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Disjoint Perfect Secure Domination in the Cartesian Product and Lexicographic Product of Graphs

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Abstract
Let G be a graph. A dominating set $D \subseteq V(G)$ is called a secure dominating set of G if for each vertex $u \in V(G) \setminus D$, there exists a vertex $v \in D$ such that $uv \in E(G)$ and the set $(D \setminus \{v\}) \cup \{u\}$ is a dominating set of G. If every $u \in V(G) \setminus D$ is adjacent to exactly one vertex in D, then D is a perfect secure dominating set of G. Let D be a minimum perfect secure dominating set of G. If $S \subseteq V(G) \setminus D$ is a perfect secure dominating set of G, then S is called an inverse perfect secure dominating set of G with respect to D. A disjoint perfect secure dominating set of G is the set $C = D \cup S \subseteq V(G)$. Furthermore, the disjoint perfect secure domination number, denoted by $\gamma_{ps}\gamma_{ps}(G)$, is the minimum cardinality of a disjoint perfect secure dominating set of G. A disjoint perfect secure dominating set of cardinality $\gamma_{ps}\gamma_{ps}(G)$ is called $\gamma_{ps}\gamma_{ps}$-set. In this paper, we extended the study on the concept of disjoint perfect secure domination in graphs. Furthermore, we characterized the disjoint perfect secure domination in the Cartesian product and lexicographic product of two graphs.
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1. Introduction

In 1736, L. Euler published a paper entitled *Solutio Problematis ad Geometriam Situs Pertinentis*, presenting his solution to the popular Königsberg Bridge Problem. Euler's significant contribution to solving the problem is by using a conceptual approach, where he used lines and letters to represent the broader scenario involving landmasses and bridges [1]. His approach to solving the said problem is widely recognized as the first to discuss and demonstrate the fundamental concepts of graph theory. Over time, graph theory as a mathematical field evolved and branched out into different subfields, including the domination theory. Berge initially introduced the concept of graph domination [2], while Ore formally defined the terms *dominating set* and *domination number* [3]. These pioneering concepts have motivated researchers to study other types of domination parameters.

A dominating set $D \subseteq V(G)$ is a set of vertices of G where every vertex in $V(G)\backslash S$ is adjacent to some vertex in D. A minimum dominating set is a dominating set such that no subset has this property. The domination number $\gamma(G)$ of a graph G is the smallest number of vertices in any minimum dominating set. Numerous types of dominating sets [4-16] have been discovered since the introduction of domination theory. One type of a dominating set is the inverse dominating set. The dominating set $S \subseteq V(G) \setminus D$ is called an *inverse dominating set* of G with respect to a minimum dominating set D. The concept of inverse domination in graphs was first introduced by Kulli [17], with further information in [18-26]. Another type is the disjoint dominating set, defined by Hedetniemi et al [27]. The *disjoint dominating set* is the set $\gamma\gamma(G) = min\{|S1| + |S2| : S1$ and $S2$ are disjoint dominating sets of $G\}$. A $\gamma\gamma$*-pair* of G consists of two disjoint dominating sets whose union has cardinality $\gamma\gamma(G)$. For further insights on disjoint domination in graphs, refer to [28-31].

Other variants of dominating sets are the perfect and secure dominating sets. A dominating set $S \subseteq V(G)$ is called a *perfect dominating set* of G if each $u \in V(G) \setminus S$ is dominated by exactly one element of S. This type of domination was introduced by Cockayne et.al [32]. Further topics on perfect dominating sets can be found in [33-35]. On the other hand, a dominating set $D \subseteq V(G)$ is called a *secure dominating set* of G if for each vertex $u \in V(G) \setminus D$, there exists a vertex $v \in D$ such that $uv \in E(G)$ and the set $(D \setminus \{v\}) \cup \{u\}$ is a dominating set of G [36]. More topics on secure dominating sets can be found in [37-39].

Rashmi et al. [40] introduced perfect secure dominating sets. In a graph G, a subset D of vertices in $V(G)$ is called a *perfect secure dominating set* of G if, for every vertex $v \in V(G) \setminus D$, there exists a unique vertex $u \in D$ such that u and v are adjacent. In addition, the set obtained by $(D \setminus \{v\}) \cup \{u\}$ still forms a dominating set. The minimum number of vertices needed for this is denoted as the perfect secure domination number, $\gamma_{ps}(G)$.

