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Disjoint Perfect Domination in the Cartesian Products of Two Graphs

Ma. Elizabeth N. Diapo[1], Grace M. Estrada[2], Marie Cris A. Bulay-og[3], Edward M. Kiunisala[4], Enrico L. Enriquez[5]

[1]MS Math, Department of Computer, Information Sciences, and Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines

[2]Associate Professor, Department of Computer, Information Sciences, and Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines

[3]Assistant Professor, Mathematics and Statistics Programs, University of the Philippines Cebu, 6000 Cebu City, Philippines

[4]Professor, Mathematics Department, College of Computing, Artificial Intelligence, and Sciences, Cebu Normal University, 6000 Cebu City, Philippines

[5]Full Professor, Department of Computer, Information Sciences, and Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines

Abstract

Let G be a nontrivial connected graph. A dominating set $D \subseteq V(G)$ is called a perfect dominating set of$G$ of each $u \in V(G) \setminus D$ is dominated by exactly one element of$D$. Let D be a minimum perfect dominating set of G. If $S \subseteq V(G) \setminus D$ is a perfect dominating set of G, then S is called an inverse perfect dominating set of G with respect to D. A disjoint perfect dominating set of G is the set $C = D \cup S \subseteq V(G)$. Furthermore, the disjoint perfect domination number, denoted by $\gamma_p\gamma_p(G)$, is the minimum cardinality of a disjoint perfect dominating set of G. A disjoint perfect dominating set of cardinalities$\gamma_p\gamma_p(G)$ is called $\gamma_p\gamma_p$-set. In this paper, we initiate a study of the concept of disjoint perfect domination in graphs and give some important results.

Mathematics Subject Classification: 05C69

Keywords: dominating, perfect, inverse, disjoint, Cartesian product

1. Introduction

Domination in graphs dates to 1960 when Berge defined the concept of domination number which was then called, the coefficient of external stability [1]. The name dominating set and domination number was used by Ore in 1962 for the same concept [2]. Accordingly, a subset D of a vertex set $V(G)$ in a graph $G = (V, E)$ is a *dominating set* if every vertex $v \in V(G)$ not in $D$is adjacent to at least one vertex $x \in D$. The *domination number*$\gamma(G)$ of G is the smallest cardinality of a dominating set D of G. There are several studies related to the concept of domination in graphs [3-15].

One of the parameters of domination in graphs is *perfect domination in graphs* for which definition was given by Cockayne, et. al., in [16] in 1993. For a graph G, a subset D of $V(G)$ is a *perfect dominating set*

of G if every vertex not in D is adjacent to exactly on vertex in D. Other studies that involve perfect domination can be found in [17-23]. Another variant of domination was introduced by Kulli and Sigarkanti[24] that is, *inverse domination in graphs*, that is, for a graph G with dominating set D, if $V \setminus D$ contains a dominating set D' of G, then D' is an *inverse dominating set* of G with respect to D. More articles that involve the concept of inverse domination can be found in [25-32].Hedetneimi, et. at., in [33]has initiated the study of *disjoint domination in graphs* which they defined as the disjoint union of two dominating sets. Some studies on disjoint domination in graphs can be found in [34-37].A combination of the parameters, perfect and inverse domination in graphs is the*inverse perfect domination*, studied by Enriquez and Salve in [38], where for a graph G with a perfect dominating set D, there exists another perfect dominating set $S \subseteq V(G)$ whose vertices are not in D then S is an inverse dominating set of G.For more graph-theoretical concepts, the readers may refer to paper [39].
The concepts of disjoint domination, perfect domination, and inverse perfect domination in graphs motivated the researchers to define a new parameter of domination that is, *disjoint perfect domination*. Let G be a nontrivial connected graph. A dominating set $D \subseteq V(G)$ is called a perfect dominating set of G of each $u \in V(G) \setminus D$ is dominated by exactly one element of D. Let D be a minimum perfect dominating set of G. If $S \subseteq V(G) \setminus D$ is a perfect dominating set of G, then S is called an inverse perfect dominating set of G with respect to D. A disjoint perfect dominating set of G is the set $C = D \cup S \subseteq V(G)$. Furthermore, the disjoint perfect domination number, denoted by $\gamma_p\gamma_p(G)$, is the minimum cardinality of a disjoint perfect dominating set of G. A disjoint perfect dominating set of cardinality $\gamma_p\gamma_p(G)$ is called $\gamma_p\gamma_p$ −set. Further, the researchers give some property of the disjoint perfect domination in the Cartesian products two graphs. The disjoint prefect domination in graphs is an NP-complete problem. Unless otherwise stated, all subsets of the vertex set in this paper are assumed to be nonempty.

