Restrained Inverse Domination in the Lexicographic and Cartesian Products of Two Graphs

Wenlin Rose E. Alabastro[1], Katrina B. Fuentes[2], Grace M. Estrada[3], Marie Cris A. Bulay-og[4], Edward M. Kiunisala[5], Enrico L. Enriquez[6]

[1]MS Math, Department of Computer, Information Sciences, and Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines
[2,3]Associate Professor, Department of Computer, Information Sciences, and Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines
[4]Assistant Professor, Mathematics and Statistics Programs, University of the Philippines Cebu, 6000 Cebu City, Philippines
[5]Professor, Mathematics Department, College of Computing, Artificial Intelligence and Sciences, Cebu Normal University, 6000 Cebu City, Philippines
[6]Full Professor, Department of Computer, Information Sciences, and Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines

Abstract
Let G be a connected simple graph and D be a minimum dominating set of G. A dominating set $S \subseteq V(G) \setminus D$ is called an inverse dominating set of G with respect to D. An inverse dominating set S is called a restrained inverse dominating set of G if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. The restrained inverse domination number of G, denoted by, $\gamma_r^{(-1)}(G)$, is the minimum cardinality of a restrained inverse dominating set of G. A restrained inverse dominating set of cardinality $\gamma_r^{(-1)}(G)$ is called $\gamma_r^{(-1)}(G)$ is called $\gamma_r^{(-1)}$-set. This study is an extension of an existing research on restrained inverse domination in graphs. In this paper, we characterized the restrained inverse domination in graphs under the lexicographic and Cartesian products of two graphs.
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1 Introduction

Let G be a connected simple graph. A set S of vertices of G is a dominating set of G if every vertex in $V(G) \setminus S$ is adjacent to some vertex in S. A minimum Let G be a connected simple graph. A set S of vertices of G is a dominating set of G if every vertex in $V(G) \setminus S$ is adjacent to some vertex in S. A minimum dominating set in a graph G is a dominating set of minimum cardinalities. The cardinality of a minimum dominating set in G is called the domination number of G and is denoted by $\gamma(G)$. The concept of domination in graphs introduced by Claude Berge in 1958 and Oystein Ore in 1962 [1] is currently receiving much attention in literature. Following the article of Ernie Cockayne and Stephen Hedetniemi [2], the domination in graphs became an area of study by many researchers [3-19].

If D is a minimum dominating set in G, then a dominating set $S \subseteq V(G) \setminus D$ is called an inverse dominating set with respect to D. The inverse domination number, denoted by $\gamma^{-1}(G)$, of G is the order of an inverse dominating set with minimum cardinality. The inverse domination in a graph was first found in the paper of Kulli [20] and studied in papers [21-29].

Another type of domination parameter is the restrained domination number in a graph. A restrained dominating set is defined to be a set $S \subseteq V(G)$ where every vertex in $V(G) \setminus S$ is adjacent to a vertex in S and to another vertex in $V(G) \setminus S$. The restrained domination number of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G. This was introduced by Telle and Proskurowski [30] indirectly as a vertex partitioning problem. One practical application of restrained domination is that of prisoners and guards. Here, each vertex not in the restrained dominating set corresponds to a position of a prisoner, and every vertex in the restrained dominating set corresponds to a position of a guard. To effect security, each prisoner's position is observed by a guard's position. To protect the rights of prisoners, each prisoner's position is seen by at least one other prisoner's position. To be cost effective, it is desirable to place as few guards as possible. Some studies on restrained domination in graphs can be found in papers [31-38].

A graph G is a pair $(V(G), E(G))$, where $V(G)$ is a nonempty finite set whose elements are called vertices and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $E(G)$ are called edges of the graph G. The number of vertices in G is called the order of G and the number of edges is called the size of G. For more graph-theoretical concepts, the readers may refer to paper [39].

