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Fair Doubly Connected Domination in the Join of two Graphs

Jan Niño C. Serrano[1], Enrico L. Enriquez[2], Grace M. Estrada[3], Marie Cris A. Bulay-og[4], Edward M. Kiunisala[5]

[1]MS Math, Department of Computer, Information Sciences, and Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines
[2]Full Professor, Department of Computer, Information Sciences, and Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines
[3]Associate Professor, Department of Computer, Information Sciences, and Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines
[4]Assistant Professor, Mathematics and Statistics Programs, University of the Philippines Cebu, 6000 Cebu City, Philippines
[5]Professor, Mathematics Department, College of Computing, Artificial Intelligence and Sciences, Cebu Normal University, 6000 Cebu City, Philippines

Abstract
Let G be a nontrivial connected graph. A dominating set $S \subseteq V(G)$ is called a doubly connected dominating set of G if both $\langle S \rangle$ and $\langle V(G) \setminus S \rangle$ are connected. If every distinct vertices u and v from $V(G) \setminus S$, $|N_G(u) \cap S| = |N_G(v) \cap S|$, then S is called a fair doubly connected dominating set of G. Furthermore, the fair doubly connected domination number, denoted by $\gamma_{fcc}(G)$, is the minimum cardinality of a fair doubly connected dominating set of G. A fair doubly connected dominating set of cardinalities $\gamma_{fcc}(G)$ is called $\gamma_{fcc} - set$. In this paper, we characterized the fair doubly connected dominating set in the join of two graphs.

Keywords: dominating set, doubly connected dominating set, fair dominating set, fair doubly connected dominating set

1. Introduction

The concept of domination in graphs has begun in the book of C. Berge in 1958 [1] and O. Ore in 1962 [2] formally defined the term *dominating set* and *domination number*. Following an article of E. Cockayne and S. Hedetniemi [3] in 1977, where they used the notation γ to represent the domination number of a graph G and their paper became an area of study by many researchers. Let G be a graph, a subset S of $V(G)$ is a dominating set of G if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The domination number of G, denoted by $\gamma(G)$ is the smallest cardinality of a dominating set of G. Some studies in domination in graphs were found in the papers [4-24].
Another parameter of domination is the doubly connected domination in graphs introduced by J. Cyman, M. Lemanska and J. Raczek [25] in 2006. A dominating set $S \subseteq V(G)$ is a doubly connected dominating set of G if both $\langle S \rangle$ and $\langle V(G) \setminus S \rangle$ are connected. The smallest cardinality of a doubly connected

dominating set of G, denoted by $\gamma_{cc}(G)$, is called the doubly connected domination number of G. A doubly connected dominating set of cardinalities $\gamma_{cc}(G)$ is called $\gamma_{cc} - set$ of G. The related study of the Doubly Connected Domination in Graphs can be found in [26-32]

In 2012, a new constraint of domination was introduced and this is the concept of fair domination in graphs initiated by Y. Caro, A. Hansberg, and M. Henning [33]. A dominating subset S of $V(G)$ is a fair dominating set in G if all vertices in $V(G)\backslash S$ are dominated by the same number of vertices from S, that is, for every two distinct vertices u and v from $V(G)\backslash S$ such that $|N_G(u) \cap S| = |N_G(v) \cap S|$. A subset S of $V(G)$ is a k −fair dominating set in G if for every vertex $v \in V(G)\backslash S, |N(v \cap\ S| = k$. The minimum cardinality of a fair dominating set of G, denoted by $\gamma_{fd}(G)$, is called the fair domination number of G. A fair dominating set of cardinalities $\gamma_{fd}(G)$ is called $\gamma_{fd} - set$. For additional insights on fair domination in graphs, refer to [34-40].

In this paper, we extend the paper [41] entitled Fair Doubly Connected in the Corona and Cartesian Product of two Graphs published in 2023. A doubly connected dominating set $S \subseteq V(G)$ is called a fair doubly connected dominating set of G if every distinct vertex u and v from $V(G)\backslash S$, then $|N_G(u) \cap S| = |N_G(v) \cap S| > 0$. The minimum cardinality of a fair doubly connected dominating set of G, denoted by $\gamma_{fcc}(G)$, is called the fair doubly connected domination number of G. A fair doubly connected dominating set of cardinalities $\gamma_{fcc}(G)$ is called $\gamma_{fcc} - set$. In this paper, we characterized the fair doubly connected domination in the join of two graphs and give some important results.