2. Results

Since the $\gamma_p^{-1}(G)$does not always exist in a connected nontrivial graph G by Salve et.al., the researchers introduce $\mathcal{DP}(G)$ as a family of all graphs with inverse perfect dominating set and disjoint perfect dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered belong to the family $\mathcal{DP}(G)$.

Remark 2.1 Let G be a connected nontrivial graph. Then$\gamma(G) = \gamma_p(G) = \gamma_p^{-1}(G) = 1$.

Definition 2.2 Let D be a minimum perfect dominating set of G and $S \subseteq V(G) \setminus D$ is an inverse perfect dominating set of G with respect to D. A disjoint perfect dominating set of G is the set $C = D \cup S \subseteq V(G)$.

Definition 2.3 The cartesian product $G \square H$ is the graph with vertex set$V(G \square H) = V(G) \times V(H)$ and edge set $E(G \square H)$ satisfying the following conditions: $(u_1, u_2)(v_1, v_2) \in E(G \times H)$ if and only if either $v_1 = v_2$ and $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

The following result shows that if a given property is attained by $C = D \cup S$, then C is a disjoint perfect dominating set of the Cartesian products of two given graphs.

Theorem 2.4 Let $G = P_m$ where $m \equiv 0(mod5)$, $m \neq 0$ and $H = P_4 = [u_1, u_2, u_3, u_4]$. Then $C = D \cup S \subseteq V(G\Box H)$ is a disjoint perfect dominating set of $G\Box H$, if

$$D = (A \times \{u_1, u_4\}) \cup (B \times \{u_2, u_3\})$$

where $A = \{v_{5i-4}, v_{5i}: i = 1,2, \dots, \frac{m}{5}\}$ and $B = \{v_{5i-2}: i = 1,2, \dots, \frac{m}{5}\}$ and

$$S = (A' \times \{u_1, u_4\}) \cup (B' \times \{u_2, u_3\})$$

where $A' = \{v_{5i-2}: i = 1,2, \dots, \frac{m}{5}\}$ and $B' = \{v_{5i-4,}v_{5i}: 1,2, \dots, \frac{m}{5}\}$.

Proof. Suppose that $D = (A \times \{u_1, u_4\}) \cup (B \times \{u_2, u_3\})$ where $A = \{v_{5i-4}, v_{5i}: i = 1,2, \dots, \frac{m}{5}\}$ and $B = \{v_{5i-1}: i = 1,2, \dots, \frac{m}{5}\}$. Let $(v, u) \in V(G\Box H) \setminus D$. Then,$(v, u) \in \left(\{v_{5i-3}, v_{5i-1}: i = 1,2, \dots, \frac{m}{5}\} \times V(H)\right) \cup \left(\{v_{5i-4}, v_{5i}: i = 1,2, \dots, \frac{m}{5}\} \times \{u_2, u_3\}\right) \cup \left(\{v_{5i-2}: i = 1,2, \dots, \frac{m}{5}\} \times \{u_1, u_4\}\right)$.

Case 1. If $(v, u) \in \{v_{5i-3}, v_{5i-1}: i = 1,2, \dots, \frac{m}{5} \times V(H)\}$, that is,$(v, u) \in \left\{(v_{5i-3}, u), (v_{5i-1}, u): i = 1,2, \dots, \frac{m}{5}, \text{and } u \in V(H)\right\} \subset V(G\Box H) \setminus D$, then $(v, u) = (v_{5i-3}, u_j)$ is dominated by exactly one vertex $(v_{5i-4}, u_j) \in D$ for $i = 1,2, \dots, \frac{m}{5}, j = 1,4$ and $(v, u) = (v_{5i-1}, u_j)$ is dominated by exactly one vertex $(v_{5i}, uj) \in D$ for $i = 1,2, \dots, \frac{m}{5}, j = 1,4$. Similarly, $(v, u) = (v_{5i-3}, u_j)$ is dominated by exactly one vertex $(v_{5i-2}, u_j) \in D$ for $i = 1,2, \dots, \frac{m}{5}, j = 2,3$ and $(v, u) = (v_{5i-1}, u_j)$ is dominated by exactly one vertex $(v_{5i-2}, u_j) \in D$ for $i = 1,2, \dots, \frac{m}{5}, j = 2,3$.