An inverse dominating set S is called a restrained inverse dominating set of G if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. The restrained inverse domination number of G, denoted by $\gamma_r^{(-1)}(G)$, is the minimum cardinality of a restrained inverse dominating set of G. A restrained inverse dominating set of cardinality $\gamma_r^{(-1)}(G)$ is called $\gamma_r^{(-1)}$-set. Following the results presented in [40], the researchers extended the study by investigating other binary graph operations. In this paper, the researchers characterized the restrained inverse dominating set of the lexicographic and Cartesian products of two graphs.

2 Results

Definition 2.1 *The lexicographic products of two graphs G and H, denoted by $G[H]$, is the graph with vertex-set $V(G[H] = V(G) \times V(H)$ and edge-set $E(G[H])$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G[H])$ if either $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.*

Remark 2.2 *Let $G = P_m, m \geq 2$ and $H = K_2$. The nonempty set $X \times \{u\}$ is a minimum dominating set of $G[H]$ if X is a minimum dominating set of G and $u \in V(H)$.*

The following result shows a property of restrained inverse dominating set of the lexicographic product of two graphs.

Theorem 2.3 *Let $G = P_m$, $m \geq 2$ and $H = K_2$. Then $S \subseteq V(G[H]) \setminus D$ is a restrained inverse dominating set of $G[H]$ with respect to a minimum dominating set D of $G[H]$, if $D = A \times \{u\}$ where A is a minimum dominating set of G, $u \in V(H)$ and one of the following is satisfied.*

(i) $S = S' \times \{u\}$ where $S' \subseteq V(G) \setminus A$, and S' is a dominating set of G.

(ii) $S = S' \times \{u'\}$ where $u' \in V(H) \setminus \{u\}, S' \subseteq V(G)$, and S' is a dominating set of G.

(iii) $S = (S' \times \{u\}) \cup (S'' \times \{u'\})$ where $u' \in V(H) \setminus \{u\}, S' \subseteq V(G) \setminus A$ is a dominating set of G, and $\emptyset \subseteq S'' \subseteq S'$.

Proof: Let $G = [v_1, v_2, \dots, v_m]$ and $H = [u_1, u_2]$. If $D = A \times \{u\}$ where A is a minimum dominating set of G, $u \in V(H)$, then D is a minimum dominating set of $G[H]$ by Remark 2.2.

Suppose that statement *(i)* is satisfied. Then $S = S' \times \{u\}$ where $u \in V(H)$, $S' \subseteq V(G) \setminus A$, and S' is a dominating set of G. Since S' is a dominating set of $G, S = S' \times \{u\}$, is a dominating set of $G[H]$ by Remark 2.2. Since

$$V(G[H]) \setminus D = V(G[H]) \setminus (A \times \{u\}), u \in V(H)$$
$$= [(V(G) \setminus A) \times \{u_1\}] \cup [V(G) \times \{u_2\}], where\ H = [u_1, u_2],$$

it follows that $(V(G) \setminus A) \times \{u_1\} \subset V(G[H]) \setminus D$, that is, $S = S' \times \{u_1\} \subseteq (V(G) \setminus A) \times \{u_1\} \subset V(G[H]) \setminus D$. Thus, $S \subset V(G[H]) \setminus D$ is an inverse dominating set of $G[H]$ with respect to D.

Note that $u \in V(H) = \{u_1, u_2\}$ and S' is a dominating set of G implies that for every $v \in V(G) \setminus S'$, there exists $v' \in S'$ such that $vv' \in E(G)$. Further, $V(G[H]) \setminus S = V(G[H]) \setminus (S' \times \{u\}) = [(V(G) \setminus S') \times \{u_1\}] \cup [V(G) \times \{u_2\}]$ where $S = S' \times \{u_1\} \subseteq (V(G) \setminus A) \times \{u_1\}$. Now, let $(v, u) \in V(G[H]) \setminus S$.

Case1. If $u = u_1$, then $(v, u_1) \in (V(G) \setminus S') \times \{u_1\}$. There exists $(v', u_1) \in S$ such that $(v, u_1)(v', u_1) \in E(G[H])$ and another $(v, u_2) \in V(G[H]) \setminus S$ such that $(v, u_1)(v, u_2) \in E(G[H])$.