Readers may refer to [42] for the general terminology in graph theory. A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the *vertex-set* of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the *edge-set* of G. The elements of $V(G)$ are called *vertices* and the cardinality $|V(G)|$ of $V(G)$ is the *order* of G. The elements of $E(G)$ are called *edges* and the cardinality $|E(G)|$ of $E(G)$ is the *size* of G. If $|V(G)| = 1$, then G is called *trivial graph*, otherwise it is called *nontrivial graph*. If $E(G) = \emptyset$, then G is called an *empty graph*, denoted as $\overline{K_n}$ of order n. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. The elements of $N_G(v)$ are called *neighbors* of v. The *closed neighborhood* of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. For no confusion, $N_G[x]$ and $N_G(x)$ will be denoted by $N[x]$ and $N(x)$, respectively. The *join* $G + H$ of two graphs G and H is the graph with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv: u \in V(G), v \in V(H)\}$.

2. Results

The following results are needed for the characterization of the fair doubly connected dominating set in the join of two graphs.

Lemma 2.1 *Let G and H be nontrivial graphs. If $\langle S\rangle$ is connected and*

a) $S = V(G)$ and H is connected, or

b) $S = V(H)$ and G is connected,

then a nonempty $S \subseteq V(G + H)$ is a fair doubly connected dominating set of $G + H$.

Proof: Suppose that $\langle S\rangle$ is connected and statement a) is satisfied. Then, $S = V(G)$ and H is connected. This implies that $\langle V(G + H)\backslash S\rangle = \langle V(H)\rangle$ is connected. Thus S is a doubly connected dominating set of $G + H$. For every $u, v \in V(H)$,

$$\begin{aligned}|N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap V(G)|\\ &= |V(G)|\end{aligned}$$

$$= |N_{G+H}(v) \cap V(G)| \ = |N_{G+H}(v) \cap S|,$$

that is, S is a fair dominating set of $G + H$. Accordingly, S is a fair doubly connected dominating set of $G + H$. Next, statement $b)$ is satisfied, then $S = V(H)$ and G is connected. By similar arguments, S is a fair doubly connected dominating set of $G + H$. ■

Lemma 2.2 *Let G and H be nontrivial graphs. If $\langle S\rangle$ is connected, S is an $|S|$-fair dominating set, and*

a) $S \cap V(H) = \emptyset$, G is connected, and $S \neq V(G)$, or

b) $S \cap V(G) = \emptyset$, H is connected, and $S \neq V(H)$,

then a nonempty $S \subseteq V(G + H)$ is a fair doubly connected dominating set of $G + H$.

Proof: Suppose that $\langle S\rangle$ is connected, S is an $|S|$-fair dominating set. If statement $a)$ is satisfied, then, $S \cap V(H) = \emptyset$, G is connected, and $S \neq V(G)$. Thus, $\langle S\rangle$ is connected in G, that is, $\langle S\rangle$ is also connected in $G + H$. Further, $S \subset V(G)$ implies $V(G)\backslash S \neq \emptyset$. Clearly, $\langle V(G + H)\backslash S\rangle$ is connected, that is, S is a doubly connected dominating set of $G + H$ by definition. Further, S is an $|S|$-fair dominating set of G, that is, $|N_G(u') \cap S| = |N_G(v') \cap S| = |S|$ for every $u', v' \in V(G)\backslash S$. Let $u'', v'' \in V(H) \subset V(G + H)$. Clearly, $|N_{G+H}(u'') \cap S| = |N_{G+H}(v'') \cap S| = |S|$. Thus, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$ for all $u, v \in V(G + H)\backslash S$, that is, S is a fair dominating set of $G + H$. Accordingly, S is a fair doubly connected dominating set of $G + H$. Next, consider that statement $b)$ is satisfied. Then $S \cap V(G) = \emptyset$, H is connected, and $S \neq V(H)$. This implies that $\langle S\rangle$ is connected in H, that is $\langle S\rangle$ is also connected in $G + H$. Further, $S \subset V(H)$ implies $V(H)\backslash S \neq \emptyset$. Thus, $\langle V(G + H)\backslash S\rangle$ is clearly connected, that is, S is a doubly connected dominating set of $G + H$ by definition. Further, S is an $|S|$-fair dominating set of H, that is, $|N_H(u') \cap S| = |N_H(v') \cap S| = |S|$ for every $u', v' \in V(H)\backslash S$. Let $u'', v'' \in V(G) \subset V(G + H)$. Clearly, $|N_{G+H}(u'') \cap S| = |N_{G+H}(v'') \cap S| = |S|$. Thus, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$ for all $u, v \in V(G + H)\backslash S$, that is, S is a fair dominating set of $G + H$. Accordingly, S is a fair doubly connected dominating set of $G + H$. ■