Integrating the properties of an inverse dominating set, Castañares and Enriquez [41] studied on the concept of inverse perfect secure dominating sets. Let D be a minimum perfect secure dominating set of G. If $S \subseteq V(G) \setminus D$ is a perfect secure dominating set of G, then S is called an *inverse perfect secure dominating set* of G with respect to D. The inverse perfect secure domination number of G, denoted by $\gamma_{ps}^{-1}(G)$ is the minimum cardinality of an inverse perfect secure dominating set of G.

This study is an extension of the study of Udtohan and Enriquez [42]. Let D be a minimum perfect secure dominating set of G. If $S \subseteq V(G) \setminus D$ is a perfect secure dominating set of G, then S is called an inverse perfect secure dominating set of G with respect to D. A *disjoint perfect secure* dominating set of G is the set formed by $C = D \cup S \subseteq V(G)$. Moreover, the disjoint perfect secure domination number, denoted by $\gamma_{ps}\gamma_{ps}(G)$, is the minimum cardinality of a disjoint perfect secure dominating set of G. A $\gamma_{ps}\gamma_{ps}$-set is a disjoint perfect secure dominating set with cardinality $\gamma_{ps}\gamma_{ps}(G)$. In this paper, the researchers had presented the characterization of the disjoint perfect secure dominating sets and give the corresponding disjoint perfect secure domination number in the Cartesian and lexicographic product of two graphs.

2. Results

Since the $\gamma_p^{-1}(G)$ does not always exist in a connected nontrivial graph G by Salve et.al. [17], the researchers introduce $\mathcal{DPS}(G)$ as a family of all graphs with inverse perfect secure dominating set and disjoint perfect secure dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered belong to the family $\mathcal{DPS}(G)$.

Definition 2.1. *Let D be a minimum perfect secure dominating set of G. If $S \subseteq V(G) \setminus D$ is a perfect secure dominating set of G, then S is called an inverse perfect secure dominating set of G with respect to D. A disjoint perfect secure dominating set of G is the set $C = D \cup S \subseteq V(G)$.*

Note that a perfect dominating set is also a minimum dominating set of a graph as seen in the following remarks.

Remark 2.2. *Let G be a nontrivial connected graph. Then $\gamma(G) = \gamma_p(G) = \gamma_p^{-1}(G) = 1$.*

Remark 2.3. $\gamma_{ps}(K_n) = 1$ for all positive integer $n \geq 2$.

Definition 2.4. *The Cartesian product $G\square H$ is the graph with vertex set $V(GH) = V(G) \times V(H)$ and edge set $E(G\square H)$, which satisfy the following conditions: $(u_1, u_2)(v_1, v_2) \in E(G \times H)$ if and only if either $v_1 = v_2$ and $u_1 u_2 \in E(G)$ or $u_1 = u_2$ and $v_1 v_2 \in E(H)$.*

The following result shows that if a given property is attained by $C = D \cup S$, then C is a disjoint perfect secure dominating set of the Cartesian products of two given graphs.

Theorem 2.5. *Let $G = P_m = [v_1, v_2, \dots, v_m]$ of order $m \equiv 0 \ (mod\ 4)$, where $m \neq 0$. Suppose that $H = P_n = [u_1, u_2, \dots, u_n], n \geq 2$. Then $C = D \cup S \subseteq V(G\square H)$ is a disjoint perfect secure dominating set of $G\square H$, if $D = A \times V(H)$ and $S = B \times V(H)$, where*

$$A = \left\{v_{4i-3}, v_{4i} : i = 1, 2, \dots, \frac{m}{4}\right\} \text{ and } B = \left\{v_{4i-2}, v_{4i-1} : i = 1, 2, \dots, \frac{m}{4}\right\}.$$

Proof. Suppose $D = A \times V(H)$ and $A = \left\{v_{4i-3}, v_{4i} : i = 1, 2, \dots, \frac{m}{4}\right\}$. Let $(v, u) \in V(G\square H) \setminus D$. Then

$$(v, u) \in \left(\left\{v_{4i-2}, v_{4i-1} : i = 1, 2, \dots, \frac{m}{4}\right\} \times V(H)\right).$$

Case 1. If $(v, u) \in \left\{v_{4i-2} : i = 1, 2, \dots, \frac{m}{4}\right\} \times V(H)$, then (v, u) is dominated by exactly one vertex $(v_{4i-3}, u) \in D$ for $i = 1, 2, \dots, \frac{m}{4}, u \in V(H)$, and

$$\left(D \setminus \left\{(v_{4i-3}, u) : i = 1, 2, \dots, \frac{m}{4}, u \in V(H)\right\}\right) \cup \{(v, u)\}$$

is a dominating set of $G\square H$.