Case2. If $u = u_2$, then $(v, u_2) \in V(G) \times \{u_2\}$. There exists $(v', u_1) \in S$ such that $(v, u_2)(v', u_1) \in E(G[H])$ and another $(v', u_2) \in V(G[H]) \setminus S$ such that $(v, u_2)(v', u_2) \in E(G[H])$.

In any case, S is a restrained dominating set of $G[H]$. Accordingly, S is a restrained inverse dominating set of $G[H]$.

Suppose that statement *(ii)* is satisfied. Then $S = S' \times \{u'\}$ where $u' \in V(H) \setminus \{u\}, S' \subseteq V(G)$, and S' is a dominating set of G. Since S' is a dominating set of $G, S = S' \times \{u'\}$ is a dominating set of $G[H]$ by Remark 2.2. Since

$$V(G[H]) \setminus D = V(G[H]) \setminus (A \times \{u\}), u \in V(H)$$
$$= [(V(G) \setminus A) \times \{u_1\}] \cup [V(G) \times \{u_2\}]\ where\ H = [u_1, u_2],$$

it follows that $V(G) \times \{u_2\} \subset V(G[H]) \setminus D$, that is, $S = S' \times \{u_2\} \subseteq V(G) \times \{u_2\} \subset V(G[H]) \setminus D$. Thus $S \subset V(G[H]) \setminus D$ is an inverse dominating set of $G[H]$ with respect to D.

Note that $u \in V(H) = \{u_1, u_2\}$ and S' is a dominating set of G implies that for every $v \in V(G) \setminus S'$, there exists $v' \in S'$ such that $vv' \in E(G)$. Further, $V(G[H]) \setminus S = V(G[H]) \setminus (S' \times \{u'\}) = [V(G) \times \{u_1\}] \cup [(V(G) \setminus S') \times \{u_2\}]$ where $S = S' \times \{u_2\} \subseteq V(G) \times \{u_2\}$. Now, let $(v, u) \in V(G[H]) \setminus S$.

Case1. If $u = u_1$, then $(v, u_1) \in V(G) \times \{u_1\}$. There exists $(v', u_2) \in S$ such that $(v, u_1)(v', u_2) \in E(G[H])$ and another $(v', u_1) \in V(G[H]) \setminus S$ such that $(v, u_1)(v', u_1) \in E(G[H])$.

Case2. If $u = u_2$, then $(v, u_2) \in (V(G) \setminus S') \times \{u_2\}$. There exists $(v', u_2) \in S$ such that $(v, u_2)(v', u_2) \in E(G[H])$ and another $(v, u_1) \in V(G[H]) \setminus S$ such that $(v, u_2)(v, u_1) \in E(G[H])$.

In any case S is a restrained dominating set of $G[H]$. Accordingly, S is a restrained inverse dominating set of $G[H]$.

Suppose that statement *(iii)* is satisfied. Then $S = (S' \times \{u\}) \cup (S'' \times \{u'\})$ where $u' \in V(H) \setminus \{u\}, S' \subseteq V(G) \setminus A$ is a dominating set of G, and $\emptyset \subseteq S'' \subseteq S'$. Since S' is a dominating set of G, $S' \times \{u\}$ is a dominating set of $G[H]$ by Remark 2.2. Thus, $S = (S' \times \{u\}) \cup (S'' \times \{u'\})$ is a dominating set of $G[H]$. Since

$$\begin{aligned} V(G[H]) \setminus D &= V(G[H]) \setminus (A \times \{u\}), u \in V(H) \\ &= [(V(G) \setminus A) \times \{u_1\}] \cup [V(G) \times \{u_2\}] \ where\ H = [u_1, u_2] \\ &\supseteq [S' \times \{u_1\}] \cup [S'' \times \{u_2\}] = S. \end{aligned}$$

Thus, $S \subseteq V(G[H]) \setminus D$ is an inverse dominating set of $G[H]$ with respect to a minimum dominating set D of $G[H]$.

