_{m-1}\} \times T_{v_{m-1}}) \text{for } j = n$$

(iii) If $(v, u) = (v_i, u_j)$ with $i \neq 1$ or $i \neq m$, then

$$N_{G[H]}[(v_i, u_j)] = \{(v_i, u_{j+1})\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})$$
$$\text{for } j = 2, 6, 10, \dots, n-3$$
$$\text{or } \{(v_i, u_{j-1}), (v_i, u_j), (v_i, u_{j+1})\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})$$
$$\text{for } j = 3, 7, 11, \dots, n-2$$
$$\text{or } \{(v_i, u_{j-1}), (v_i, u_j)\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})$$
$$\text{for } j = 4, 8, 12, \dots, n-1$$
$$\text{or } \{(v_i, u_{j-1}), (v_i, u_{j+1})\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})$$
$$\text{for } j = 5, 9, 13, \dots, n-4, (n \neq 5)$$
$$\text{or } \{(v_i, u_2)\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}}) \text{ for } j = 1$$
$$\text{or } \{(v_i, u_{n-1})\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}}) \text{ for } j = n$$

Subcase 1. Consider $(v, u) = (v_1, u_j)$ for some $j \in \{2, 6, 10, \dots, n-3\}$ and $q \in \{5, 9, \dots, n-4\}$. Then

$$N_{G[H]}[(v, u)] \cap C = N_{G[H]}[(v_1, u_j)] \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)$$
$$= \{(v_1, u_j), (v_1, u_{j+1})\} \cup (\{v_2\} \times T_{v_2}) \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)$$
$$= \left[\{(v_1, u_j)\} \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)\right]$$
$$\cup \left[\{(v_1, u_{j+1})\} \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)\right] \cup \left[(\{v_2\} \times T_{v_2}) \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)\right]$$

$$= \{(v_1, u_j)\} \cup \{(v_1, u_{j+1})\} \cup (\{v_2\} \times T_{v_2})$$
$$\neq \{(v_1, u_{q-1})\} \cup \{(v_1, u_{q+1})\} \cup (\{v_2\} \times T_{v_2})$$
$$= \left\{\{(v_1, u_{q-1})\} \cup \{(v_1, u_{q+1})\} \cup (\{v_2\} \times T_{v_2})\right\} \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)$$
$$= N_{G[H]}[(v_1, u_q)] \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)$$
$$= N_{G[H]}[(v, u')] \cap C \text{ with } (v, u') = (v_1, u_q)$$

This implies that $N_{G[H]}[(v,u)] \cap C \neq N_{G[H]}[(v,u')] \cap C$ for $(v,u) \neq (v,u')$.

Similarly, if $N_{G[H]}[(v_1, u_j)] = \{(v_1, u_{j-1}), (v_1, u_j), (v_1, u_{j+1})\}$ for $j = 3, 7, 11, \ldots, n-2$, or $N_{G[H]}[(v_1, u_j)] = \{(v_1, u_{j-1}), (v_1, u_j)\} \cup (\{v_2\} \times T_{v_2})$ for $j = 4, 8, 12, \ldots, n-1$ or $N_{G[H]}[(v_1, u_j)] = \{(v_1, u_{j-1}), (v_1, u_{j+1})\} \cup (\{v_2\} \times T_{v_2})$ for $j = 5, 9, 13, \ldots, n-4$, $(n \neq 5)$, then $N_{G[H]}[(v,u)] \cap C \neq N_{G[H]}[(v,u')] \cap C$ for $(v,u) \neq (v,u')$.

