International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

Outer-restrained Domination in the Lexicographic Product of Two Graphs

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Abstract
Let G be a connected simple graph. A set $S \subseteq V(G)$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $(G) \setminus S$. A set S of vertices of a graph G is an outer-restrained dominating set if every vertex not in S is adjacent to some vertex in S and $V(G) \setminus S$ is a restrained set. The outer-restrained domination number of G, denoted by $\widetilde{\gamma_r}(G)$ is the minimum cardinality of an outer-restrained dominating set of G. An outer-restrained set of cardinality $\widetilde{\gamma_r}(G)$ will be called a $\widetilde{\gamma_r}(G)$ *−set*. This study is an extension of an existing research on outer-restrained domination in graphs. In this paper, we characterized the outer-restrained domination in graphs under the lexicographic product of two graphs.

Mathematics Subject Classification: 05C69

Keywords: dominating, restrained, outer-connected, outer-restrained, lexicographic

1. Introduction

Let G be a connected simple graph. A set S of vertices of G is termed a dominating set of G if each vertex in $V(G)\setminus S$ is adjacent to at some vertex in S. A minimum dominating set in a graph G is one with the fewest vertices possible. The cardinality of this set is referred to as the *domination number* of G and is symbolized by $\gamma(G)$. The notion of domination in graphs was originally introduced by Claude Berge in 1958 and Oystein Ore in 1962 [1]. Since then, it has garnered significant attention in the literature. Following the article by Ernie Cockayne and Stephen Hedetniemi [2], *domination* in graphs has become a prominent area of study for numerous researchers [3–17]. As researchers continue to delve into this area, we can expect further advancement and insights that will enrich our understanding of graphs.

Another type of domination parameter in a graph is the *restrained domination number*. A *restrained dominating set* is defined as a set $S \subseteq V(G)$ where every vertex in $V(G) \setminus S$ is adjacent to a vertex in S

and to another vertex in $V(G) \setminus S$. The restrained domination number of G, denoted by $\gamma_r(G)$, represents the smallest cardinality of a restrained dominating set of G. This concept was introduced by Telle and Proskurowski [18] indirectly as a vertex partitioning problem. It's noteworthy that the set $S = V(G)$ constitutes a restrained dominating set, and determining is $\gamma_r(G)$, an NP-complete decision problem [19]. Several studies on restrained domination in graphs can be found in papers [20–30].

A graph G is considered connected if there exists at least one path that connects every pair of vertices $x, y \in V(G)$; otherwise, G is termed disconnected. A set S of vertices of a graph G is defined as an *outer-connected dominating set* if every vertex not in S is adjacent to at least one vertex in S, and the subgraph induced by $V(G) \setminus S$ is connected. The *outer-connected domination number*, denoted by $\widetilde{\gamma_c}(G)$ signifies the minimum cardinality of such an outer connected dominating set S of a graph G. Cyman introduced the notion of outer-connected domination in graphs [31]. Various extensions and alternative formulations of outer-connected domination can be observed in papers [32- 41].

A graph G is represented as a pair $(V(G), E(G))$, where $V(G)$ is a nonempty finite set whose elements are referred to as vertices, and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$, termed edges of the graph G. The number of vertices in G is termed the order of G, and the number of edges is termed the size of G. For further exploration of graph-theoretical concepts, readers may refer to additional resources.

Motivated by the concepts of restrained domination and outer-connected domination in graphs, researchers introduced a new graph domination variant: outer-restrained domination. A set S of vertices of a graph G is an outer-restrained dominating set if every vertex not in S is adjacent to some vertex in S and $V(G) \setminus S$ is a restrained set. The outer-restrained domination number of G, denoted by $\widetilde{\gamma_r}(G)$ is the minimum cardinality of an outer-restrained dominating set of G. An outer-restrained set of cardinality $\widetilde{\gamma_r}(G)$ will be called a $\widetilde{\gamma_r}(G)$ *−set*. Following the results presented in [42], the researchers extended the study by investigating other binary graph operation, which is the lexicographic product of two graphs.