Lemma 2.3 *Let G and H be nontrivial graphs. If $S_H = S \cap V(H) \neq \emptyset$ and $S_G = S \cap V(G) \neq \emptyset$, and*

a) S_H is an $|S_H|$-fair dominating set of H and S_G is an $|S_G|$-fair dominating set of G, or

b) S_H is a k-fair dominating set of H and S_G is an k-fair dominating set of G with $|S_H| = |S_G|$, or

c) S_H is a $(k + m)$-fair dominating set of H and S_G is an k-fair dominating set of G with $|S_H| - |S_G| = m$.

d) S_H is a k-fair dominating set of H and S_G is an $(k + m)$-fair dominating set of G with $|S_G| - |S_H| = m$,

then a nonempty $S \subseteq V(G + H)$ is a fair doubly connected dominating set of $G + H$.

Proof: Suppose that $S_H = S \cap V(H) \neq \emptyset$ and $S_G = S \cap V(G) \neq \emptyset$. That is,

$$\begin{aligned} S_G \cup S_H &= \big(S \cap V(G)\big) \cup (S \cap V(H)) \\ &= S \cap (V(G) \cup V(H)) \\ &= S \cap V(G + H) \\ &= S \end{aligned}$$

Since $S = S_G \cup S_H$, it follows that $\langle S\rangle$ is connected in $G + H$ and $\langle V(G + H)\backslash S\rangle$ is connected in $G + H$. Thus, S is a doubly connected dominating set of $G + H$.

If statement $a)$ is satisfied, then S_H is an $|S_H|$-fair dominating set of H and S_G is an $|S_G|$-fair dominating set of G. This implies that every $u, v \in V(G + H)\backslash S$ is dominated by S_G and S_H, that is, $|N_{G+H}(u) \cap (S_G \cup S_H)| = |S_G \cup S_H|$ and $|N_{G+H}(v) \cap (S_G \cup S_H) = |S_G \cup S_H|$. Thus,

$$\begin{aligned} |N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap (S_G \cup S_H)| \\ &= |S_G \cup S_H| \end{aligned}$$

$$= |N_{G+H}(v) \cap (S_G \cup S_H)| = |N_{G+H}(v) \cap S|,$$

that is, S is a fair dominating set of $G + H$. Accordingly, S is a fair doubly connected dominating set of $G + H$.

If statement $b)$ is satisfied, then S_H is a k-fair dominating set of H and S_G is an k-fair dominating set of G with $|S_H| = |S_G|$. Let $u, v \in V(G + H) \backslash S$.

Case 1. If $u \in V(G) \backslash S_G$ and $v \in V(H) \backslash S_H$. Then

$$|N_{G+H}(u) \cap (S_G \cup S_H)| = |N_G(u) \cap S_G| + |S_H| = k + |S_H|.$$

and

$$|N_{G+H}(v) \cap (S_G \cup S_H)| = |S_G| + |N_H(v) \cap S_H| = |S_G| + k.$$

That is,

$$\begin{aligned} |N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap (S_G \cup S_H)| \\ &= k + |S_H| \\ &= |S_G| + k \text{ , since } |S_H| = |S_G| \\ &= |N_{G+H}(v) \cap (S_G \cup S_H)| = |N_{G+H}(v) \cap S| \end{aligned}$$

Case 2. If $u, v \in V(G) \backslash S_G$. Then

$$|N_{G+H}(u) \cap (S_G \cup S_H)| = |N_G(u) \cap S_G| + |S_H| = k + |S_H|.$$

and

$$|N_{G+H}(v) \cap (S_G \cup S_H)| = |N_G(v) \cap S_G| + |S_H| = k + |S_H|$$

That is,

$$\begin{aligned} |N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap (S_G \cup S_H)| \\ &= k + |S_H| \\ &= |N_{G+H}(v) \cap (S_G \cup S_H)| = |N_{G+H}(v) \cap S|. \end{aligned}$$

Case 3. If $u, v \in V(H) \backslash S_H$. Then

$$|N_{G+H}(u) \cap (S_G \cup S_H)| = |S_G| + |N_G(u) \cap S_H| = |S_G| + k.$$

and

$$|N_{G+H}(v) \cap (S_G \cup S_H)| = |S_G| + |N_G(v) \cap S_H| = |S_G| + k$$

That is,

$$\begin{aligned} |N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap (S_G \cup S_H)| \\ &= |S_G| + k \\ &= |N_{G+H}(v) \cap (S_G \cup S_H)| = |N_{G+H}(v) \cap S|. \end{aligned}$$

In any case, S is a fair dominating set of $G + H$. Accordingly, S is a fair doubly connected dominating set of $G + H$.