Note that $u \in V(H) = \{u_1, u_2\}$ and S' is a dominating set of G implies that for every $v \in V(G) \setminus S'$, there exists $v' \in S'$ such that $vv' \in E(G)$. Further,

$$V(G[H]) \setminus S = V(G[H]) \setminus ((S' \times \{u\}) \cup (S'' \times \{u'\})),$$

where $u' \in V(H) \setminus \{u\}, S' \subseteq V(G) \setminus A$ is a dominating set of G, and $\emptyset \subseteq S'' \subseteq S'$.

Case1. If $S'' = \emptyset$, then $S = (S' \times \{u\}) \cup (S'' \times \{u'\}) = S' \times \{u\}$. By the proof of *(i)*, S is a restrained inverse dominating set of $G[H]$.

Case2. If $S'' = S'$, then

$$\begin{aligned} S &= (S' \times \{u\}) \cup (S'' \times \{u'\}) \\ &= (S' \times \{u\}) \cup (S' \times \{u'\}) \\ &= S' \times \{u, u'\} \\ &= S' \times V(H). \end{aligned}$$

Let $(v, u) \in V(G[H]) \setminus S = V(G[H]) \setminus (S' \times V(H)) = (V(G) \setminus S') \times V(H)$. Then there exists $(v', u) \in S$ such thst $(v, u)(v', u) \in E(G[H])$ and there exists another $(v, u') \in V(G[H]) \setminus S$ such that $(v, u)(v, u') \in E(G[H])$. Hence, S is a restrained dominating set of $G[H]$, that is, S is a restrained inverse dominating set of $G[H]$.

Case3. If $\emptyset \subset S'' \subset S'$, then $S = (S' \times \{u\}) \cup (S'' \times \{u'\})$. Let $(v, u) \in V(G[H]) \setminus S$. Then

$$\begin{aligned} (v, u) \in V(G[H]) \setminus S &= V(G[H]) \setminus [(S' \times \{u_1\}) \cup (S'' \times \{u_2\})] \\ &= [(V(G) \setminus S') \times \{u_1\}] \cup [(V(G) \setminus S'') \times \{u_2\}]. \end{aligned}$$

If $u = u_1$, then $(v, u_1) \in (V(G) \setminus S') \times \{u_1\} \subset V(G[H]) \setminus S$. There exists $(v', u_1) \in S$ such that $(v, u_1)(v', u_1) \in E(G[H])$ and there exists another $(v, u_2) \in V(G[H]) \setminus S$ such that $(v, u_1)(v, u_2) \in E(G[H])$.

If $u = u_2$, then $(v, u_2) \in (V(G) \setminus S'') \times \{u_2\} \subset V(G[H]) \setminus S$. There exists $(v'', u_2) \in S$ such that $(v, u_2)(v'', u_2) \in E(G[H])$ and there exists another $(v, u_1) \in V(G[H]) \setminus S$ such that $(v, u_2)(v, u_1) \in E(G[H])$.

Hence, S is a restrained dominating set of $G[H]$, that is, S is a restrained inverse dominating set of $G[H]$. ■

The following result is an immediate consequence of Theorem 2.3.

Corollary 2.4 *Let $G = P_m, m \geq 2$ and $H = K_2 = [u, u']$. Then*

$$\gamma_r^{(-1)}(G[H]) = \gamma(G).$$

Proof: Let A be a minimum dominating set of G. Then $A \times \{u\}$ is a minimum dominating set of $G[H]$, $u \in V(H)$ by Remark 2.2. Suppose that $S = S' \times \{u'\}$ where $u' \in V(H) \setminus \{u\}$, $S' \subseteq V(G)$, and S' is a dominating set of G. Then by Theorem 2.3*(ii)*, S is a restrained inverse dominating set of $G[H]$. Thus,

$$\gamma_r^{(-1)}(G[H]) \leq |S| = |S' \times \{u'\}| = |S'| \cdot 1 = |S'|,$$

that is, $\gamma_r^{(-1)}(G[H]) \leq |S'|$ for all $S' \subset V(G)$. Thus, $\gamma_r^{(-1)}(G[H]) \leq \gamma(G)$. Since,