2. Results

The graphs G considered here are simple, finite, nontrivial, undirected and without isolated vertices.

Definition 2.1 *A set S of vertices of a graph G is an outer-restrained dominating set if every vertex not in S is adjacent to some vertex in S and $V(G) \setminus S$ is a restrained set. The outer-restrained domination number of G, denoted by $\widetilde{\gamma_r}(G)$ is the minimum cardinality of an outer-restrained dominating set of G. An outer-restrained set of cardinality e $\widetilde{\gamma_r}(G)$ will be called $\widetilde{\gamma_r}(G)$ -set.*

Definition 2.2 *The lexicographic product of two graphs G and H, is the graph $G[H]$ with vertex-set $V(G[H] = V(G) \times V(H)$ and edge set $E(G[H])$ satisfying the following conditions: $(x,u)(y,v) \in E(G[H])$ if and only if either $x, y \in E(H)$ or $x = y$ and $uv \in E(H)$.*

Remark 2.3 *Let G be a nontrivial connected graph and $H = K_2$. Then $V(G)$ x $\{u\}$ is a dominating set of $G[H])$ for all $u \in V(H)$.*

International Journal for Multidisciplinary Research (IJFMR)

E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

The following results are needed for the characterization of an outer-restrained dominating set in the lexicographic product of two graphs.

Lemma 2.4 *Let G be a nontrivial connected graph and $H = K_2$. If $S = S' \times V(H)$, where $S' \subseteq V(G)$ is a dominating of G, then a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $(G[H])$.*
Proof: Let G be a nontrivial connected graph and $H = K_2 = [u_1, u_2]$. Suppose that $S = S' \times V(H)$, where $S' \subseteq V(G)$ is a dominating of G. If $S' = V(G)$, then $S = V(G) \times V(H) = V(G[H])$ is an outer-restrained dominating set (immediate). If $S \neq V(G)$, then $S \subset V(G)$, that is, $V(G) \setminus S' = \phi$. Let $v \in V(G)\setminus S'$. Then there exists $v' \in S'$ such that $vv' \in E(G)$, that is, for every $(v,u) \in V(G[H]) \setminus S$, there exists $(v',u) \in S$ such that $(v,u)(v',u) \in E(G[H])$. Thus, S is a dominating set of $G[H]$. Let $(v',u_1) \in S = S' \times V(H)$. Then there exists $(v,u_1) \in V(G[H]) \setminus S$ such that $(v',u_1)(v,u_1) \in E(G[H])$ and another $(v',u_2) \in S$ such that $(v',u_1)(v',u_2) \in E(G[H])$, that is, $V(G[H]) \setminus S$ is a restrained set. Hence, a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $G[H]$. ■

Lemma 2.5 *Let G be a nontrivial connected graph and $H = K_2$. If $S = S' \times \{u\}$, where $u \in V(H)$ and S' is a dominating set of G and $V(G) \setminus S'$ is a restrained set, then a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $G[H]$.*

Proof: Let G be a nontrivial connected graph and $H = K_2 = [u_1, u_2]$. Suppose that $S = S' \times \{u\}$, where $u \in V(H)$ and S' is a dominating set of G and $V(G)\setminus S'$ is a restrained set. If $S' = V(G)$, then $S = V(G) \times \{u_1\}$ is clearly a dominating set of $G[H]$. For every $(v,u_2) \in V(G[H])\setminus S$, there exists $(v,u_1) \in S$ such that $(v,u_2)(v,u_1) \in E(G[H])$ and there exists another $(v',u_1) \in S$ such that $(v,u_1)(v',u_1) \in E(G[H])$, that is, $V(G[H]) \setminus S$ is a restrained set. Hence, a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $G[H]$. If $S' = V(G)$, then $S' \subset V(G)$. Let $v \in V(G)\setminus S'$. Since S' is a dominating set, there exists $v' \in S'$ such that $vv' \in E(G)$, and since $V(G) \setminus S'$ is a restrained set, there is another $v'' \in S'$ such that $v'v'' \in E(G)$. Thus, for every $(v,u) \in V(G[H]) \setminus S$, there exists $(v',u) \in S$ such that $(v,u)(v',u) \in E(G[H])$, that is, S is a dominating set of $G[H]$, and there exists another $(v'',u) \in S$ such that $(v',u)(v'',u) \in E(G[H])$, that is, $V(G[H])\setminus S$ is a restrained set. Hence, a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $G[H]$. ■