If statement $c)$ is satisfied, then S_H is a $(k + m)$-fair dominating set of H and S_G is an k-fair dominating set of G with $|S_H| - |S_G| = m$. Let $u, v \in V(G + H) \backslash S$.

Case 1. If $u \in V(G) \backslash S_G$ and $v \in V(H) \backslash S_H$. Then

$$|N_{G+H}(u) \cap (S_G \cup S_H)| = |N_G(u) \cap S_G| + |S_H| = k + |S_H|$$

and

$$|N_{G+H}(v) \cap (S_G \cup S_H)| = |S_G| + |N_H(v) \cap S_H| = |S_G| + (k + m).$$

That is,

$$\begin{aligned} |N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap (S_G \cup S_H)| \\ &= k + |S_H| \\ &= k + (m + |S_G|) \text{ since } |S_H| = m + |S_G| \end{aligned}$$

$$= (k+m) + |S_G|$$
$$= |N_{G+H}(v) \cap (S_G \cup S_H) = |N_{G+H}(v) \cap S|.$$

Case 2. If $u, v \in V(G)\backslash S_G$. Then

$$|N_{G+H}(u) \cap (S_G \cup S_H)| = |N_G(u) \cap S_G| + |S_H| = k + |S_H|.$$

and

$$|N_{G+H}(v) \cap (S_G \cup S_H)| = |N_G(v) \cap S_G| + |S_H| = k + |S_H|$$

That is,

$$|N_{G+H}(u) \cap S| = |N_{G+H}(u) \cap (S_G \cup S_H)|$$
$$= k + |S_H|$$
$$= |N_{G+H}(v) \cap (S_G \cup S_H)| = |N_{G+H}(v) \cap S|$$

Case 3. If $u, v \in V(H)\backslash S_H$. Then

$$|N_{G+H}(u) \cap (S_G \cup S_H)| = |S_G| + |N_G(u) \cap S_H| = |S_G| + (k+m).$$

and

$$|N_{G+H}(v) \cap (S_G \cup S_H)| = |S_G| + |N_G(v) \cap S_H| = |S_G| + (k+m)$$

That is,

$$|N_{G+H}(u) \cap S| = |N_{G+H}(u) \cap (S_G \cup S_H)|$$
$$= |S_G| + (k+m)$$
$$= |N_{G+H}(v) \cap (S_G \cup S_H)| = |N_{G+H}(v) \cap S|.$$

In any case, S is a fair dominating set of $G+H$. Accordingly, S is a fair doubly connected dominating set of $G+H$.

If statement $d)$ is satisfied, then S_H is a k-fair dominating set of H and S_G is an $k+m$-fair dominating set of G with $|S_G| - |S_H| = m$. By similar arguments in proving statement $c)$, S is a fair doubly connected dominating set of $G+H$. ■

Lemma 2.4 *Let G and H be nontrivial graphs. If $G = G_1 \cup \overline{K}_p$, $H = H_1 \cup \overline{K'}_p$, where G_1 is connected and H_1 is connected, for all integer $p \geq 1$, and $S = V(\overline{K}_m) \cup (\overline{K'}_m)$ for all $m \in \{1,2,\dots,p\}$, then a nonempty $S \subseteq V(G+H)$ is a fair doubly connected dominating set of $G+H$.*

Proof: Suppose that $G = G_1 \cup \overline{K}_p$, $H = H_1 \cup \overline{K'}_p$ for all integer $p \geq 1$ where G_1 is connected and H_1 is connected, and $S = V(\overline{K}_m) \cup V(\overline{K'}_m)$ with $m \in \{1,2,\dots,p\}$. *Let* $V(\overline{K}_m) = \{v_1, v_2, \dots, v_m\}$ and $V(\overline{K'}_m) = \{u_1, u_2, \dots, u_m\}$ for all $m \in \{1,2,\dots,p\}$. By definition of the join of two graphs G and H, v_i is adjacent to $u_1, u_2, \dots, u_m$ for all $i \in \{1,2,\dots,m\}$. This implies that $\langle S \rangle$ is connected. Similarly, $vu \in E(G+H)$ for each $v \in V(G)\backslash V(\overline{K}_m)$ and for each $u \in V(H)\backslash V(\overline{K'}_m)$. This implies that $\langle V(G+H)\backslash S \rangle$ is connected. Thus, S is a doubly connected dominating set of $G+H$. Now, let $x, y \in V(G+H)\backslash S$. Consider the following.

