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Inverse Doubly Connected Domination in the Lexicographic Product of Two Graphs

Khaty M. Cruz[1], Enrico L. Enriquez[2], Katrina B. Fuentes[3], Grace M. Estrada[4], Marie Cris A. Bulay-og[5]

[1]MS Math, Department of Computer, Information Sciences, and Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines
[2]Full Professor, Department of Computer, Information Sciences, and Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines
[3,4]Associate Professor, Department of Computer, Information Sciences, and Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines
[5]Assistant Professor, Mathematics and Statistics Programs, University of the Philippines Cebu, 6000 Cebu City, Philippines

Abstract
Let G be a nontrivial connected graph. A dominating set $D \subseteq V(G)$ is called a doubly connected dominating set of G if both $\langle D \rangle$ and $\langle V(G)\backslash D \rangle$ are connected. Let D be a minimum connected dominating set of G. If $\mathrm{S} \subseteq \mathrm{V}(\mathrm{G})\backslash\mathrm{D}$ is a connected dominating set of G, then S is called an inverse doubly connected dominating set of G with respect to D. Furthermore, the inverse doubly connected domination number, denoted by $\gamma_{cc}^{-1}(\mathrm{G})$ is the minimum cardinality of an inverse doubly connected dominating set of G. An inverse doubly connected dominating set of cardinalities $\gamma_{cc}^{-1}(\mathrm{G})$ is called γ_{cc}^{-1}-set. In this paper, we characterized the inverse doubly connected domination in the lexicographic product of two graphs and give some important results.

Mathematics Subject Classification: 05C69

Keywords: dominating, doubly connected, inverse, lexicographic

1. Introduction

The graphs G considered here are simple, finite, nontrivial, undirected and without isolated vertices. Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [1]. Following an article [2] by Ernie Cockayne and Stephen Hedetniemi in 1977, the domination in graphs became an area of study by many researchers. A subset S of $V(\mathrm{G})$ is a *dominating set* of G if for every $v \in V(\mathrm{G})\backslash\mathrm{S}$, there exists $x \in \mathrm{S}$ such that $xv \in E(\mathrm{G})$, that is, $N[\mathrm{S}] = \mathrm{V}(\mathrm{G})$. The *domination number* $\gamma(\mathrm{G})$ of G is the smallest cardinality of a dominating set of G. Some studies on domination in graphs were found in the paper [3-24].

One variant of domination is the doubly connected domination in graphs. A dominating set $S \subseteq V(G)$ is called a doubly connected dominating set of G if both $\langle S \rangle$ and $\langle V(G)\backslash S \rangle$ are connected. The minimum cardinality of a doubly connected dominating set of G, denoted by $\gamma_{cc}(\mathrm{G})$, is called *doubly connected*

domination number of G. A doubly connected dominating set of cardinalities γ_{cc}(G) is called a γ_{cc}-set of G. Doubly connected domination in graphs is found in the papers [25-30].
The inverse domination in a graph was first found in the paper of Kulli [31] and studied in [32-38]. If D is a minimum dominating set in G, then a dominating set $S \subseteq V(G)\backslash D$ is called an inverse dominating set with respect to D. The inverse domination number, denoted by, $\gamma^{-1}(G)$ of G is the order of an inverse dominating set with minimum cardinality.
A dominating set $D \subseteq V(G)$ is called a doubly connected dominating set of G if both $\langle D\rangle$ and $\langle V(G)\backslash D\rangle$ are connected. Let D be a minimum connected dominating set of G. If $\mathrm{S} \subseteq \mathrm{V(G)}\backslash \mathrm{D}$ is a connected dominating set of G, then S is called an inverse doubly connected dominating set of G with respect to D. Furthermore, the inverse doubly connected domination number, denoted by $\gamma_{cc}^{-1}(\mathrm{G})$ is the minimum cardinality of an inverse doubly connected dominating set of G. An inverse doubly connected dominating set of cardinalities $\gamma_{cc}^{-1}(\mathrm{G})$ is called γ_{cc}^{-1}-set. This paper is an extension of [39], hence, the researchers' characterized the inverse doubly connected domination in the lexicographic product of two graphs and give some important results.
For the general terminology in graph theory, readers may refer to [40]. A *graph* G is a pair $\big(V(G), E(G)\big)$, where $V(G)$ is a finite nonempty set called the *vertex-set* of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the *edge-set* of G. The elements of $V(G)$ are called *vertices* and the cardinality $|V(G)|$ of $V(G)$ is the *order* of G. The elements of $E(G)$ are called *edges* and the cardinality $|E(G)|$ of $E(G)$ is the *size* of G. If $|V(G)| = 1$, then G is called a trivial graph. If $E(G) = \emptyset$, then G is called an empty graph. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. The elements of $N_G(v)$ are called *neighbors* of v. The *closed neighborhood* of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{v\in X} N_G(v)$. The *closed neighborhood* of X in G is the set $N_G[X] = \bigcup_{v\in X} N_G[v] = N_G(X) \cup X$. When no confusion arises, $N_G[x]$ [resp. $N_G(x)$] will be denote by $N[x]$ [resp. $N(x)$].