Lemma 2.6 *Let G be a nontrivial connected graph and $H = K_2$. If $S = (V(G) \times \{u_1\}) \cup (V(G)\setminus S' \times \{u_2\})$, where $\phi \subset S' \subseteq V(G)$, then a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $G[H]$.*

Proof: Let G be a nontrivial connected graph and $H = K_2 = [u_1, u_2]$. Suppose that $S = (V(G) \times \{u_1\}) \cup (V(G) \setminus S' \times \{u_2\})$, where $\phi \subset S' \subseteq V(G)$. If $S' = V(G)$, then $S = (V(G) \times \{u_1\}) \cup (V(G)\setminus V(G) \times \{u_2\}) = (V(G) \times \{u_1\})$, that is, $S = V(G) \times \{u1\}$. By Lemma 2.5, a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $G[H]$. If $S' \neq V(G)$, then $S' \subset V(G)$, that is, $V(G) \setminus S' \neq \phi$. Since $V(G) \times \{u_1\} \subset V(G[H])$, S is a dominating set of $(G[H])$, by Remark 2.3. Let $v \in V(G)\setminus S'$, that is, $(v,u) \in S$.

Case 1. If $u = u_1$, then $(v, u_1) \in S$ and there exists $(v, u_2) \in V(G[H]) \setminus S$ such that $(v, u_1)(v, u_2) \in E(G[H])$ and another $(v', u_1) \in S$ such that $(v, u_1)(v', u_1) \in E(G[H])$.

Case 2. If $u = u_2$, then $(v, u_2) \in S$ and there exists $(v', u_2) \in V(G[H]) \setminus S$ such that $(v, u_2)(v', u_2) \in E(G[H])$ and another $(v, u_1) \in S$ such that $(v, u_2)(v, u_1) \in E(G[H])$.

In any case, $V(G[H]) \setminus S$ is a restrained set. Since S is a dominating set of $G[H]$, it follows that a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $G[H]$. ■

Lemma 2.7 *Let G be a nontrivial connected graph and $H = K_2$. If $S = (V(G) \setminus S' \times \{u_1\}) \cup (V(G) \times \{u_2\})$, where $\emptyset \subset S' \subseteq V(G)$, then a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $G[H]$.*

Proof: Let G be a nontrivial connected graph and $H = K_2 = [u_1, u_2]$. Suppose that $S = (V(G) \setminus S' \times \{u_1\}) \cup (V(G) \times \{u_2\})$, where $\phi \subset S' \subseteq V(G)$. If $S' = V(G)$, then $S = (V(G) \setminus V(G) \times \{u_1\}) \cup (V(G) \times \{u_2\})$, that is, $S = V(G) \times \{u_2\}$. By Lemma 2.5, a nonempty set $S \subseteq V(G[H])$ is an outer restrained dominating set of $G[H]$. If $S' = V(G)$, then $S' \subset V(G)$, that is, $V(G) \setminus S' =$. Since $V(G) \times \{u_2\} \subset V(G[H]), S$ is a dominating set of $(G[H])$, by Remark 2.3. Let $v \in V(G) \setminus S'$, that is, $(v, u) \in S$.

Case 1. If $u = u_1$, then $(v, u_1) \in S$ and there exists $(v', u_1) \in V(G[H]) \setminus S$ such that $(v, u_1)(v', u_1) \in E(G[H])$ and another $(v, u_2) \in S$ such that $(v, u_1)(v, u_2) \in E(G[H])$.

Case 2. If $u = u_2$, then $(v, u_2) \in S$ and there exists $(v, u_1) \in V(G[H]) \setminus S$ such that $(v, u_2)(v, u_1) \in E(G[H])$and another $(v', u_2) \in S$ such that $(v, u_2)(v', u_2) \in E(G[H])$.