Case 1. If $x, y \in V(G)$, then $N_{G+H}(x) \cap S = \{u_1, u_2, \dots, u_p\} = N_{G+H}(y) \cap S$.

Case 2. If $x, y \in V(H)$, then $N_{G+H}(x) \cap S = \{v_1, v_2, \dots, v_p\} = N_{G+H}(y) \cap S$.

Case 3. If $x \in V(G)$ and $y \in V(H)$, then $N_{G+H}(x) \cap S = \{u_1, u_2, \dots, u_p\}$ and $N_{G+H}(y) \cap S = \{v_1, v_2, \dots, v_p\}$.

In any case, $|N_{G+H}(x) \cap S| = p = |N_{G+H}(y) \cap S|$ for all $x, y \in V(G+H)\backslash S$. Thus, S is a fair dominating set of $G+H$. Accordingly, S is a fair doubly connected dominating set of $G+H$. ■

The following result shows the characterization of the fair doubly connected dominating set in the join of two graphs.

Theorem 2.5 *Let G and H be nontrivial graphs. Then a nonempty subset $S \subseteq V(G+H)$ is a fair doubly connected dominating set of $G+H$ if and only if one of the following is satisfied.*

(i) $\langle S \rangle$ is connected and

a) $S = V(G)$ and H is connected, or

b) $S = V(H)$ and G is connected.

(ii) $\langle S \rangle$ is connected, S is an $|S|$ −fair dominating set, and

a) $S \cap V(H) = \emptyset, G$ is connected, and $S \neq V(G)$, or

b) $S \cap V(G) = \emptyset$, H is connected, and $S \neq V(H)$.

(iii) $S_H = S \cap V(H) \neq \emptyset$ and $S_G = S \cap V(G) \neq \emptyset$, and

a) S_H is an $|S_H|$-fair dominating set of H and S_G is an $|S_G|$-fair dominating set of G, or

b) S_H is a k-fair dominating set of H and S_G is an k-fair dominating set of G with $|S_H| = |S_G|$, or

c) S_H is a $(k+m)$-fair dominating set of H and S_G is an k-fair dominating set of G with $|S_H| - |S_G| = m$.

d) S_H is a k-fair dominating set of H and S_G is an $(k+m)$-fair dominating set of G with $|S_G| - |S_H| = m$.

(iv) $G = G_1 \cup \overline{K}_p, H = H_1 \cup \overline{K'}_p$, where G_1 is connected and H_1 is connected, for all integer $p \geq 1$, and $S = V(\overline{K}_m) \cup V(\overline{K'}_m)$ for all $m \in \{1,2,\dots,p\}$.

Proof: Suppose that a nonempty $S \subseteq V(G+H)$ is a fair doubly connected dominating set of $G+H$. Then, both $\langle S \rangle$ and $\langle V(G+H) \backslash S \rangle$ are connected and every distinct vertices u and v from $V(G+H) \backslash S$, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$. Consider the following cases.

Case 1. If $S \cap V(H) = \emptyset$, then $S \subseteq V(G)$. First, consider that $S = V(G)$, then $V(G+H) \backslash S = V(H)$. This implies that H is connected. Thus, $\langle S \rangle$ is connected and $S = V(G)$ and H is connected, showing statement $(i)a)$.

Next, consider that $S \neq V(G)$. Then $S \subset V(G)$. Since S is a fair doubly connected dominating set of $G+H$, S must be a fair dominating set of G. Let $u, v \in V(G+H) \backslash S$. If $u, v \in V(H)$, then $|N_{G+H}(u) \cap S| = |S| = |N_{G+H}(v) \cap S|$ is clear. Since $S \subset V(G)$, $V(G) \backslash S \neq \emptyset$. Let $x \in V(G) \backslash S$. Then $|N_{G+H}(u) \cap S| = |S|$ and $|N_{G+H}(x) \cap S|$ must be equal to $|S|$ because S is a fair dominating set of $G+H$. Thus, $|N_G(x) \cap S| = |N_{G+H}(x) \cap S| = |S|$. This implies that S is an $|S|$-fair dominating set and G is connected. Hence, $\langle S \rangle$ is connected, S is an $|S|$-fair dominating set, $S \cap V(H) = \emptyset$, G is connected, and $S \neq V(G)$ showing statement $(ii)a)$.