2. Results

Definition 2.1 *A dominating set $D \subseteq V(G)$ is called a doubly connected dominating set of G if both $\langle D\rangle$ and $\langle V(G)\backslash D\rangle$ are connected. Let D be a minimum doubly connected dominating set of G. If $S \subseteq V(G)\backslash D$ is a doubly connected dominating set of G, then S is called an inverse doubly connected dominating set of G with respect to D. Furthermore, the inverse doubly connected domination number, denoted by $\gamma_{cc}^{-1}(G)$ is the minimum cardinality of an inverse doubly connected dominating set of G. An inverse doubly connected dominating set of cardinalities $\gamma_{cc}^{-1}(G)$ is called γ_{cc}^{-1}-set.*

Definition 2.2 *The lexicographic products of two graphs G and H is the graph $G[H]$ with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ satisfying the following condition: $(x, u)(y, v) \in E(G[H])$ if and only if either $xy \in E(G)$ or $x = y$ and $uv \in E(H)$.*

Proposition 2.3 Let $\mathrm{G} = \mathrm{P_m} = [\text{ v\_1,\textbackslash dots, v\_m}]$, $\mathrm{m} \geq 2$ and $\mathrm{H} = \mathrm{P_n} = [\mathrm{u_1}, \dots, \mathrm{v_m}]\mathrm{n} \geq 2$. If S' is a connected dominating set of G and $\emptyset \neq S'' \subset \mathrm{V(H)}$, then $S = S' \times S''$ is a connected dominating set of $G[H]$.

Proof. Suppose that S' is a connected dominating set of G and $\emptyset \neq S'' \subset V(H)$. If $\mathrm{S}'' = \{\mathrm{u}''\}$, then $\mathrm{S} = \mathrm{S}' \times \mathrm{S}'' =$\lbrace $\mathrm{v_2}, \mathrm{v_3}, \dots,$ v_{m − 1}} $\times \{\mathrm{u}''\} = \{(\mathrm{v_2}, \mathrm{u}''), \dots, (\mathrm{v_{m-1}}, \mathrm{u}'')\}$ implies $\langle \mathrm{S}\rangle$ is connected. Let $(\mathrm{v}, \mathrm{u}) \in \mathrm{V(G[H])} \setminus \mathrm{S}$. Then

$$V \in (G[H]) \setminus S = V(G[H]) \setminus \{(v_2, u''), \dots, (v_{m-1}, u'')\}$$

$$= \{(v_1, u''), (v_m, u'')\} \cup \{V(G) \times V(H) \setminus \{u''\}\}$$
$$= \{(v_1, u''), (v_m, u'')\} \cup \{(v_i, u_j) : i = 1, \dots, m;\ j = 1, \dots, n-1, \text{ and } u_j \neq u''\}.$$

There exists $(v', u'') \in S$ such that,

Case 1. If $(v, u) \in \{(v_1, u''), (v_m, u'')\}$, then $(v, u)(v', u'') = (v_1, u'')(v_2, u'') \in E(G[H])$, or $(v, u)(v', u'') = (v_m, u'')(v_{m-1}, u'') \in E(G[H])$.