In any case,$V(G[H]) \setminus S$ *is* a restrained set. Since S is a dominating set of $G[H]$, it follows that a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $G[H]$. ■

The following result is the characterization of an outer-restrained dominating set of the lexicographic products of two graphs.

Theorem 2.8 *Let G be a nontrivial connected graph and $H = K_2$. Then a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $G[H]$ if and only if one of the following is satisfied.*
(i) $S = S' \times V(H)$, where $S' \subseteq V(G)$ is a dominating of G.
(ii) $S = S' \times \{u\}$, where $u \in V(H)$ and S' is a dominating set of G and $V(G) \setminus S'$ is a restrained set.
(iii) $S = (V(G) \times \{u_1\}) \cup (V(G) \setminus S' \times \{u_2\})$, where $\phi \subset S' \subseteq V(G)$.
(iv) $S = (V(G) \setminus S' \times \{u_1\}) \cup (V(G) \times \{u_2\})$, where $\phi \subset S' \subseteq V(G)$.

Proof: Suppose that a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $G[H]$. Then every vertex not in S is adjacent to some vertex in S and $V(G[H]) \setminus S$ is a restrained set.

Case1. Consider that $S = S' \times S''$ where $S' \subseteq V(G)$ and $S'' \subseteq V(H)$. *Subcase 1*. Let $S'' = V(H)$.If $S' = V(G)$, then $S = S' \times V(H)$, where $S' = V(G)$ is a dominating of G. This satisfy statement (i). If $S' \subset V(G)$ and suppose that S' in not a dominating set of G, then there exists $x \in V(G)$ such that $xx' \notin E(G)$ for all $x' \in S'$, that is, there exists $(x,u) \in V(G[H]) \setminus S$ such that$(x,u)(x',u) \notin E(G[H])$ for all $(x',u) \in S$. This contradict to our assumption that S is a dominating set of $G[H]$. Thus, $S = S' \times V(H)$, where $S' \subset V(G)$ and S' is a dominating set of G. This satisfy statement (i).

Subcase 2. Let $S'' \subset V(H)$. If $S' \neq V(G)$, then $S' \subset V(G)$ and $S = S' \times \{u\}$, where $u \in V(H)$. If S' in not a dominating set of G, then there exists $x \in V(G) \setminus S'$ such that $xx' \notin E(G)$ for all $x' \in S'$, that is, there exists $(x,u) \in V(G[H]) \setminus S$ such that $(x,u)(x',u) \notin E(G[H])$ for all $(x',u) \in S$. This contradict to our assumption that S is a dominating set of $G[H]$. Thus, $S = S' \times \{u\}$, where $u \in V(H)$ and S' is a dominating set of G. Note that $V(G[H]) \setminus S$ is a retrained set if for every $(y,u) \in V(G[H]) \setminus S$, there exists $(x,u) \in S$ such that $(y,u)(x,u) \in E(G[H])$ and another $(x',u) \in S$ such that $(x,u)(x',u) \in E(G[H])$. Suppose that $V(G) \setminus S'$ is not a restrained set. Then there exists $x \in S'$ such that $xx' \notin E(G)$ for all $x' \in S'$, that is there exists $(x,u) \in S = S' \times \{u\}$, such that $(x,u)(x',u) \notin E(G[H])$ for all $(x',u) \in S \setminus \{(x,u)\}$. This implies that $V(G[H]) \setminus S$ is not a restrained set contrary to our assumption. Thus, $S = S' \times \{u\}$, where $u \in V(H)$ and S' is a dominating set of G and $V(G) \setminus S'$ is a restrained set. This satisfy statement (ii).

If $S' = V(G)$, then $S = S' \times S''$. Let $S = V(G) \times \{u\}$ where $u \in S'' \subset V(H) = V(K_2) = \{u_1, u_2\}$. Thus, $S = V(G) \times \{u\} = (V(G) \times \{u_1\}) \cup (V(G) \setminus S' \times \{u_2\})$, where $\emptyset \subset S' = V(G)$. This satisfy statement (iii). Or $S = V(G) \times \{u\} = (V(G) \setminus S' \times \{u_1\}) \cup (V(G) \times \{u_2\})$, where $\emptyset \subset S' = V(G)$. This satisfy statement (iv).