Case 2. If $(v, u) \in \left\{v_{4i-1} : i = 1, 2, \dots, \frac{m}{4}\right\} \times V(H)$, then (v, u) is dominated by exactly one vertex $(v_{4i}, u) \in D$ for $i = 1, 2, \dots, \frac{m}{4}, u \in V(H)$, and

$$\left(D \setminus \left\{(v_{4i}, u) : i = 1, 2, \dots, \frac{m}{4}, u \in V(H)\right\}\right) \cup \{(v, u)\}$$

is a dominating set of $G\square H$.

In any case, D is a perfect secure dominating set of $G\square H$. By Remark 2.2, D is a minimum perfect secure dominating set of $G\square H$.

Suppose that $S = B \times V(H)$, where $B = \left\{v_{4i-2}, v_{4i-1} : i = 1, 2, \dots, \frac{m}{4}\right\}$. Let $(v, u) \in V(G\square H) \setminus S$. Then

$$(v, u) \in \left(\left\{v_{4i-3}, v_{4i} : i = 1, 2, \dots, \frac{m}{4}\right\} \times V(H)\right).$$

Case 1. If $(v,u) \in \left\{v_{4i-3}: i = 1, 2, \dots, \frac{m}{4}\right\} \times V(H)$, then (v,u) is dominated by exactly one vertex $(v_{4i-2},u) \in S$ for $i = 1, 2, \dots, \frac{m}{4}, u \in V(H)$, and

$$\left(S \setminus \left\{(v_{4i-2},u): i = 1, 2, \dots, \frac{m}{4}, u \in V(H)\right\}\right) \cup \{(v,u)\}$$

is a dominating set of $G \square H$.

Case 2. If $(v,u) \in \left\{v_{4i}: i = 1, 2, \dots, \frac{m}{4}\right\} \times V(H)$, then (v,u) is dominated by exactly one vertex $(v_{4i-1},u) \in S$ for $i = 1, 2, \dots, \frac{m}{4}, u \in V(H)$, and

$$\left(S \setminus \left\{(v_{4i-1},u): i = 1, 2, \dots, \frac{m}{4}, u \in V(H)\right\}\right) \cup \{(v,u)\}$$

is a dominating set of $G \square H$.

In any case, S is a perfect secure dominating set of $G \square H$. Since

$$\begin{aligned} D \cap S &= \left(A \times V(H)\right) \cap \left(B \times V(H)\right) \\ &= \left(\left\{v_{4i-3}, v_{4i}: i = 1, 2, \dots, \frac{m}{4}\right\} \times V(H)\right) \bigcap \left(\left\{v_{4i-2}, v_{4i-1}: i = 1, 2, \dots, \frac{m}{4}\right\} \times V(H)\right) \\ &= \emptyset, \end{aligned}$$

it follows that $D \cap S = \emptyset$. Thus, $S \subseteq V(G \square H) \setminus D$ is an inverse perfect secure dominating set of $G \square H$ with respect to D. Hence, $C = D \cup S$ is a disjoint perfect secure dominating set of $G \square H$.■

The following result is an immediate consequence of Theorem 2.5.

Corollary 2.6. *Let* $G = P_m$ *where* $m \equiv 0 \ (mod\ 4)$, $m \neq 0$, *and* $H = P_n, n \geq 2$. *Then* $\gamma_{ps}\gamma_{ps}(G \square H) = mn$.