Case 2. If $(v, u) \in \{V(G) \times (V(H) \setminus \{u''\})$, then $(v, u)(v', u'') = (v_i, u_j)(v_k, u'') \in E(G[H])$ where $v_k \in S'$, $i \in \{k-1, k+1\}$, and $u_j \neq u''$.

In either case, $S' \times S''$ is a connected dominating set of $G[H]$. By using similar arguments, if $\{u''\} \subset S'' \subset V(H)$, then $S = S' \times S''$ is a connected dominating set of $G[H]$. ■

The following result is the characterization of an inverse doubly connected dominating set in lexicographic product of two graphs.

Theorem 2.4 Let $G = P_m = [v_1, \dots, v_m]$, $m \geq 2$, and $H = P_n = [u_1, \dots, u_n]$, $n \geq 2$. Then $S \subseteq V(G[H]) \setminus D$ is an inverse doubly connected dominating set of $G[H]$ with respect to a minimum doubly connected dominating set D of $G[H]$, if S' is a connected dominatig set of G, $\emptyset \neq S'' \subset V(H)$, and $S = S' \times S''$.

Proof. Let $G = P_m = [v_1, \dots, v_m]$, $m \geq 2$ and $H = P_n = [u_1, \dots, u_n], n \geq 2$. Suppose that $D = \{(v_i, u_1): i = 2, 3, \dots, m-1\}$. Then $D = D' \times D''$ where $D' = \{v_2, v_3, \dots, v_{m-1}\} \subset V(G)$ and $D'' = \{u_1\} \subset V(H)$. This implies that D' is a connected dominating set of G and $D'' \subset V(H)$. By Proposition 2.3, D is a connected dominating set of $G[H]$. Let $D \setminus \{(v, u_1)\}$ for any $v \in D'$. If $v \in \{v_2, v_{m-1}\}$, then D is not a dominating set of $G[H]$. If $v \notin \{v_2, v_{m-1}\}$, then $\langle D \rangle$ is not connected. This implies that D is a minimum connected dominating set of $G[H]$.

Further,

$$\begin{aligned} V(G[H]) \setminus D &= V(G[H]) \setminus \{(v_2, u_1), \dots, (v_{m-1}, u_1)\} \\ &= [V(G \setminus D') \times \{u_1\}] \cup [V(G) \times (V(H) \setminus \{u_1\})] \\ &= [\{v_1, v_m\} \times \{u_1\}] \cup [V(G) \times \{u_2, \dots, u_n\}] \\ &= \{(v_1, u_1), (v_m, u_1)\} \cup \{(v_1, u_2), \dots, (v_1, u_n), \dots, (v_m, u_2), \dots, (v_m, u_n)\}. \end{aligned}$$

Clearly, $\langle V(G) \times \{u_2, \dots, u_n\} \rangle$ is connected. Since $(v_1, u_1)(v_1, u_2) \in E(G[H])$ and $(v_m, u_1)(v_m, u_2) \in E(G[H])$, it follows that $V(G[H]) \setminus D$ is connected. Hence, D is a minimum doubly connected dominating set of $G[H]$.

If $S = S' \times S''$, where S' is a connected dominating set of G, and $\emptyset \neq S'' \subset V(H)$, S is a connected dominating set of $G[H]$ by Proposition 2.3. consider the following cases.

Case 1. If $S'' = \{u''\}$, then $S = S' \times S'' = \{v_2, v_3, \dots, v_{m-1}\} \times \{u''\} = \{(v_2, u''), \dots, (v_{m-1}, u'')\}$.