$$\gamma(G) = |A| = |A| \cdot 1 = |A \times \{u\}| = \gamma(G[H]) \leq \gamma_r^{(-1)}(G[H]) \leq \gamma(G),$$

it follows that $\gamma_r^{(-1)}(G[H]) = \gamma(G)$. ■

Definition 2.5 *The Cartesian product $G \square H$ is the graph with vertex set $V(G \square H) = V(G) \times V(H)$ and edge set $E(G \square H)$ satisfying the following conditions: $(u_1, u_2)(v_1, v_2) \in E(G \times H)$ if and only if either $v_1 = v_2$ and $u_1, u_2 \in E(G)$ or $u_1 = u_2$ and $v_1 v_2 \in E(H)$.*

Remark 2.6 *Let $G = P_m = [u_1, u_2, \dots, u_m]$ where $m \equiv 1(mod 4), m \neq 1$ and $H = P_4 = [v_1, v_2, v_3, v_4]$. Then $D = (\{(u_{4i-3}, v_1), (u_{4i-3}, v_4): i = 1, 2, \dots, \frac{m+3}{4}\}) \cup (\{(u_{4i-1}, v_2), (u_{4i-1}, v_3): i = 1, 2, \dots, \frac{m-1}{4}\})$ is a minimum dominating set of $G \square H$.*

The following result shows a property of restrained inverse dominating set of the Cartesian products of two graphs.

Theorem 2.7 *Let $G = P_m = [u_1, u_2, \dots, u_m]$ where $m \equiv 1(mod 4)$, $m \neq 1$ and $H = P_4 = [v_1, v_2, v_3, v_4]$. Then $S \subseteq V(G \square H) \setminus D$ is restrained inverse dominating set of $G \square H$ with respect to a minimum dominating set of D of $G \square H$, if $D = (X_1 \times Y_1) \cup (X_2 \times \{V(H) \setminus Y_1\})$, and $Y_1 = \{v_1, v_4\}$,*

$$X_1 = \{u_{4i-3}: i = 1, 2, \dots, \frac{m+3}{4}\}, X_2 = \{u_{4i-1}: i = 1, 2, \dots, \frac{m-1}{4}\},$$

and $S = (X_2 \times Y_1) \cup (X_1 \times (V(H) \times Y_1))$.

Proof: Suppose that $D = (X_1 \times Y_1) \cup (X_2 \times \{V(H) \setminus Y_1\})$, and $Y_1 = \{v_1, v_4\}, X_1 = \{u_{4i-3}: i = 1, 2, \dots, \frac{m+3}{4}\}, X_2 = \{u_{4i-1}: i = 1, 2, \dots, \frac{m-1}{4}\}$, and $X_3 = \{u_{2i}: i = 1, 2, \dots, \frac{m+3}{4}\}$. Then

$$\begin{aligned} D &= (X_1 \times Y_1) \cup (X_2 \times \{V(H) \setminus Y_1\}) \\ &= (\{u_{4i-3}: i = 1, 2, \dots, \frac{m+3}{4}\} \times \{v_1, v_4\}) \\ &\quad \cup (\{u_{4i-1}: i = 1, 2, \dots, \frac{m-1}{4}\} \times \{V(H) \setminus \{v_1, v_4\}\}) \\ &= (\{(u_{4i-3}, v_1), (u_{4i-3}, v_4): i = 1, 2, \dots, \frac{m+3}{4}\}) \\ &\quad \cup (\{u_{4i-1}: i = 1, 2, \dots, \frac{m-1}{4}\} \times \{v_2, v_3\}) \end{aligned}$$

$$= (\{(u_{4i-3}, v_1), (u_{4i-3}, v_4) : i = 1, 2, \dots, \frac{m+3}{4}\})$$
$$\cup (\{(u_{4i-1}, v_2), (u_{4i-1}, v_3) : i = 1, 2, \dots, \frac{m-1}{4}\})$$

$$\text{Thus, } D = (\{(u_{4i-3}, v_1), (u_{4i-3}, v_4) : i = 1, 2, \dots, \frac{m+3}{4}\})$$
$$\cup (\{(u_{4i-1}, v_2), (u_{4i-1}, v_3) : i = 1, 2, \dots, \frac{m-1}{4}\})$$