Case 2. If $(v, u) \in \{v_{5i-4}, v_{5i}: i = 1,2, \dots, \frac{m}{5}\} \times \{u_1, u_4\}$, that is, $(v, u) \in \left\{(v_{5i-4}, u), (v_{5i}, u): i = 1,2, \dots, \frac{m}{5}, \text{and } u \in \{u_1, u_4\}\right\} \subset V(G\Box H) \setminus S$ then $(v, u) = (v_{5i-4}, u_1)$ is dominated by exactly one vertex $(v_{5i-4}, u_2) \in S$ for $i = 1,2, \dots, \frac{m}{5}, (v, u) = (v_{5i-4}, u_4)$ is dominated by exactly one vertex $(v_{5i-4}, u_3) \in S$ for $i = 1,2, \dots, \frac{m}{5}$, and $(v, u) = (v_{5i}, u_1)$ is dominated by exactly one vertex $(v_{5i}, u_2) \in S$ for $i = 1,2, \dots, \frac{m}{5}, (v, u) = (v_{5i}, u_4)$ is dominated by exactly one vertex $(v_{5i}, u_3) \in S$ for $i = 1,2, \dots, \frac{m}{5}$.

Case 3. If $(v, u) \in \{v_{5i-2}: i = 1,2, \dots, \frac{m}{5}\} \times \{u_2, u_3\}$, that is, $(v, u) \in \left\{(v_{5i-2}, u): i = 1,2, \dots, \frac{m}{5}, \text{and } u \in \{u_2, u_3\}\right\} \subset V(G\Box H) \setminus S$, then $(v, u) = (v_{5i-2}, u_2)$ is dominated by exactly one vertex $(v_{5i-2}, u_1) \in S$ for $i = 1,2, \dots, \frac{m}{5}, (v, u) = (v_{5i-2}, u_3)$ is dominated by exactly one vertex $(v_{5i-2}, u_4) \in S$ for $i = 1,2, \dots, \frac{m}{5}$.

In any case, S is a perfect dominating set of $G\Box H$. Since,

$$D \cap S = (A \times \{u_1, u_4\}) \cup (B \times \{u_2, u_3) \cap (A^{\wedge\prime} \times \{u_1, u_4\}) \cup (B' \times \{u_2, u_3\})$$

where $A = \{v_{5i-4}, v_{5i}: i = 1,2,\dots,\frac{m}{5}\}$ and $B = \{v_{5i-2}: i = 1,2,\dots,\frac{m}{5}\}$ and $A' = \{v_{5i-2}: i = 1,2,\dots,\frac{m}{5}\}$ and $B' = \{v_{5i-4}, v_{5i}: i = 1,2,\dots,\frac{m}{5}\}$, it follows that $D \cup S = \emptyset$. Thus, $S \subset V(G \times H) \setminus D$, that is S is an inverse perfect dominating set of $G\square H$ with respect to D. Hence, $C = D \cup S$ is a disjoint perfect dominating set of $G\square H$.

□

The following result is an immediate consequence of Theorem 2.4.

Corollary 2.5 Let $G = P_m$ where $m \equiv 0(mod5), m \neq 0$ and $H = P_4$. Then, $\gamma_p\gamma_p(G\square H) = \frac{12m}{5}$.

