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Inverse Fair Restrained Domination in the Join of Two Graphs

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Abstract
Let G be a connected simple graph. A dominating subset S of $V(G)$ is a fair dominating set in G if all the vertices not in S are dominated by the same number of vertices from S. A fair dominating set $S \subseteq V(G)$ is a *fair restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Alternately, a fair dominating set $S \subseteq V(G)$ is a *fair restrained dominating* set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. Let D be a minimum fair restrained dominating set of G. A fair restrained dominating set $S \subseteq (V(G) \setminus D)$ is called an *inverse fair restrained dominating set* of G with respect to D. The *inverse fair restrained domination number* of G denoted by $\gamma_{frd}^{-1}(G)$ is the minimum cardinality of an inverse fair restrained dominating set of G. An inverse fair restrained dominating set of cardinality $\gamma_{frd}^{-1}(G)$ is called γ_{frd}^{-1}-set. In this paper, we investigate the concept and give some important results on inverse fair restrained dominating sets under the join of two graphs.

Keywords: dominating set, fair dominating set, fair restrained dominating set, inverse fair restrained dominating set, join of two graphs

1. Introduction

A subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. *The domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G [1]. Some studies on domination in graphs were found in the papers [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

In 2011, Caro, Hansberg and Henning [14] introduced fair domination and k-fair domination in graphs. A dominating subset S of $V(G)$ is a fair dominating set in G if all the vertices not in S are dominated by the same number of vertices from S, that is, $|N(u) \cap S| = |N(v) \cap S|$ for every two distinct vertices u and

v from $V(G) \setminus S$ and a subset S of $V(G)$ is a k-fair dominating set in G if for every vertex $v \in V(G) \setminus S$, $|N(v) \cap S| = k$. The minimum cardinality of a fair dominating set of G, denoted by $\gamma_{fd}(G)$, is called the fair domination number of G. A fair dominating set of cardinality $\gamma_{fd}(G)$ is called γ_{fd}-set. Some studies on fair domination in graphs were found in the paper [15-18].

The restrained domination in graphs was introduced by Telle and Proskurowski [19] indirectly as a vertex partitioning problem. Accordingly, a set $S \subseteq V(G)$ is a *restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Alternately, a subset S of $V(G)$ is a restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. The minimum cardinality of a restrained dominating set of G, denoted by $\gamma_r(G)$, is called the restrained domination number of G. A restrained dominating set of cardinality $\gamma_r(G)$ is called γ_r-set. Restrained domination in graphs was also found in the papers [20-27].

The study of fair restrained dominating set is found in [28]. A fair dominating set $S \subseteq V(G)$ is a *fair restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. The minimum cardinality of a fair restrained dominating set of G, denoted by $\gamma_{frd}(G)$, is called the fair restrained domination number of G. A fair restrained dominating set of cardinality $\gamma_{frd}(G)$-is called γ_{frd}-set.

Let D be a minimum dominating set in G. The dominating set $S \subseteq V(G) \setminus D$ is called an *inverse dominating set* with respect to D. The minimum cardinality of inverse dominating set is called an *inverse domination number* of G and is denoted by $\gamma^{-1}(G)$. An inverse dominating set of cardinality $\gamma^{-1}(G)$ is called γ^{-1}-set of G. Inverse domination in graphs is found in in [29-34].

The inverse fair restrained domination in graphs was introduced in [35]. Let D be a minimum fair restrained dominating set of G. A fair restrained dominating set $S \subseteq (V(G) \setminus D)$ is called an *inverse fair restrained dominating set* of G with respect to D. The *inverse fair restrained domination number* of G denoted by $\gamma_{frd}^{-1}(G)$ is the minimum cardinality of an inverse fair restrained dominating set of G. An inverse fair restrained dominating set of cardinality $\gamma_{frd}^{-1}(G)$ is called γ_{frd}^{-1}-set. In this paper, we investigate the concept and give some important results on inverse fair restrained dominating sets under the join of two graphs.