Case 2. Consider that $S = (V(G) \times S_1'') \cup (V(G) \setminus S' \times S_2'')$ *or* $S = (V(G) \setminus S' \times S_1'') \cup (V(G) \times S_2'')$ where $S' \subset V(G)$ and $S_1'', S_2'' \subset V(H) = \{u_1, u_2\}$. Then $S = (V(G) \times \{u_1\}) \cup (V(G) \setminus S' \times \{u_2\})$ or $S = (V(G) \setminus S' \times \{u_1\}) \cup (V(G) \times u_2)$ where $\emptyset \subset S' \subset V(G)$. This satisfy statement (iii) and statement (iv) respectively.

For the converse, suppose that statement (i) is satisfied. Then $S = S' \times V(H)$, where $S' \subseteq V(G)$ is a dominating of G. In view of Lemma 2.4, a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $G[H]$.

Suppose that statement (ii) is satisfied. In view of Lemma 2.5, a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $G[H]$.

Suppose that statement (iii) is satisfied. In view of Lemma 2.6, a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $G[H]$.

Suppose that statement (iv) is satisfied. In view of Lemma 2.7, a nonempty set $S \subseteq V(G[H])$ is an outer-restrained dominating set of $G[H]$. ■

The following result is an immediate consequence of Theorem 2.8.

Corollary 2.9 *Let* $G = Pn$ *of order* $n \geq 2$ *and* $H = K_2$. *Then*

$$\widetilde{\gamma_r}(G[H]) = \begin{cases} \frac{n}{2}, if\ n \equiv 0(mod\ 4) \\ \frac{n+1}{2}, if\ n \equiv 1(mod\ 4)\ or\ n \equiv 3(mod\ 4) \\ \frac{n+2}{2}, if\ n \equiv 2\ (mod\ 4). \end{cases}$$

Proof: Let $G = Pn = [v_1, v_2, \dots, v_n]$ where $n \geq 2$ and $H = K_2 = [u_1, u_2]$. Suppose that $S = S' \times \{u\}$, where $u \in V(H)$ and S' is a dominating set of G and $V(G)\backslash S'$ is a restrained set. Then S is an outer-restrained dominating set of $G[H]$ by Theorem 2.8. Thus, $\widetilde{\gamma_r}(G[H])\ |S| = |S' \times \{u\}| = |S'|$.

Case 1. If $n \equiv 0(mod4)$, then $S' = \left\{v_{4i-2}, v_{4i-1} : i = 1, 2, \dots, \frac{n}{4}\right\}$. Thus, $|S'| = 2 \cdot \left(\frac{n}{4}\right) = \frac{n}{2}$, that is, $\widetilde{\gamma_r}(G[H]) \leq |S'| = \frac{n}{2}$. Suppose that $S'' = S''\backslash\{v\}$ is a minimum outer-restrained dominating set of G for some $v \in S'$. Then $S'' = \{v_2, v_3, v_6, v_{7'} \dots, v_{n-2}, v_{n-1}\} \backslash\{v\}$ for some $v \in S'$. However, for any $v \in S'$, the S'' is not a dominating set of G. This implies that S' must be a minimum outer-restrained dominating set of G. Therefore, $\frac{n}{2} = |S'| = |S'| \cdot |\{u\}| = \widetilde{\gamma_r}(G) \cdot 1 \leq \widetilde{\gamma_r}(G[H]) \leq |S'| = \frac{n}{2}$,
that is, $\widetilde{\gamma_r}(G[H]) = \frac{n}{2}$.