Hence, D is a minimum dominating set of $G\square H$ by Remark 2.6. Suppose that statement *(i)* is satisfied. Then $S = (X_2 \times Y_1) \cup (X_1 \times (V(H) \setminus Y_1))$.

$$\begin{aligned} S &= (X_2 \times Y_1) \cup (X_1 \times (V(H) \setminus Y_1)) \\ &= (\{u_{4i-1} : i = 1, 2, \dots, \frac{m-1}{4}\} \times \{v_1, v_4\}) \\ &\quad \cup (\{u_{4i-3} : i = 1, 2, \dots, \frac{m+3}{4}\} \times \{v_2, v_3\}) \\ &= \{(u_3, v_1), (u_3, v_4), \dots, (u_{m-1}, v_1), (u_{m-1}, v_4)\} \\ &\quad \cup \{(u_1, v_2), (u_1, v_3), \dots, (u_{m+3}, v_2), (u_{m+3}, v_3)\} \end{aligned}$$

Let $(u, v) \in V(G\square H) \setminus S$.

Case1. If $u = u_{4i-1}$, then $(u_{4i-1}, v_k) \in V(G\square H) \setminus S$, where $k = 2$ or $k = 3$. Then there exists $(u_{4i-1}, v_j) \in S$ $(j = 1 \text{ or } j = 4)$ such that $(u_{4i-1}, v_k)(u_{4i-1}, v_j) \in E(G\square H)$.

Case2. If $u = u_{4i-3}$, then $(u_{4i-3}, v_k) \in V(G\square H) \setminus S$, where $k = 1$ or $k = 4$. Then there exists $(u_{4i-3}, v_j) \in S$ $(j = 2 \text{ or } j = 3)$ such that $(u_{4i-3}, v_k)(u_{4i-3}, v_j) \in E(G\square H)$.

In any case, S is a dominating set of $G\square H$. Let $A = (X_1 \times Y_1)$, $B = (X_2 \times (V(H) \setminus Y_1))$, $C = (X_2 \times Y_1)$, $D = (X_1 \times (V(H) \setminus Y_1))$. Then

$$\begin{aligned} D \cap S &= [(X_1 \times Y_1) \cup (X_2 \times (V(H) \setminus Y_1))] \cup [(X_2 \times Y_1) \cup (X_1 \times (V(H) \setminus Y_1))] \\ &= [A \cup B] \cap [C \cup D] \\ &= [(A \cup B) \cap C] \cup [(A \cup B) \cap D] \\ &= [(A \cap C) \cup (B \cap C)] \cup [(A \cap D) \cup (B \cap D)]. \end{aligned}$$

Now,

$$\begin{aligned} A \cap C &= (X_1 \times Y_1) \cap (X_2 \times Y_1) \\ &= (\{u_{4i-3} : i = 1, 2, \dots, \frac{m+3}{4}\} \times \{v_1, v_4\}) \cap (\{u_{4i-1} : i = 1, 2, \dots, \frac{m-1}{4}\} \times \{v_1, v_4\}) \\ &= \{(u_1, v_1), (u_5, v_1), \dots, (u_m, v_1), (u_1, v_4), (u_5, v_4), \dots, (u_m, v_4)\} \\ &\quad \cap \{(u_3, v_1), (u_7, v_1), \dots, (u_{m-2}, v_1), (u_3, v_4), (u_7, v_4), \dots, (u_{m-2}, v_4)\} \\ &= \emptyset. \end{aligned}$$

$$\begin{aligned} B \cap C &= (X_2 \times (V(H) \setminus Y_1)) \cap (X_2 \times Y_1) \\ &= (\{u_{4i-1} : i = 1, 2, \dots, \frac{m-1}{4}\} \times \{v_2, v_3\}) \cap (\{u_{4i-1} : i = 1, 2, \dots, \frac{m-1}{4}\} \times \{v_1, v_4\}) \\ &= \{(u_3, v_2), (u_7, v_2), \dots, (u_{m-2}, v_2), (u_3, v_3), (u_7, v_3), \dots, (u_{m-2}, v_3)\} \\ &\quad \cap \{(u_3, v_1), (u_7, v_1), \dots, (u_{m-2}, v_1), (u_3, v_4), (u_7, v_4), \dots, (u_{m-2}, v_4)\} \end{aligned}$$