Further,

$$\begin{aligned} V(G[H]) \setminus S &= V(G[H]) \setminus \{(v_2, u''), \dots, (v_{m-1}, u'')\} \\ &= [(V(G) \setminus S') \times \{u''\}] \cup [V(G) \times (V(H) \setminus \{u''\})] \\ &= [\{v_1, v_m\} \times \{u''\}] \cup [V(G) \times \{u_i : i = 1, \dots, n\} \setminus \{u''\}] \\ &= \{(v_1, u''), (v_m, u'')\} \cup \{(v_i, u_j): i = 1, \dots, m;\ j = 1, \dots, n\} \setminus \{(v_i\ u'')\}. \end{aligned}$$

Clearly, $\langle V(G) \times \{u_i, \dots, u_n\} \setminus \{u''\}\rangle$ is connected in $G[H]$. Let $u'' = u_k$ for some positive integer k. Since $(v_1, u_{k-1})(v_m, u''), (v_m, u'')(v_m, u_{k+1}) \in E(G[H])$, it follows that $\langle V(G[H]) \setminus S\rangle$ is connected. Hence, S is a doubly connected dominating set of $G[H]$. If $u_1 \neq u''$, then

$$D \cap S = (D' \times \{u_1\}) \cap (S' \times \{u''\}) = \emptyset.$$

Thus, $S \subseteq V(G[H]) \setminus D$ is an inverse dominating set of G. Accordingly, $S \subseteq V(G[H]) \setminus D$ is an inverse doubly connected dominating set of $G[H]$ with respect to a minimu doubly connected dominating set D of $G[H]$.

Case 2. If $\{u''\} \subset S'' \subset V(H)$, then using the similar arguments in Case 1, $S \subseteq V(G[H]) \setminus D$ is an inverse doubly connected dominating set of $G[H]$ with respect to a minimum doubly connected dominating set D of $G[H]$.■

The following result is an immediate consequence of Theorem 2.4.

Corollary 2.5 Let $G = P_m = [v_1, \dots, v_m]$, $m \geq 2$ and $H = P_n = [u_1, \dots, u_2]$, $n \geq 2$. Then $\gamma_{cc}^{-1}(G[H]) = m - 2$.

Proof. Suppose that S' is a connected dominating set of G, $\emptyset \neq S'' \subset V(H)$, and $S = S' \times S''$. By Theorem 2.4, S is an inverse doubly connected dominating set of $G[H]$ with respect to a minimum doubly connected dominating set D. Thus, $\gamma_{cc}^{-1}(G[H]) \leq |S| = |S' \times S''|$. If $S'' = \{i''\}$, then

$$S = S' \times S'' = \{v_2, v_3, \dots, v_{m-1}\} \times \{u''\} = \{(v_2, u''), \dots, (v_{m-1}, u'')\}, \text{ that is,}$$

$$|S| = |S' \times S''| = |\{v_2, v_3, \dots, v_{m-1}\}| \cdot |\{u''\}| = \big((m-1) - 1\big) \cdot 1 = m - 2.$$

Thus, $\gamma_{cc}^{-1}(G[H]) \leq m - 2$. In the proof of Theorem 2.4, $D = \{(v_i, u_1) : i = 1, 2, 3, \dots, m-1\}$ is a minimum doubly connected dominating set of $G[H]$. Thus,

$$m - 2 = |D| = \gamma_{cc}(G[H]) \leq \gamma_{cc}^{-1}(G[H]) \leq m - 2,$$

implies that $\gamma_{cc}^{-1}(G[H]) = m - 2$.■

3. Conclusion

In this paper, we introduced a new parameter of domination in graphs - the inverse doubly connected domination in graphs. The inverse doubly connected domination in the lexicographic products of two graphs was characterized. The exact inverse doubly connected domination number resulting from the lexicographic product of two graphs was computed. This study will pave the way to new researches such as bounds and other binary operations of two connected graphs. Other parameters involving inverse doubly connected domination in graphs and its bounds may also be explored.

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