Proof. Suppose that $D = A \times V(H)$ and $S = B \times V(H)$, where

$$A = \left\{v_{4i-3}, v_{4i}: i = 1, 2, \dots, \frac{m}{4}\right\} \text{ and } B = \left\{v_{4i-2}, v_{4i-1}: i = 1, 2, \dots, \frac{m}{4}\right\}.$$

Then by Theorem 2.5, $C = D \cup S$ is a disjoint perfect secure dominating set of $G \square H$. Thus, $\gamma_{ps}\gamma_{ps}(G \square H) \leq |C|$. Now, $|C| = |D \cup S| = |D| + |S|$.

Since $D = A \times V(H)$ where $A = \left\{v_{4i-3}, v_{4i}: i = 1, 2, \dots, \frac{m}{4}\right\}$,

$$\begin{aligned} |D| &= \left|\left(\left\{v_{4i-3}, v_{4i}: i = 1,2, \dots, \frac{m}{4}\right\} \times V(H)\right)\right| \\ &= \left|\left\{v_{4i-3}, v_{4i}: i = 1,2, \dots, \frac{m}{4}\right\}\right| \cdot |V(H)| \\ &= \left[2 \cdot \frac{m}{4}\right] \cdot n \\ &= \frac{mn}{2}. \end{aligned}$$

Since $S = B \times V(H)$, where $B = \left\{v_{4i-2}, v_{4i-1}: i = 1, 2, \dots, \frac{m}{4}\right\}$,

$$\begin{aligned} |S| &= \left|\left(\left\{v_{4i-2}, v_{4i-1}: i = 1,2, \dots, \frac{m}{4}\right\} \times V(H)\right)\right| \\ &= \left|\left\{v_{4i-2}, v_{4i-1}: i = 1,2, \dots, \frac{m}{4}\right\}\right| \cdot |V(H)| \\ &= \left[2 \cdot \frac{m}{4}\right] \cdot n \\ &= \frac{mn}{2}. \end{aligned}$$

Thus, $|C| = |D \cup S| = |D| + |S| = \frac{mn}{2} + \frac{mn}{2} = mn$. Since D and $S$are minimum perfect secure dominating sets by Remark 2.2, it follows that

$$mn = \frac{mn}{2} + \frac{mn}{2} = \gamma_{ps}(G\square H) + \gamma_{ps}(G\square H) = \gamma_{ps}\gamma_{ps}(G\square H) \leq |C| = mn,$$

that is, $\gamma_{ps}\gamma_{ps}(G\square H) = mn$. ■

Definition 2.7. The lexicographic product of two graphs G and H, denoted by $G[H]$, is the graph with $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ satisfying the following conditions:

$(u_1, v_1)(u_2, v_2) \in E(G[H])$ if either $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

Theorem 2.8. *Let $G = \bigcup_{k=1}^{\frac{n}{4}} P_4^k$ of order $n \equiv 0 \ (mod\ 4)$, where $n \geq 4$. Suppose that $P_4^k = [v_1, v_2, v_3, v_4]$ for each k, and $H = K_2 = [u_1, u_2]$. Then $C = D \cup S \subseteq V(G[H])$ is a disjoint perfect secure dominating set of $G[H]$ if $D = \left(\bigcup_{k=1}^{\frac{n}{4}} A_k\right) \times \{u_1\}$ and $S = \left(\bigcup_{k=1}^{\frac{n}{4}} A_k\right) \times \{u_2\}$ where $A_k = \{v_{4k-3}, v_{4k}\}$ for each k.*

Proof. Suppose $D = \left(\bigcup_{k=1}^{\frac{n}{4}} A_k\right) \times \{u_1\}$ where $A_k = \{v_{4k-3}, v_{4k}\}$ for each k. Let $(v, u) \in V(G[H]) \setminus D$. Then

$$(v, u) = \left(\left\{v_{4i-2}, v_{4i-1} : i = 1,2, \dots, \frac{n}{4}\right\} \times \{u_1\}\right) \cup (V(G) \times \{u_2\}),$$

that is,

$$(v, u) = \left\{(v_{4i-2}, u_1), (v_{4i-1}, u_1) : i = 1,2, \dots, \frac{n}{4}\right\} \cup \{(v, u_2) : v \in V(G)\}.$$

Case 1. If $(v, u) \in \left\{(v_{4i-2}, u_1) : i = 1,2, \dots, \frac{n}{4}\right\}$, then (v, u) is dominated by exactly one vertex $(v_{4i-3}, u_1) \in D$ for $i = 1,2, \dots, \frac{n}{4}$. Further,

$$\left(D \setminus \left\{(v_{4i-3}, u_1) : i = 1,2, \dots, \frac{n}{4}\right\}\right) \cup \{(v, u)\}$$

is a dominating set of $G[H]$.