$= \emptyset.$

$$A \cap D = (X_1 \times Y_1) \cap (X_1 \times (V(H) \setminus Y_1))$$
$$= (\{u_{4i-3} : i = 1,2,\dots,\frac{m+3}{4}\} \times \{v_1, v_4\}) \cap (\{u_{4i-3} : i = 1,2,\dots,\frac{m+3}{4}\} \times \{v_2, v_3\})$$
$$= \{(u_1,v_1),(u_5,v_1),\dots,(u_m,v_1),(u_1,v_4),(u_5,v_4),\dots,(u_m,v_4)\}$$
$$\cap \{(u_1,v_2),(u_5,v_2),\dots,(u_m,v_2),(u_1,v_3),(u_5,v_3),\dots,(u_m,v_3)\}$$
$$= \emptyset.$$

$$B \cap D = (X_2 \times (V(H) \setminus Y_1)) \cap (X_1 \times (V(H) \setminus Y_1))$$
$$= (\{u_{4i-1} : i = 1,2,\dots,\frac{m-1}{4}\} \times \{v_2, v_3\}) \cap (\{u_{4i-3} : i = 1,2,\dots,\frac{m+3}{4}\} \times \{v_2, v_3\})$$
$$= \{(u_3,v_2),(u_7,v_2),\dots,(u_{m-2},v_2),(u_3,v_3),(u_7,v_3),\dots,(u_{m-2},v_3)\}$$
$$\cap \{(u_1,v_2),(u_5,v_2),\dots,(u_m,v_2),(u_1,v_3),(u_5,v_3),\dots,(u_m,v_3)\}$$
$$= \emptyset.$$

Thus,

$$D \cap S = [(A \cap C) \cup (B \cap C)] \cup [(A \cap D) \cup (B \cap D)]$$
$$= [(\emptyset) \cup (\emptyset)] \cup [(\emptyset) \cup (\emptyset)]$$
$$= \emptyset,$$

implies that $S \subseteq V(G\square H) \setminus D$ is an inverse dominating set of $G\square H$ with respect to D.

Let $(u,v) \in V(G\square H) \setminus S$.

Case1. If $u = u_{4i-1}$, then there exists $(u_{4i-1}, v_k) \in S$, (where $k = 1$ or $k = 4$) such that $(u_{4i-1}, v_k)(u_{4i-1}, v_j) \in E(G\square H)$ (where $j = 2$ or $j = 3$) and there exists another $(u_{4i-p}, v_j) \in V(G\square H) \setminus S$ ($p = 0$ or $p = 2$) such that $(u_{4i-1}, v_k)(u_{4i-p}, v_j) \in E(G\square H)$.

Case2. If $u = u_{4i-3}$, then there exists $(u_{4i-3}, v_k) \in S$, (where $k = 2$ or $k = 3$) such that $(u_{4i-3}, v_k)(u_{4i-3}, v_j) \in E(G\square H)$ (where $j = 1$ or $j = 2$) and there exists another $(u_{4i-p}, v_j) \in V(G\square H) \setminus S$ ($p = 2$ or $p = 0$) such that $(u_{4i-3}, v_k)(u_{4i-p}, v_j) \in E(G\square H)$.

In any case, S is a restrained dominating set of $G\square H$. Accordingly, S is a restrained inverse dominating set of $G\square H$. ■

The following result is an immediate consequence of Theorem 2.7

Corollary 2.8 *Let* $G = P_m = [u_1, u_2, \dots, u_m]$ *where* $m \equiv 1(mod 4), m \neq 1$ *and* $H = P_4 = [v_1, v_2, v_3, v_4]$. *Then* $\gamma_r^{(-1)}(G\square H) = m + 1$.