Proof. Let $G = P_m$ where $m \equiv 0(mod5), m \neq 0$ and $H = P_4$. Suppose that

$$D = (A \times \{u_1, u_4\}) \cup (B \times \{u_2, u_3\})$$

where $A = \{v_{5i-4}, v_{5i}: i = 1,2,\dots,\frac{m}{5}\}$ and $B = \{v_{5i-2}: i = 1,2,\dots,\frac{m}{5}\}$ and

$$S = (A' \times \{u_1, u_4\}) \cup (B' \times \{u_2, u_3\})$$

where $A' = \{v_{5i-2}: i = 1,2,\dots,\frac{m}{5}\}$ and $B' = \{v_{5i-4}, v_{5i}: i = 1,2,\dots,\frac{m}{5}\}$. Then by Theorem 2.4, $C = D \cup S$ is a disjoint perfect dominating set of $G\square H$. Thus, $\gamma_p\gamma_p(G\square H) \leq |C|$. Now, $|C| = |D \cup S| = |D| + |S|$.

Since $D = (A \times \{u_1, u_4\}) \cup (B \times \{u_2, u_3\})$ where $A = \{v_{5i-4}, v_{5i}: i = 1,2,\dots,\frac{m}{5}\}$ and $B = \{v_{5i-2}: i = 1,2,\dots,\frac{m}{5}\}$, it follows that

$$|D| = \left|\left(\{v_{5i-4}, v_{5i}: i = 1,2,\dots,\frac{m}{5}\} \times \{u_1, u_4\}\right) \cup \left(\{v_{5i-2}: i = 1,2,\dots,\frac{m}{5}\} \times \{u_2, u_3\}\right)\right|.$$

Thus,

$$\begin{aligned}|D| &= \left|\left(\{v_{5i-4}, v_{5i}: i = 1,2,\dots,\frac{m}{5}\} \times \{u_1, u_4\}\right)\right| \\ &\quad + \left|\left(\{v_{5i-2}: i = 1,2,\dots,\frac{m}{5}\} \times \{u_2, u_3\}\right)\right| \\ &= \left[2 \cdot \left(\frac{m}{5}\right) \cdot 2\right] + \left(\frac{m}{5}\right) \cdot 2 \\ &= \frac{6m}{5}.\end{aligned}$$

Since $S = (A' \times \{u_1, u_4\}) \cup (B' \times \{u_2, u_3\})$ where $A' = \{v_{5i-2}: i = 1,2,\dots,\frac{m}{5}\}$ and $B' = \{v_{5i-4}, v_{5i}: i = 1,2,\dots,\frac{m}{5}\}$, it follows that

$$|S| = \left|\left(\{v_{5i-2}: i = 1,2,\dots,\frac{m}{5}\} \times \{u_1, u_4\}\right) \cup \left(\{v_{5i-4}, V_{5i}: i = 1,2,\dots,\frac{m}{5}\} \times \{u_2, u_3\}\right)\right|.$$

Thus,

$$\begin{aligned}|S| &= \left|\left(\{v_{5i-2}: i = 1,2,\dots,\frac{m}{5}\} \times \{u_1, u_4\}\right)\right| \\ &\quad + \left|\left(\{v_{5i-4}, v_{5i}: i = 1,2,\dots,\frac{m}{5}\} \times \{u_2, u_3\}\right)\right| \\ &= \left(\frac{m}{5} \cdot 2\right) + \left(2 \cdot \frac{m}{5} \cdot 2\right) \\ &= \frac{6m}{5}.\end{aligned}$$

Therefore, $|C| = |D \cup S| = |D| + |S| = \frac{6m}{5} + \frac{6m}{5} = \frac{12m}{5}$. Since D and S are minimum perfect dominating sets, by Remark 2.1, it follows that $\frac{12m}{5} = \frac{6m}{5} + \frac{6m}{5} = \gamma_p(G\Box H) + \gamma_p(G\Box H) = \gamma_p\gamma_p(G\Box H) \leq |C| = \frac{12m}{5}$, that is, $\gamma_p\gamma_p(G\Box H) = \frac{12m}{5}$.

3. Conclusion

In this paper, we introduced a new parameter of domination of graphs - the disjoint perfect domination in graphs. Some property of the disjoint perfect in the Cartesian product of two graphs were proven and the exact values of the disjoint perfect domination number resulting from the Cartesian product of two graphs were computed. This study will pave a way to new research such bounds and other binary operations of two connected graphs. Other parameters involving the disjoint perfect domination in graphs may also be explored. Finally, the characterization of a disjoint perfect domination in graphs in the lexicographic product, and its bounds are promising extension of this study.

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