For the general terminology in graph theory, readers may refer to [36]. A graph G is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the *vertex-set* of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the *edge-set* of G. The elements of $V(G)$ are called *vertices* and the cardinality $|V(G)|$ of $V(G)$ is the *order* of G. The elements of $E(G)$ are called *edges* and the cardinality $|E(G)|$ of $E(G)$ is the *size* of G. If $|V(G)| = 1$, then G is called a trivial graph. If $E(G) = \emptyset$, then G is called an empty graph. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. The elements of $N_G(v)$ are called *neighbors* of v. The *closed neighborhood* of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set

$N_G(X) = \bigcup_{v \in X} N_G(v)$. The *closed neighborhood* of X in G is the set $N_G[X] = \bigcup_{v \in X} N_G[v] = N_G(X) \cup X$.

When no confusion arises, $N_G[x]$ [resp. $N_G(x)$] will be denoted by $N[x]$ [resp. $N(x)$].

2. Results

The join of two graphs G and H is the graph $G+H$ with vertex-set $V(G+H)=V(G)\cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Remark 2.1 *Let G and H be connected graphs. Then $V(G)$ and $V(H)$ are fair dominating sets of $G+H$.*

Remark 2.2 *Let $G = P_n$ and $H = P_m$ where $n, m \geq 2$. Then $S \subset V(G)$ or $S \subset V(H)$ is 1-fair or 2-fair dominating set of $G+H$.*

We need the following Lemma for our next Theorem.

Lemma 2.3 *Let G and H be nontrivial connected graphs. If $S = S_G \cup S_H$ where S_G is an r-fair dominating set of G, S_H is an s-fair dominating set of H, and $r - s = |S_G| - |S_H|$ then S is a fair restrained dominating set of $G+H$.*

Proof: Since S_G is an r-fair dominating set of G, for every $u \in V(G) \setminus S_G$, $|N_G(u) \cap S_G| = r$. Since S_H is an s-fair dominating set of H, for every $v \in V(H) \setminus S_H$, $|N_H(v) \cap S_H| = s$. Now, $S_G \subset V(G)$ implies that $V(G) \setminus S_G \neq \emptyset$. Let $u \in V(G) \setminus S_G$. Then $u \in V(G+H) \setminus S$, $|(N_{G+H}(u) \cap S_H)| = |S_H|$, and

$$\begin{aligned} |(N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap (S_G \cup S_H)| \\ &= |N_{G+H}(u) \cap S_G) \cup (N_{G+H}(u) \cap S_H)| \\ &= |N_{G+H}(u) \cap S_G| + \left|(N_{G+H}(u) \cap S_{H)}\right| \\ &= |(N_G(u) \cap S_G)| + |S_H| \\ &= r + |S_H|. \end{aligned}$$

Similarly, since $S_H \subset V(H)$, $V(H) \setminus S_H \neq \emptyset$. Let $v \in V(H) \setminus S_H$. Then $v \in V(G+H) \setminus S$, $|(N_{G+H}(v) \cap S_G)| = |S_G|$, and

$$\begin{aligned} |N_{G+H}(v) \cap S| &= |N_{G+H}(v) \cap (S_G \cup S_H)| \\ &= |(N_{G+H}(v) \cap S_G) \cup (N_{G+H}(v) \cap S_H)| \\ &= |(N_{G+H}(v) \cap S_G)| + |(N_{G+H}(v) \cap S_H)| \\ &= |S_G| + |(N_H(v) \cap S_H)| \\ &= |S_G| + s. \end{aligned}$$

Since $r - s = |S_G| - |S_H|$ implies that $r + |S_H| = |S_G| + s$. It follows that, for every $u, v \in V(G+H) \setminus S$, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$. Hence, S is a fair dominating set of $G+H$. Now, let $u \in V(G) \setminus S_G$ and $v \in V(H) \setminus S_H$. Then $u, v \in V(G+H) \setminus S$ and $uv \in E(G+H)$. Since S is a dominating set, there exists $x \in S$ such that $xu \in E(G+H)$ or $xv \in E(G+H)$. Thus, every vertex in $V(G+H) \setminus S$ is adjacent to a vertex in S and to another vertex in $V(G+H) \setminus S$. Hence, S is a fair restrained dominating set of $G+H$. ■