Subcase 2. Consider $(v,u) = (v_m, u_j)$ for some $j \in \{2, 6, 10, \ldots, n-3\}$ and $q \in \{5, 9, \ldots, n-4\}$. Then

$$N_{G[H]}[(v,u)] \cap C = N_{G[H]}[(v_m, u_j)] \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)$$
$$= \{(v_m, u_j), (v_m, u_{j+1})\} \cup (\{v_{m-1}\} \times T_{v_{m-1}}) \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)$$
$$= \left[\{(v_m, u_j)\} \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)\right] \cup \left[\{(v_m, u_{j+1})\} \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)\right]$$
$$\cup \left[(\{v_{m-1}\} \times T_{v_{m-1}}) \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)\right]$$
$$= \{(v_m, u_j)\} \cup \{(v_m, u_{j+1})\} \cup (\{v_{m-1}\} \times T_{v_{m-1}})$$
$$\neq \{(v_m, u_{q-1})\} \cup \{(v_m, u_{q+1})\} \cup (\{v_{m-1}\} \times T_{v_{m-1}})$$
$$= \left\{\{(v_m, u_{q-1})\} \cup \{(v_m, u_{q+1})\} \cup (\{v_{m-1}\} \times T_{v_{m-1}})\right\} \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)$$
$$= N_{G[H]}[(v_m, u_q)] \cap \left(\bigcup_{x \in V(G)} (\{x\} \times T_x)\right)$$
$$= N_{G[H]}[(v, u')] \cap C \text{ with } (v, u') = (v_m, u_q)$$

This implies that $N_{G[H]}[(v,u)] \cap C \neq N_{G[H]}[(v,u')] \cap C$ for $(v,u) \neq (v,u')$.

Similarly, if $N_{G[H]}[(v_m, u_j)] = \{(v_m, u_{j-1}), (v_m, u_j), (v_m, u_{j+1})\}$ for $j = 3, 7, 11, \ldots, n-2$, or $N_{G[H]}[(v_m, u_j)] = \{(v_m, u_{j-1}), (v_m, u_j)\} \cup (\{v_{m-1}\} \times T_{v_{m-1}})$ for $j = 4, 8, 12, \ldots, n-1$ or

$N_{G[H]}[(v_m, u_j)] = \{(v_m, u_{j-1}), (v_m, u_{j+1})\} \cup (\{v_{m-1}\} \times T_{v_{m-1}})$ for $j = 5, 9, 13, \dots, n-4$, $(n \neq 5)$, then $N_{G[H]}[(v,u)] \cap C \neq N_{G[H]}[(v,u')] \cap C$ for $(v,u) \neq (v,u')$.

Subcase 3. Consider $(v,u) = (v_i, u_j)$ for some $j \in \{2, 6, 10, \dots, n-3\}$ and $q \in \{5, 9, \dots, n-4\}$. Then

$$N_{G[H]}[(v,u)] \cap C = N_{G[H]}[(v_i,u_j)] \cap \left(\bigcup_{x\in V(G)} (\{x\} \times T_x)\right)$$

$$= \left[\{(v_i,u_j),(v_i,u_{j+1})\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})\right] \cap \left(\bigcup_{x\in V(G)} (\{x\} \times T_x)\right)$$

$$= \left[\{(v_i,u_j)\} \cap \left(\bigcup_{x\in V(G)} (\{x\} \times T_x)\right)\right] \cup \left[\{(v_i,u_{j+1})\} \cap \left(\bigcup_{x\in V(G)} (\{x\} \times T_x)\right)\right]$$

$$\cup \left[(\{v_{i-1}\} \times T_{v_{i-1}}) \cap \left(\bigcup_{x\in V(G)} (\{x\} \times T_x)\right)\right] \cup \left[(\{v_{i+1}\} \times T_{v_{i+1}}) \cap \left(\bigcup_{x\in V(G)} (\{x\} \times T_x)\right)\right]$$

$$= \{(v_i,u_j)\} \cup \{(v_i,u_{j+1})\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})$$

$$\neq \{(v_i,u_{q-1})\} \cup \{(v_i,u_{q+1})\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})$$

$$= \left\{\{(v_i,u_{q-1})\} \cup \{(v_i,u_{q+1})\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})\right\}$$

$$\cap \left(\bigcup_{x\in V(G)} (\{x\} \times T_x)\right)$$

$$= N_{G[H]}[(v_i,u_q)] \cap \left(\bigcup_{x\in V(G)} (\{x\} \times T_x)\right)$$

$$= N_{G[H]}[(v,u')] \cap C \text{ with } (v,u') = (v_i,u_q)$$

This implies that $N_{G[H]}[(v,u)] \cap C \neq N_{G[H]}[(v,u')] \cap C$ for $(v,u) \neq (v,u')$.