Case 2. If $S \cap V(G) = \emptyset$, then $S \subseteq V(H)$. First, consider that $S = V(H)$, then $V(G+H) \backslash S = V(G)$. This implies that G is connected. Thus, $\langle S \rangle$ is connected and $S = V(H)$ and G is connected, showing statement $(i)b)$.

Next, consider that $S \neq V(H)$. Then $S \subset V(H)$. Since S is a fair doubly connected dominating set of $G+H$, S must be a fair dominating set of H. Let $u, v \in V(G+H) \backslash S$. If $u, v \in V(G)$, then $|N_{G+H}(u) \cap S| = |S| = |N_{G+H}(v) \cap S|$ is clear. Since $S \subset V(H), V(H) \backslash S \neq \emptyset$. Let $y \in V(H) \backslash S$. Then $|N_{G+H}(u) \cap S| = |S|$ and $|N_{G+H}(y) \cap S|$ must be equal to $|S|$ because S is a fair dominating set of $G+H$. Thus, $|N_G(y) \cap S| = |N_{G+H}(y) \cap S| = |S|$. This implies that S is an $|S|$-fair dominating set and H is connected. Hence, $\langle S \rangle$ is connected, S is an $|S|$-fair dominating set, $S \cap V(G) = \emptyset$, H is connected, and $S \neq V(H)$ showing statement $(ii)b)$.

Case 3. If $S \cap V(H) \neq \emptyset$ and $S \cap V(G) \neq \emptyset$, then assign $S_H = S \cap V(H)$ and $S_G = S \cap V(G)$. Since

$$\begin{aligned} S_G \cup S_H &= \big(S \cap V(G)\big) \cup \big(S \cap V(H)\big) \\ &= S \cap (V(G) \cup V(H)) \\ &= S \cap V(G+H) \\ &= S \end{aligned}$$

is a fair dominating set of $G+H$, it follows that S_H and S_G must be fair dominating sets of H and G, respectively. Since $S_H \subset V(H)$ and $S_G \subset V(G)$, let $u \in V(H)\backslash S_H$ and $v \in V(G)\backslash S_G$. Now, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$ because S is a fair dominating set of $G+H$. For all $u \in V(H)\backslash S_H$ and $v \in V(G)\backslash S_G$,

$$\begin{aligned} |N_H(u) \cap S_H| + |S_G| &= |(N_H(u) \cap S_H) \cup S_G| \\ &= |(N_{G+H}(u) \cap S_H) \cup (N_{G+H}(u) \cap S_G)| \\ &= |N_{G+H}(u) \cap (S_H \cup S_G)| \\ &= |N_{G+H}(u) \cap S| \\ &= |N_{G+H}(v) \cap S| \\ &= |N_{G+H}(v) \cap (S_H \cup S_G)| \\ &= |(N_{G+H}(v) \cap S_G) \cup (N_{G+H}(v) \cap S_H)| \\ &= |(N_G(v) \cap S_G) \cup S_H| \\ |N_H(u) \cap S_H| + |S_G| &= |N_G(v) \cap S_G| + |S_H| \\ |N_H(u) \cap S_H| - |N_G(v) \cap S_G| &= |S_H| - |S_G|, or \\ |S_G| - |S_H| &= |N_G(v) \cap S_G| - |N_H(u) \cap S_H| \end{aligned}$$

Subcase 1. If $|S_G| - |S_H| = 0$, then $|N_G(v) \cap S_G| = |N_H(u) \cap S_H| = k$ for some positive integer k. Thus, S_H is a k-fair dominating set of H and S_G is an k-fair dominating set of G with $|S_H| = |S_G|$, showing statement $(iii)b$). If $|S_G| = |S_H| = k$ with $|N_G(v) \cap S_G| = |N_H(u) \cap S_H| = k$ for some positive integer k, then S_H is an $|S_H|$-fair dominating set of H and S_G is an $|S_G|$-fair dominating set of G, showing statement $(iii)a$).