Lemma 2.4 *Let $G = P_n = [x_1, x_2, \ldots, x_n]$ and $H = P_n = [y_1, y_2, \ldots, y_n]$ for $n \geq 4$. Then D is a minimum fair restrained dominating sets of $G+H$ and S is an inverse fair restrained dominating set of $G+H$ with respect to D such as:*

i) $D = \left\{x_{3i-2}, y_{3i-2} : i = 1, 2, 3, \ldots, \frac{n+2}{3}\right\}$ and $S = \left\{x_2, y_2, x_{3i}, y_{3i} : i = 1, 2, 3, \ldots, \frac{n-1}{3}\right\}$ if $n = 3k+1$ for all positive integer k.

ii) $D = \left\{x_{3i-2}, y_{3i-2} : i = 1, 2, 3, \ldots, \frac{n+1}{3}\right\}$ and $S = \left\{x_2, y_2, x_{3i+2}, y_{3i+2} : i = 1, 2, 3, \ldots, \frac{n-2}{3}\right\}$ if $n = 3k+2$ for all positive integer k.

iii) $D = \left\{x_{3i-1}, y_{3i-1} : i = 1,2,3,\ldots, \frac{n}{3}\right\}$ *and* $S = V(G+H) \setminus D$ *if* $n = 3k + 3$ *for all positive integer* k.

Proof: Suppose that $G = P_n = [x_1, x_2, \ldots, x_n]$ and $H = P_n = [y_1, y_2, \ldots, y_n]$ for $n \geq 4$. If $n = 3k + 1$ for all positive integer k, then the set $D_G = \left\{x_{3i-2} : i = 1,2,3, \ldots, \frac{n+2}{3}\right\}$ of order $\frac{n+2}{3}$ is a minimum fair dominating set of G and $D_H = \left\{y_{3i-2}: i = 1,2,3, \ldots, \frac{n+2}{3}\right\}$ of order $\frac{n+2}{3}$ *is* a minimum fair dominating set of H. This implies that $D = D_G \cup D_H$ of order $\frac{n+2}{3} + \frac{n+2}{3} = \frac{2n+4}{3}$ is a minimum fair dominating set of $G + H$. Observe that for every $u \in V(G+H) \setminus D$, there exists $v \in D$ and $u' \in V(G+H) \setminus D$ such that $uv, uu' \in E(G+H)$. Thus, D is a minimum fair restrained dominating set of $G + H$. Using the same arguments for $n = 3k + 1$, $S = \left\{x_2, y_2, x_{\{3i\}}, y_{\{3i\}}: i = 1,2,3, \ldots, \frac{n-1}{3}\right\}$ is also a minimum fair restrained dominating set of $G + H$ of order $2 + \frac{n-1}{3} + \frac{n-1}{3} = 2 + \frac{(n-1)+(n-1)}{3} = 2 + \frac{2n-2}{3} = \frac{6+(2n-2)}{3} = \frac{2n+4}{3}$. Next, if $n = 3k + 2$ for all positive integer k, then $D_G = \left\{x_{3i-2} : i = 1,2,3, \ldots, \frac{n+1}{3}\right\}$ of order $\frac{n+1}{3}$ is a minimum fair dominating set of G and $D_H = \left\{y_{3i-2} : i = 1,2,3, \ldots, \frac{n+1}{3}\right\}$ of order $\frac{n+1}{3}$ is a minimum fair dominating set of H. This implies that $D = D_G \cup D_H$ of order $\frac{n+1}{3} + \frac{n+1}{3} = \frac{2n+2}{3}$ is a minimum fair dominating set of $G + H$. Observe that for every $u \in V(G+H) \setminus D$, there exists $v \in D$ and $u' \in V(G+H) \setminus D$ such that $uv, uu' \in E(G+H)$. Thus, D is a minimum fair restrained dominating set of $G + H$. Using the same arguments for $n = 3k + 2$, $S = \left\{x_2, y_2, x_{3i+2}, y_{3i+2}: i = 1, 2, 3, \ldots, \frac{n-2}{3}\right\}$ is also a minimum fair restrained dominating set of $G + H$ of order $2 + \frac{n-2}{3} + \frac{n-2}{3} = 2 + \frac{(n-2)+(n-2)}{3} = 2 + \frac{2n-4}{3} = \frac{6+(2n-4)}{3} = \frac{2n+2}{3}$. Now, if $n = 3k + 3$ for all positive integer k, then $D_G = \left\{x_{3i-1} : i = 1,2,3, \ldots, \frac{n}{3}\right\}$ of order $\frac{n}{3}$ is a minimum fair dominating set of G and $D_H = \left\{y_{3i-1} : i = 1,2,3, \ldots, \frac{n}{3}\right\}$ of order $\frac{n}{3}$ is a minimum fair dominating set of H. This implies that $D = D_G \cup D_H$ of order $\frac{n}{3} + \frac{n}{3} = \frac{2n}{3}$ is a minimum fair dominating set of $G + H$. Observe that for every $u \in V(G+H) \setminus D$, there exists $v \in D$ and $u' \in V(G+H) \setminus D$ such that $uv, uu' \in E(G+H)$. Thus, D is a minimum fair restrained dominating set of $G + H$. Using the same arguments for $n = 3k + 3$, $S = V(G+H) \setminus D$ is also a fair restrained dominating set of $G + H$ of order $|S| = |V(G+H) \setminus D| = |V(G+H)| - |D| = 2n - \frac{2n}{3} = \frac{6n-2n}{3} = \frac{4n}{3}$. This complete the proofs. ■