Similarly, if $N_{G[H]}[(v_i,u_j)] = \{(v_i,u_{j-1}), (v_i,u_j), (v_i,u_{j+1})\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})$ for $j = 3, 7, 11, \dots, n-2$, or $N_{G[H]}[(v_i,u_j)] = \{(v_i,u_{j-1}), (v_i,u_j)\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})$ for $j = 4, 8, 12, \dots, n-1$, or $N_{G[H]}[(v_i,u_j)] = \{(v_i,u_{j-1}), (v_i,u_{j+1})\} \cup (\{v_{i-1}\} \times T_{v_{i-1}}) \cup (\{v_{i+1}\} \times T_{v_{i+1}})$ for $j = 5, 9, 13, \dots, n-4$, $(n \neq 5)$, then $N_{G[H]}[(v,u)] \cap C \neq N_{G[H]}[(v,u')] \cap C$ for $(v,u) \neq (v,u')$.

Thus, $N_{G[H]}[(v,u)] \cap C$ is distinct. Hence, C is an identifying code of $G[H]$. Since C is also a secure dominating set of $G[H]$, it follows that C is an identifying secure domination set of $G[H]$. ■

The following result is an immediate consequence of **Theorem 2.5**.

Corollary 2.6 *Let $G = P_m$ and $H = P_n$ with $m \geq 3$ and $n = 2k + 3$ for some positive integer k. Then*

$$\gamma_s^{ID}(G[H]) \leq m \cdot \gamma_s^{ID}(H).$$

Proof: Given that $G = P_m$ and $H = P_n$ with $m \geq 3$ and $n = 2k + 3$ for some positive integer k. Suppose that $C = \bigcup_{x\in V(G)}(\{x\} \times T_x)$ where T_x is an identifying secure dominating set of H and $N_H[u] \cap T_x \neq T_x$ for each $x \in V(G)$ and for some $u \in V(H)$. Then C is an identifying secure dominating set of $G[H]$ by Theorem 2.5. Thus,

$$\begin{aligned}
\gamma_s^{ID}(G[H]) &\leq |C| \\
&= \left|\bigcup_{x\in V(G)}(\{x\} \times T_x)\right|, T_x \in V(H) \text{ for all } x \in V(G) \\
&= |V(G) \times T_x|, T_x \in V(H) \text{ for all } x \in V(G) \\
&= |V(G)| \cdot |T_x|, T_x \in V(H) \text{ for all } x \in V(G) \\
&= m \cdot \gamma_s^{ID}(H).
\end{aligned}$$

Hence, $\gamma_s^{ID}(G[H]) \leq m \cdot \gamma_s^{ID}(H)$. ■

3 Conclusion

This study showed that the identifying secure domination number of a graph exists, and the characterization of this domination parameter resulting from the lexicographic product of two graphs was presented. This study will result in new research such as bounds and other binary operations of two graphs. Other parameters involving the identifying secure domination in graphs may also be explored. Finally, the characterization of an identifying secure domination in graphs and its bounds is a promising extension of this study.