If $(v, u) \in \left\{(v_{4i-1}, u_1) : i = 1,2, \dots, \frac{n}{4}\right\}$, then (v, u) is dominated by only one vertex $(v_{4i}, u_1) \in D$ for $i = 1,2, \dots, \frac{n}{4}$. Further,

$$\left(D \setminus \left\{(v_{4i}, u_1) : i = 1,2, \dots, \frac{n}{4}\right\}\right) \cup \{(v, u)\}$$

is a dominating set of $G[H]$.

Case 2. If $(v, u) \in \{(v, u_2) : v \in V(G)\}$, then (v, u) is dominated by exactly one vertex $(v_{4i-3}, u_1) \in D$ or $(v_{4i}, u_1) \in D$ for $i = 1,2, \dots, \frac{n}{4}$. Further,

$$\left(D \setminus \left\{(v_{4i-3}, u_1) : i = 1,2, \dots, \frac{n}{4}\right\}\right) \cup \{(v, u)\}$$

is a dominating set of $G[H]$ or

$$\left(D \setminus \left\{(v_{4i}, u_1) : i = 1,2, \dots, \frac{n}{4}\right\}\right) \cup \{(v, u)\}$$

is a dominating set of $G[H]$.

In any case, D is a perfect secure dominating set of $G[H]$. By Remark 2.2, D is a minimum perfect secure dominating set of $G[H]$.

Suppose $S = \left(\bigcup_{k=1}^{\frac{n}{4}} A_k\right) \times \{u_2\}$ where $A_k = \{v_{4k-3}, v_{4k}\}$ for each k. Let $(v, u) \in V(G[H]) \setminus S$. Then

$$(v,u)=\left(\left\{v_{4i-2},v_{4i-1}: i=1,2,\dots,\frac{n}{4}\right\}\times\{u_2\}\right)\cup\left(V(G)\times\{u_1\}\right),$$

that is,

$$(v,u)=\left\{(v_{4i-2},u_2),(v_{4i-1},u_2) : i=1,2,\dots,\frac{n}{4}\right\}\cup\{(v,u_1): v\in V(G)\}.$$

Case 1. If $(v,u)\in\left\{(v_{4i-2},u_2): i=1,2,\dots,\frac{n}{4}\right\}$, then (v,u) is dominated by exactly one vertex $(v_{4i-3},u_2)\in S$ for $i=1,2,\dots,\frac{n}{4}$. Further,

$$\left(S\setminus\left\{(v_{4i-3},u_2): i=1,2,\dots,\frac{n}{4}\right\}\right)\cup\{(v,u)\}$$

is a dominating set of $G[H]$.

If $(v,u)\in\left\{(v_{4i-1},u_2): i=1,2,\dots,\frac{n}{4}\right\}$, then (v,u) is dominated by only one vertex $(v_{4i},u_2)\in S$ for $i=1,2,\dots,\frac{n}{4}$. Further,

$$\left(S\setminus\left\{(v_{4i},u_2): i=1,2,\dots,\frac{n}{4}\right\}\right)\cup\{(v,u)\}$$

is a dominating set of $G[H]$.

Case 2. If $(v,u)\in\{(v,u_1): v\in V(G)\}$, then (v,u) is dominated by exactly one vertex $(v_{4i-3},u_2)\in S$ or $(v_{4i},u_2)\in S$ for $i=1,2,\dots,\frac{n}{4}$. Further,

$$\left(S\setminus\left\{(v_{4i-3},u_2): i=1,2,\dots,\frac{n}{4}\right\}\right)\cup\{(v,u)\}$$

is a dominating set of $G[H]$ or

$$\left(S\setminus\left\{(v_{4i},u_2): i=1,2,\dots,\frac{n}{4}\right\}\right)\cup\{(v,u)\}$$

is a dominating set of $G[H]$.