Lemma 2.5 *Let* $G = P_n = [x_1, x_2, \ldots, x_n]$ *and* $H = P_m = [y_1, y_2, \ldots, y_m]$ *for* $n \geq 4$ *and* $m \geq 5$ *with* $m = n + 1$. *The minimum fair restrained dominating sets* D *and* S *of* $G + H$ *are the following:*

i) $D = \left\{x_{3i-2} : i = 1,2,3,\ldots, \frac{n+2}{3}\right\} \cup \left\{y_{3i-2} : i = 1,2,3,\ldots, \frac{m+1}{3}\right\}$ *and*
$S = \left\{x_2, x_{3i} : i = 1,2,3,\ldots, \frac{n-1}{3}\right\} \cup \left\{y_{3i}: i = 1,2,3,\ldots, \frac{m+1}{3}\right\}$ *if* $n = 3k + 1$ *for all positive integer* k.

ii) $D = \left\{x_{3i-1}, x_{n-1} : i = 1,2,3,\ldots, \frac{n}{3}\right\} \cup \left\{y_{3i-2}: i = 1,2,3, \ldots, \frac{m+2}{3}\right\}$ *and*

$S = \left\{ x_2, x_{3i} : i = 1,2,3,\ldots, \frac{n}{3}\right\} \cup \left\{ y_2, y_{3i} : i = 1,2,3,\ldots, \frac{m-1}{3}\right\}$ *if* $n = 3k + 3$ *for all positive integer* k.