References

1. O. Ore. *Theory of Graphs*. American Mathematical Society, Provedence, R.I., 1962.
2. E.J. Cockayne, and S.T. Hedetniemi, *Towards a theory of domination in graphs*, Networks, (1977) 247-261.
3. N.A. Goles, E.L. Enriquez, C.M. Loquias, G.M. Estrada, R.C. Alota, *z-Domination in Graphs*, Journal of Global Research in Mathematical Archives, 5(11), 2018, pp 7-12.
4. E.L. Enriquez, V.V. Fernandez, J.N. Ravina, *Outer-clique Domination in the Corona and Cartesian Product of Graphs*, Journal of Global Research in Mathematical Archives, 5(8), 2018, pp 1-7.
5. E.L. Enriquez, G.M. Estrada, V.V. Fernandez, C.M. Loquias, A.D. Ngujo, *Clique Doubly Connected Domination in the Corona and Cartesian Product of Graphs*, Journal of Global Research in Mathematical Archives, 6(9), 2019, pp 1-5.
6. E.L. Enriquez, G.M. Estrada, C.M. Loquias, *Weakly Convex Doubly Connected Domination in the Join and Corona of Graphs*, Journal of Global Research in Mathematical Archives, 5(6), 2018, pp 1-6.
7. J.A. Dayap, E.L. Enriquez, *Outer-convex Domination in Graphs in the Composition and Cartesian Product of Graphs*, Journal of Global Research in Mathematical Archives, 6(3), 2019, pp 34-42.
8. D.P. Salve, E.L. Enriquez, *Inverse Perfect Domination in the Composition and Cartesian Product of Graphs*, Global Journal of Pure and Applied Mathematics, 12(1), 2016, pp 1-10.
9. E.L. Enriquez, B.P. Fedellaga, C.M. Loquias, G.M. Estrada, M.L. Baterna, *Super Connected Domination in Graphs*, Journal of Global Research in Mathematical Archives, 6(8), 2019, pp 1-7.
10. E.L. Enriquez, *On Restrained Clique Domination in Graphs*, Journal of Global Research in

Mathematical Archives, Vol. 4, 2017, no. 12, 73-77.
11. E.L. Enriquez, *Super Restrained Domination in the Corona of Graphs*, International Journal of Latest Engineering Research and Applications, Vol. 3, 2018, no. 5, 1-6.
12. T.J. Punzalan, and E.L. Enriquez, *Inverse Restrained Domination in Graphs*, Global Journal of Pure and Applied Mathematics, Vol. 3, 2016, pp 1-6.
13. R.C. Alota, and E.L. Enriquez, *On Disjoint Restrained Domination in Graphs*, Global Journal of Pure and Applied Mathematics, Vol. 12, 2016, no. 3 pp 2385-2394.
14. E.L. Enriquez, and S.R. Canoy,Jr., *On a Variant of Convex Domination in a Graph*. International Journal of Mathematical Analysis, Vol. 9, 2015, no. 32, 1585-1592.
15. E.L. Enriquez, and S.R. Canoy,Jr., *Restrained Convex Dominating Sets in the Corona and the Products of Graphs*, Applied Mathematical Sciences, Vol. 9, 2015, no. 78, 3867-3873.
16. SP.G. Cajigas, E.L. Enriquez, K.E. Belleza, G.M. Estrada, C.M. Loquias, *Disjoint Restrained Domination in the Join and Corona of Graphs*, International Journal of Mathematics Trends and Technology, 67(12), 2021, pp 57-61.
17. J.A. Ortega, E.L. Enriquez, *Super Inverse Domination in Graphs*, International Journal of Mathematics Trends and Technology, 67(7), 2021, pp 135-140.
18. R.T. Aunzo, E.L. Enriquez, *Convex Doubly Connected Domination in Graphs*, Applied Mathematical Sciences, 9(135), 2015, pp 6723-6734.
19. E.L. Enriquez, *Convex Doubly Connected Domination in Graphs Under Some Binary Operations*, Ansari Journal of Ultra Scientist of Engineering and Management, 1(1), 2017, pp 13-18.
20. Cristina S Castañares, Enrico L Enriquez, *Inverse Perfect Secure Domination in Graphs*, International Journal of Mathematics Trends and Technology 67(8), 2022, pp 150-156.
21. Hanna Rachelle A Gohil, Enrico L Enriquez, *Inverse Perfect Restrained Domination in Graphs*, International Journal of Mathematics Trends and Technology, 67(8), 2022, pp 164-170.
22. T.J. Punzalan and E.L. Enriquez, *Inverse Restrained domination in graphs*, Global Journal of Pure and Applied Mathematics, 12, No. 3(2016), pp. 2001-2009.
23. E.L. Enriquez, *Inverse fair domination in the join and corona of graphs*, Discrete Mathematics, Algorithms and Applications, 16(01), 2024, pp 2350003.
24. D.P. Salve and E.L. Enriquez, *Inverse perfect domination in graphs*, Global Journal of Pure and Applied Mathematics, 12, No. 1(2016) 1-10.
25. W.R.E. Alabastro and E.L. Enriquez, *Restrained Inverse Domination in the Join and Corona of Two Graphs*, International Journal of Latest Engineering Research and Applications, 9(1), 2024, pp 57-62.
26. C.A. Tuble and E.L. Enriquez, *Outer-restrained Domination in the Join and Corona of Graphs*, International Journal of Latest Engineering Research and Applications, 9(1), 2024, pp 50-56.
27. K.M. Cruz and E.L. Enriquez, *Inverse Doubly Connected Domination in the Join and Cartesian Product of Two Graphs*, International Journal of Latest Engineering Research and Applications, 9(1), 2024, pp 20-25.
28. R.B. Udtohan and E.L. Enriquez, *Disjoint Perfect Secure Domination in the Join and Corona of Graphs*, International Journal of Latest Engineering Research and Applications, 9(1), 2024, pp 63-71.
29. M.E.N. Diapo and E.L. Enriquez, *Disjoint Perfect Domination in Join and Corona of Two Graphs*, International Journal of Latest Engineering Research and Applications, 9(1), 2024, pp 43-49.
30. J.N.C. Serrano and E.L. Enriquez, *Fair Doubly Connected Domination in the Corona and the Cartesian Product of Two Graphs*, International Journal of Mathematics Trends and Technology ,