Moreover, supposed that G is not a connected graph and H is not a connected graph. Let $G = G_1 \cup \overline{K}_p$, $H = H_1 \cup \overline{K'}_p$, where G_1 is connected and H_1 is connected for all integer $p \geq 1$. Assign $S_G = V(\overline{K}_m)$ and $S_H = V(\overline{K'}_m)$ for all $m \in \{1,2,\dots,p\}$. Then $\langle S \rangle = \langle S_H \cup S_G \rangle$ is clearly a connected subgraph of $G+H$ and S is an m-fair dominating set of $G+H$ for all $m \in \{1,2,\dots,p\}$. Thus, $S = V(\overline{K}_m) \cup V(\overline{K'}_m)$ for all $m \in \{1,2,\dots,p\}$, showing statement (iv).

Subcase 2. If $|S_G| - |S_H| \neq \emptyset$, then $|N_G(v) \cap S_G| - |N_H(u) \cap S_H| \neq 0$. This implies that there exists a positive integer m such that

$$|S_G| - |S_H| = |N_G(v) \cap S_G| - |N_H(u) \cap S_H| = m.$$

Let k be a positive integer such that $k = |N_H(u) \cap S_H|$.

First, consider that $k = |S_H|$. Then $|N_H(u) \cap S_H| = k = |S_H|$ and

$$\begin{aligned} |S_H| - |S_G| &= |N_H(u) \cap S_H| - |N_G(v) \cap S_G| \\ k - |S_G| &= k - |N_G(v) \cap S_G| \\ |S_G| &= |N_G(v) \cap S_G|. \end{aligned}$$

Thus, S_H is an $|S_H|$-fair dominating set of H and S_G is an $|S_G|$-fair dominating set of G, satisfying statement $(iii)a$)

Next, consider that $k \neq |S_H|$. Then $|N_G(v) \cap S_G| = k + m$. Thus, S_H is a k-fair dominating set of H and S_G is an $(k+m)$-fair dominating set of G with $|S_G| - |S_H| = m$. This shows statement $(iii)d$).

Similarly, if $|S_H| - |S_G| \neq 0$, then $|N_H(u) \cap S_H| - |N_G(v) \cap S_G| \neq 0$. This implies that there exists a positive integer m such that

$$|S_H| - |S_G| = |N_H(u) \cap S_H| - |N_G(v) \cap S_G| = m.$$

Let k be appositive integer such that $k = |N_H(v) \cap S_G|$.

First, consider that $k = |S_G|$, then $|N_G(v) \cap S_G| = k = |S_G|$ and

$$|S_H| - |S_G| = |N_H(u) \cap S_H| - |N_G(v) \cap S_G|$$
$$|S_H| - k = |N_H(u) \cap S_H| - k$$
$$|S_H| = |N_H(u) \cap S_H|.$$

Thus, S_H is an $|S_H|$-fair dominating set of H and S_G is an $|S_G|$-fair dominating set of G, satisfying statement $(iii)a$)

Next, consider $k \neq |S_G|$. Then $|N_H(u) \cap S_H| = k + m$. Thus, S_H is an $(k+m)$-fair dominating set of H and S_G is a k-fair dominating set of G with $|S_H| - |S_G| = m$. This shows statement $(iii)c$).

For the converse, if (i) is satisfied, then by Lemma 2.1, S is a fair doubly connected dominating set of $G + H$. If (ii) is satisfied, then by Lemma 2.2, S is a fair doubly connected dominating set of $G + H$. If (iii) is satisfied, then by Lemma 2.3, S is a fair doubly connected dominating set of $G + H$. If (iv) is satisfied, then by Lemma 2.4, S is a fair doubly connected dominating set of $G + H$. This completes the proof. ■

Corollary 2.6 *Let G and H be nontrivial graphs, G_1 and H_1 are connected subgraphs of G and H, respectively. Then*

$$\gamma_{fcc}(G+H) = \begin{cases} 1, & if\ \gamma(G) = 1\ or\ \gamma(H) = 1 \\ 2, & if\ G = G_1 \cup \overline{K_P}\ and\ H = H_1 \cup \overline{K'_P}\ for\ all\ p \geq 1 \\ |S|, & if\ \langle S \rangle\ is\ connected, and\ S\ is\ a\ minimum \\ & |S| - fair\ dominating\ set\ of\ G\ or\ H. \end{cases}$$

Proof: Suppose that G and H are nontrivial graphs, G_1 and H_1 are connected subgraphs of G and H, respectively.