Proof: Suppose that $G = P_n = [x_1, x_2, \ldots, x_n]$ and $H = P_m = [y_1, y_2, \ldots, y_m]$ for $n \geq 4$ and $m \geq 5$ with $m = n + 1$. If $n = 3k + 1$ for all positive integer k, then the set $D_G = \left\{x_{3i-2}: i = 1,2,3,\ldots, \frac{n+2}{3}\right\}$ of order $\frac{n+2}{3}$ is a minimum fair dominating set of G and $D_H = \left\{y_{3i-2} : i = 1,2,3,\ldots, \frac{m+1}{3}\right\}$ of order $\frac{m+1}{3}$ is a minimum fair dominating set of H. Since $\frac{m+1}{3} = \frac{(n+1)+1}{3} = \frac{n+2}{3}$, it follows that $D = D_G \cup D_H$ of order $\frac{n+2}{3} + \frac{n+2}{3} = \frac{2n+4}{3}$ is a minimum fair dominating set of $G + H$. Observe that for every $u \in V(G+H) \setminus D$, there exists $v \in D$ and $u' \in V(G+H) \setminus D$ such that $uv, uu' \in E(G+H)$. Thus, D is a minimum fair restrained dominating set of $G + H$. Using the same arguments for $n = 3k + 1$, $S = \left\{x_2, x_{3i} : i = 1,2,3,\ldots, \frac{n-1}{3}\right\} \cup \left\{y_{3i} : i = 1,2,3,\ldots, \frac{m+1}{3}\right\}$ is also a minimum fair restrained dominating set of $G + H$ of order $\left[1 + \frac{n-1}{3}\right] + \frac{m+1}{3} = \frac{3+(n-1)+((n+1)+1)}{3} = \frac{2n+4}{3}$. Now, if $n = 3k + 3$ for all positive integer k, then $D_G = \left\{x_{3i-1}, x_{n-1} : i = 1,2,3,\ldots, \frac{n}{3}\right\}$ of order $\frac{n}{3} + 1$ is a minimum fair dominating set of G and $D_H = \left\{y_{3i-2} : i = 1,2,3,\ldots, \frac{m+2}{3}\right\}$ of order $\frac{m+2}{3}$ is a minimum fair dominating set of H. Since $\frac{m+2}{3} = \frac{(n+1)+2}{3} = \frac{n+3}{3} = \frac{n}{3} + 1$, it follows that $D = D_G \cup D_H$ of order $\left[\frac{n}{3} + 1\right] + \left[\frac{n}{3} + 1\right] = \frac{2n}{3} + 2$ is a minimum fair dominating set of $G + H$. Observe that for every $x \in V(G+H) \setminus D$, there exists $y \in D$ and $x' \in V(G+H) \setminus D$ such that $xy, xx' \in E(G+H)$. Thus, D is a minimum fair restrained dominating set of $G + H$. Using the same arguments for $n = 3k + 3$, $S = \left\{x_2, x_{3i} : i = 1,2,3,\ldots, \frac{n}{3}\right\} \cup \left\{y_2, y_{3i} : i = 1,2,3,\ldots, \frac{m-1}{3}\right\}$ is also a minimum fair restrained dominating set of $G + H$ of order $[1 + \frac{n}{3}] + [1 + \frac{m-1}{3}] = [1 + \frac{n}{3}] + [1 + \frac{(n+1)-1}{3}] = 2 + \frac{2n}{3}$. This complete the proofs. ∎

The following result is the characterization of the inverse fair restrained domination in the join of two paths.

Theorem 2.6 *Let* $G = P_n$ *and* $H = P_m$ *where* $n, m \geq 2$. *Then a nonempty proper subset* S *of* $V(G+H)$ *is an inverse fair restrained dominating set of* $G + H$ *if and only if one of the following statement is satisfied.*

(i) $S = V(G)$ *and* $\gamma(H) = 1$.
(ii) $S = V(H)$ *and* $\gamma(G) = 1$.
(iii) *S is a 1-fair dominating set of G and* $n = 2$
or S is a 2-fair dominating set of G and $n = 3$.
(iv) *S is a 1-fair dominating set of H and* $m = 2$
or S is a 2-fair dominating set of H and $m = 3$.
(v) $S = S_G \cup S_H$ *where* $S_G \subset V(G)$ *is an r-fair dominating set of G,* $S_H \subset V(H)$ *is an s-fair dominating set of H,* $|S_G| + s = r + |S_H|$, *and D is a minimum fair dominating set of* $G + H$.