In any case, S is a perfect secure dominating set of $G[H]$. Since

$$D\cap S=\left(\left(\bigcup_{k=1}^{n/4}A_k\right)\times\{u_1\}\right)\bigcap\left(\left(\bigcup_{k=1}^{n/4}A_k\right)\times\{u_2\}\right)=\emptyset,$$

$$A_k=\{v_{4k-3},v_{4k}\}\text{ for each }k,$$

it follows that $D\cap S=\emptyset$. Thus, $S\subseteq V(G[H])\setminus D$ is an inverse perfect secure dominating set of $G[H]$ with respect to D. Hence, $C=D\cup S$ is a disjoint perfect secure dominating set of $G[H]$. ■

The following result is an immediate consequence of Theorem 2.8.

Corollary 2.9. *Let* $G=\bigcup_{k=1}^{\frac{n}{4}}P_4^k$ *of order* $n\equiv 0\ (mod\ 4)$, *where* $n\geq 4$. *Suppose* $P_4^k=[v_1,v_2,v_3,v_4]$ *for each* k, *and* $H=K_2=[u_1,u_2]$. *Then* $\gamma_{ps}\gamma_{ps}(G[H])=n$.

Proof. Suppose that $D=\left(\bigcup_{k=1}^{\frac{n}{4}}A_k\right)\times\{u_1\}$ and that $S=\left(\bigcup_{k=1}^{\frac{n}{4}}A_k\right)\times\{u_2\}$ where $A_k=\{v_{4k-3},v_{4k}\}$ for each k. Then by Theorem 2.8, $C=D\cup S$ is a disjoint perfect secure dominating set of $G[H]$. Thus, $\gamma_{ps}\gamma_{ps}(G[H])\leq|C|$. Now, $|C|=|D\cup S|=|D|+|S|$.

Since, $D=\left(\bigcup_{k=1}^{\frac{n}{4}}A_k\right)\times\{u_1\}$ where $A_k=\{v_{4k-3},v_{4k}\}$ for each k,

$$
\begin{aligned}
|D| &= \left|\left(\bigcup_{k=1}^{n/4} A_k\right) \times \{u_1\}\right| \\
&= \left|\bigcup_{k=1}^{n/4} A_k\right| \cdot |\{u_1\}| \\
&= \left|\left\{v_{4i-3}, v_{4i}: i = 1,2,\dots,\frac{n}{4}\right\} \cdot |\{u_1\}|\right| \\
&= \left[2 \cdot \left(\frac{n}{4}\right)\right] \cdot 1 \\
&= \frac{n}{2}.
\end{aligned}
$$

Since if $S = \left(\bigcup_{k=1}^{\frac{n}{4}} A_k\right) \times \{u_2\}$ where $A_k = \{v_{4k-3}, v_{4k}\}$ for each k,

$$
\begin{aligned}
|S| &= \left|\left(\bigcup_{k=1}^{n/4} A_k\right) \times \{u_2\}\right| \\
&= \left|\bigcup_{k=1}^{n/4} A_k\right| \cdot |\{u_2\}| \\
&= \left|\left\{v_{4i-3}, v_{4i}: i = 1,2,\dots,\frac{n}{4}\right\} \cdot |\{u_2\}|\right| \\
&= \left[2 \cdot \left(\frac{n}{4}\right)\right] \cdot 1 \\
&= \frac{n}{2}.
\end{aligned}
$$

Thus, $|C| = |D \cup S| = |D| + |S| = \frac{n}{2} + \frac{n}{2} = n$. Since D and $S$are minimum perfect secure dominating sets by Remark 2.2, it follows that

$$n = \frac{n}{2} + \frac{n}{2} = \gamma_{ps}(G[H]) + \gamma_{ps}(G[H]) = \gamma_{ps}\gamma_{ps}(G[H]) \leq |C| = n,$$

that is, $\gamma_{ps}\gamma_{ps}(G[H]) = n$. ■

3. Conclusion

In this paper, we provided some important results on the disjoint perfect secure domination in graphs under some binary operations. Specifically, we characterized the disjoint perfect secure dominating set in the lexicographic product and Cartesian product of two graphs. Moreover, the disjoint perfect secure domination number on graphs under such binary operations was determined. Future studies on other parameters involving disjoint perfect secure domination are possible for expanding the understanding of this new parameter. Moreover, the disjoint perfect secure domination on graphs under other binary operations, such as the zig-zag product and the parallel graph composition, is a potential topic for future research.

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