Case 1. If $\gamma(G) = 1$, then G is connected. Let $S = \{x\}$ be a dominating set of G. Then $\langle S \rangle$ is trivially connected. Since G is nontrivial and connected graph and H is nontrivial graph, $G + H$ is connected. Clearly, $\langle V(G+H) \backslash S \rangle$ is connected. Hence, S is a doubly connected dominating set of $G + H$. Let $u, v \in V(G+H) \backslash S$. Then

$$|N_{G+H}(u) \cap S| = |S| = |N_{G+H}(v) \cap S|,$$

That is, S is a fair dominating set of $G + H$. Thus, S is a fair doubly connected dominating set of $G + H$, that is, $1 \leq \gamma_{fcc}(G+H) \leq |S| = 1$. Therefore, $\gamma_{fcc}(G+H) = 1$. Similarly, if $\gamma(H) = 1$, then $\gamma_{fcc}(G+H) = 1$.

Case 2. If $G = G_1 \cup \overline{K}_p$ and $H = H_1 \cup \overline{K'}_p$ for all $p \geq 1$, then assign $S_G = \{x\} \subseteq V(\overline{K}_p)$ and $S_H = \{y\} \subseteq V(\overline{K'}_p)$ such that $S = S_G \cup S_H = \{x, y\}$. Since $x \in V(G)$ and $y \in V(H)$, it follows that $xy \in E(G+H)$. Thus, the subgraph $\langle S \rangle = \langle \{x, y\} \rangle$ is connected in $G + H$. Let $v \in V(G) \backslash S_G$ and $u \in V(H) \backslash S_H$. Then $v, u \in V(G+H) \backslash S$ and $vu \in E(G+H)$. Thus, $\langle V(G+H) \backslash S \rangle$ is connected, that is, S is a doubly connected graph in $G + H$. Since $S_H = \{y\} \subset$

$V(H), vy \in E(G+H)$ for all $v \in V(G) \backslash S_G$. This implies that $|N_{G+H}(v) \cap S| = |S_H| = 1$ for all $v \in V(G) \backslash S_G$. Similarly, since $S_G = \{x\} \subset V(G), xu \in E(G+H)$ for all $u \in V(H) \backslash S_H$. This implies that $|N_{G+H}(u) \cap S| = |S_G| = 1$ for all $u \in V(H) \backslash S_H$. Thus, $|N_{G+H}(v) \cap S| = |N_{G+H}(u) \cap S|$ for all $u, v \in V(G+H)$, that is, S is a fair dominating set of $G+H$. Therefore, S is a fair doubly connected dominating set, that is, $\gamma_{fcc}(G+H) \leq |S| = |\{x,y\}| = 2$. Since $\gamma(G) = \gamma(G_1 \cup \overline{K}_p) \neq 1$ and $\gamma(H) = \gamma(H_1 \cup \overline{K'}_p) \neq 1$, clearly $\gamma_{fcc}(G+H) \leq 1$. Thus, $\gamma_{fcc}(G+H) \geq 2$. Consequently, $2 \leq \gamma_{fcc}(G+H) \leq |S| = 2$, implies that $\gamma_{fcc}(G+H) = 2$.

Case 3. If $\langle S \rangle$ is connected and S is a minimum $|S|$-fair dominating set of G (or H), then S is a minimum $|S|$-fair dominating set of $G+H$. Since $S \subset V(G)$ (or $S \subset V(H)$), $V(G) \backslash S \neq \emptyset$ (or $V(H) \backslash S \neq \emptyset$). Let $x \in V(G) \backslash S$ (or $x \in V(H) \backslash S$). Then $xy \in E(G+H)$ for all $y \in V(H)$ (or $y \in V(G)$). Thus, $\langle V(G+H) \backslash S \rangle$ is connected, that is, S is a doubly connected dominating set of $G+H$. Since S is a fair dominating set, it follows that S is a fair doubly connected dominating set of $G+H$, that is, $\gamma_{fcc}(G+H) \leq |S|$. Since S is a minimum $|S|$-fair dominating set of G (or H), it implies that $|S| = \gamma_{fcc}(G) \leq \gamma_{fcc}(G+H) \leq |S|$ (or $|S| = \gamma_{fcc}(H) \leq \gamma_{fcc}(G+H) \leq |S|$). Hence, $\gamma_{fcc}(G+H) = |S|$. ■

3. Conclusion

In this paper, we characterized the fair doubly connected dominating set in the join of two graphs and the exact fair doubly connected domination number in the join of two graphs was determined. This paper will used as a reference material to new research such as new parameter involving fair doubly connected domination in graphs and characterization of other binary operations of two connected graphs. Finally, the characterization of the fair doubly connected dominating set in composition of two graphs is the promising extension of this research.

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