Case 2. If $n \equiv 1(mod4)$, then $S' = \left\{v_2, v_{4i-1}, v_{4i} : i = 1, 2, \dots, \frac{n-1}{4}\right\}$. Thus, $|S'| = 1 + 2 \cdot \left(\frac{n-1}{4}\right) = \frac{n+1}{2}$, that is, $\widetilde{\gamma_r}(G[H]) \leq |S'| = \frac{n+1}{2}$. Suppose that $S'' = S'\backslash\{v\}$ is a minimum outer-restrained dominating set of G for some $v \in S'$. Then $S'' = \{v_2, v_3, v_4, v_7 v_{8'} \dots, v_{n-2}, v_{n-1}\} \backslash\{v\}$ for some $v \in S'$. However, for any $v \in S'$ the S'' is not a dominating set of G. This implies that S' must be a minimum outer-restrained dominating set of G. Therefore, $\frac{n+1}{2} = |S'| = |S'| \cdot |\{u\}| =, \widetilde{\gamma_r}(G) \cdot 1 \leq \widetilde{\gamma_r}(G[H]) \leq |S'| = \frac{n+1}{2}$, that is, $\widetilde{\gamma_r}(G[H]) = \frac{n+1}{2}$.

Case 3. If $n \equiv 3(mod4)$, then $S' = \left\{v_{4i-2}, v_{4i-1} : i = 1, 2, \dots, \frac{n+1}{4}\right\}$. Thus, $|S'| = 2 \cdot \left(\frac{n+1}{4}\right) = \frac{n+1}{2}$, that is, $\widetilde{\gamma_r}(G[H]) \leq |S'| = \frac{n+1}{2}$. Suppose that $S'' = S'\backslash\{v\}$ is a minimum outer-restrained dominating set of G for some $v \in S'$. Then $S'' = \{v_2, v_3, v_6, v_{7'} \dots, v_{n-1}, v_n\} \backslash\{v\}$ for some $v \in S'$. However, for any $v \in S'$, the S'' is not a dominating set of G. This implies that S' must be a minimum outer-restrained dominating set of G. Therefore, $\frac{n+1}{2} = |S'| = |S'| \cdot |\{u\}| =, \widetilde{\gamma_r}(G) \cdot 1 \leq \widetilde{\gamma_r}(G[H]) \leq |S'| = \frac{n+1}{2}$,
that is, $\widetilde{\gamma_r}(G[H]) = \frac{n+1}{2}$.

Case 4. If $n \equiv 2(mod4)$, then $S' = \left\{v_{4i-3}, v_{4i-2} : i = 1, 2, \dots, \frac{n+2}{4}\right\}$. Thus, $|S'| = 2 \cdot \left(\frac{n+2}{4}\right) = \frac{n+2}{2}$, that is, $\widetilde{\gamma_r}(G[H]) \leq |S'| = \frac{n+2}{2}$. Suppose that $S'' = S'\backslash\{v\}$ is a minimum outer-restrained dominating set of G for some $v \in S'$. Then $S'' = \{v_1, v_2, v_5, v_{6'} \dots, v_{n-1}, v_n\} \backslash\{v\}$ for some $v \in S'$. However, for any $v \in S'$, the S'' is not a dominating set of G. This implies that S' must be a minimum outer-restrained dominating set of G. Therefore, $\frac{n+2}{2} = |S'| = |S'| \cdot |\{u\}| =, \widetilde{\gamma_r}(G) \cdot 1 \leq \widetilde{\gamma_r}(G[H]) \leq |S'| = \frac{n+2}{2}$, that is, $\widetilde{\gamma_r}(G[H]) = \frac{n+2}{2}$.

3. Conclusion

In this paper, we extended the study on outer restrained domination in graphs by dealing with another binary graph operation – the lexicographic product of two graphs. The outer-restrained domination in the lexicographic products of two graphs were characterized and the exact value of outer-restrained domination number resulting from the lexicographic of two graphs were computed. This study will pave the way to new researches such as bounds and other binary operations of two connected graphs. Other parameters involving outer-restrained domination in graphs may also be explored. Finally, the characterization of an outer-restrained domination in graphs of the other binary operations of two graphs, and its bounds are promising extension of this study.

4. Acknowledgement

The researchers would like to thank the Department of Science and Technology – Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP), under its accredited university, University of San Carlos in Cebu City, Philippines, for funding and supporting this research.

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