69(12), 2023, pp 46-41.
31. V.S. Verdad and E.L. Enriquez, *Inverse Fair Restrained Domination in the Corona of Two Graphs,* International Journal of Mathematics Trends and Technology , 69(12), 2023, pp 27-35.
32. M.G. Karpovsky, K. Chakrabarty and L.B. Levitin, *On a new class of codes for identifying vertices in graphs*. IEEE Transactions on Information Theory, 44(2), 1998, pp 599-611.
33. J.L. Ranara, C.M. Loquias, G.M. Estrada, T.J. Punzalan, E.L. Enriquez, *Identifying Code of Some Special Graphs*, Journal of Global Research in Mathematical Archives, 5(9), 2018, pp 1-8.
34. E.J. Cockayne, P.J.P. Grobler, W.R. Gründlingh, J.Munganga, J.H. Van Vuuren, Protection of a graph, Util Math. 67, 2005, pp 19-32.
35. E.L. Enriquez, E.S. Enriquez, *Convex Secure Domination in the Join and Cartesian Product of Graphs,* Journal of Global Research in Mathematical Archives, 6(5), 2019, pp 1-7.
36. E.L. Enriquez, and S.R. Canoy,Jr., *Secure Convex Domination in a Graph,* International Journal of Mathematical Analysis, Vol. 9, 2015, no. 7, 317-325.
37. C.M. Loquias, and E.L. Enriquez, *On Secure Convex and Restrained Convex Domination in Graphs,* International Journal of Applied Engineering Research, Vol. 11, 2016, no. 7, 4707-4710.
38. E.L. Enriquez, *Secure Restrained Convex Domination in Graphs*, International Journal of Mathematical Archive, Vol. 8, 2017, no. 7, 1-5.
39. M.P. Baldado, Jr. and E.L. Enriquez, *Super Secure Domination in Graphs*, International Journal of Mathematical Archive-8(12), 2017, pp. 145-149.
40. E.M. Kiunisala, and E.L. Enriquez, *Inverse Secure Restrained Domination in the Join and Corona of Graphs,* International Journal of Applied Engineering Research, Vol. 11, 2016, no. 9, 6676-6679.
41. HL.M. Maravillas, E.L. Enriquez, *Secure Super Domination in Graphs*, International Journal of Mathematics Trends and Technology, 67(8), 2021, pp 38-44.
42. J.P. Dagodog and E.L. Enriquez, *Secure Inverse Domination in the Join of Graphs,* International Journal of Latest Engineering Research and Applications 9(1), 2024, pp 105-112.
43. G. Chartrand and P. Zhang, *A First Course in Graph Theory*. Dover Publication, Inc., New York, 2012.