Proof: Suppose that $D = (X_1 \times Y_1) \cup (X_2 \times (V(H) \setminus Y_1))$, and $Y_1 = \{v_1, v_4\}, X_1 = \{u_{4i-3} : i = 1,2,\dots,\frac{m+3}{4}\}, X_2 = \{u_{4i-1} : i = 1,2,\dots,\frac{m-1}{4}\}$, and $S = (X_2 \times Y_1) \cup (X_1 \times (V(H) \setminus Y_1))$. Then, by Theorem 2.7, $S \subseteq V(G\square H) \setminus D$ is a restrained inverse dominating set of $G\square H$ with respect to a minimum dominating set D of $G\square H$. Thus,

$$\gamma_r^{(-1)}(G\square H) \leq |S|$$
$$= |(X_2 \times Y_1) \cup (X_1 \times (V(H) \setminus Y_1))|$$
$$= |(X_2 \times Y_1)| + |(X_1 \times (V(H) \setminus Y_1))|$$

$$= |(\{u_{4i-1} : i = 1, 2, \dots, \frac{m-1}{4}\} \times \{v_1, v_4\})| + |(\{u_{4i-3} : i = 1, 2, \dots, \frac{m+3}{4}\} \times \{v_2, v_3\})|$$
$$= |\{u_{4i-1} : i = 1, 2, \dots, \frac{m-1}{4}\}| \cdot |\{v_1, v_4\}| + |\{u_{4i-3} : i = 1, 2, \dots, \frac{m+3}{4}\}| \cdot |\{v_2, v_3\}|$$
$$= (\frac{m-1}{4}) \cdot 2 + (\frac{m+3}{4}) \cdot 2$$
$$= \frac{2m+2}{2}$$
$$= m + 1.$$

Since $D = (\{(u_{4i-3}, v_1), (u_{4i-3}, v_4) : i = 1, 2, \dots, \frac{m+3}{4}\}) \cup (\{(u_{4i-1}, v_2)(u_{4i-1}, v_3) : i = 1, 2, \dots, \frac{m-1}{4}\})$ is a minimum dominating set of $G \square H$, by Remark 2.6. It follows that

$$|D| = |(\{(u_{4i-3}, v_1), (u_{4i-3}, v_4) : i = 1, 2, \dots, \frac{m+3}{4}\}) \cup (\{(u_{4i-1}, v_2)(u_{4i-1}, v_3) : i = 1, 2, \dots, \frac{m-1}{4}\})|$$
$$= |(\{(u_{4i-3}, v_1), (u_{4i-3}, v_4) : i = 1, 2, \dots, \frac{m+3}{4}\})| + |(\{(u_{4i-1}, v_2), (u_{4i-1}, v_3) : i = 1, 2, \dots, \frac{m-1}{4}\})|$$
$$= [2 \cdot (\frac{m+3}{4})] + [2 \cdot (\frac{m-1}{4})]$$
$$= \frac{2m+2}{3}$$
$$= m + 1$$

Hence, $m + 1 = |D| = \gamma(G \square H) \leq \gamma_r^{(-1)}(G \square H) \leq m + 1$, that is,
$$\gamma_r^{(-1)}(G \square H) = m + 1. \blacksquare$$

3 Conclusion

In this paper, we extended the study on restrained inverse domination in graphs by investigating two binary graph operations – the lexicographic product and Cartesian product of two graphs. Some properties of the restrained inverse domination in the lexicographic product and Cartesian product of two graphs were proven and the exact values of the restrained inverse domination number of graphs resulting from these two binary graph operations were computed. This study will pave a way to new and relevant research concepts such as bounds and other binary operations of two connected graphs. Other parameters involving the restrained inverse domination in graphs may also be explored. Finally, the characterization of a restrained inverse domination in graphs in the tensor product, and its bounds are promising extension of this study.

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