Proof: Suppose a nonempty proper subset S of $V(G+H)$ is an inverse fair restrained dominating set of $G + H$. Consider the following cases:

Case 1. Consider that $S \cap V(H) = \emptyset$. Then $S \subseteq V(G)$. Suppose that $S = V(G)$, then S is a fair dominating set of $G + H$ by Remark 2.1. If $\gamma(H) \neq 1$, say $\gamma(H) = 2$, then let D be the minimum fair dominating set of H. Clearly, $m = 4$ or $m = 5$ or $m = 6$ since $\gamma(H) = 2$ and $H = P_m$. For every $u \in V(G) \subset V(G + H)$ and $v \in V(H) \setminus D \subset V(G + H)$, $N_{G+H}(u) \cap D = 2 \neq 1 = N_{G+H}(v) \cap D$, that is D is not a fair dominating set of $G + H$. Hence, $S \subset V(G + H) \setminus D$ is not an inverse fair dominating set of $G + H$ with respect to D, a contradiction. Hence, $\gamma(H) = 1$, and the proof of statement (i) is satisfied. Suppose that $S \neq V(G)$. Let $u \in V(G) \setminus S$ and $v \neq u$ such that $u, v \in V(G + H) \setminus S$. Then, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$ since S is a fair dominating set of $G + H$. If $v \in V(G) \setminus S$, then

$$|N_G(u) \cap S| = |N_G(v) \cap S| = k \text{ for some positive integer } k.$$

This implies that S is a k-fair dominating set of G. By Remark 2.2, $n, m \geq 2$ and $k = 1$ or $k = 2$ since $G = P_n$, a path. If $n = 2$ or $n = 3$, then statement (iii) is satisfied. Clearly if $n \geq 4$, then S is not an inverse fair restrained dominating set of $G + H$ since $H = P_m$ and $m \geq 2$.

Case 2. Consider that $S \cap V(G) = \emptyset$. Then $S \subseteq V(H)$. If $S = V(H)$, then S is a fair dominating set of $G + H$ by Remark 2.1. If $\gamma(G) \neq 1$, say $\gamma(G) = 2$, then let D be the minimum fair dominating set of G. Clearly, $n = 4$ or $n = 5$ or $n = 6$ since $G = P_n$. For every $u \in V(H) \subset V(G + H)$ and $v \in V(G) \setminus D \subset V(G + H)$, $N_{G+H}(u) \cap D = 2 \neq 1 = N_{G+H}(v) \cap D$, that is D is not a fair dominating set of $G + H$. Hence, $S \subset V(G + H) \setminus D$ is not an inverse fair dominating set of $G + H$ with respect to D, a contradiction. Hence, $\gamma(G) = 1$, and the proof of statement (ii) is satisfied. Suppose that $S \neq V(H)$. Let $u \in V(H) \setminus S$ and $v \neq u$ such that $u, v \in V(G + H) \setminus S$. Then, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$ since S is a fair dominating set of $G + H$. If $v \in V(H) \setminus S$, then $|N_H(u) \cap S| = |N_H(v) \cap S| = k$ for some positive integer k.

This implies that S is a k-fair dominating set of H. By Remark 2.2, $n, m \geq 2$ and $k = 1$ or $k = 2$ since $H = P_m$, a path. If $m = 2$ or $m = 3$, then statement (iv) is satisfied. Clearly if $m \geq 4$, then S is not an inverse fair restrained dominating set of $G + H$ since $G = P_n$ and $n \geq 2$.

Case 3. Consider that $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Then $S = S_G \cup S_H$ where $S_G \subset V(G)$ and $S_H \subset V(H)$. Suppose that to the contrary, S_G is not a fair dominating set of G. Then there exists distinct vertices u and v in $V(G) \setminus S_G$ such that

$$|N_G(u) \cap S_G| \neq |N_G(v) \cap S_G|.$$

Thus,

$$\begin{aligned}
|N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap (S_G \cup S_H)| \\
&= |(N_{G+H}(u) \cap S_G) \cup (N_{G+H}(u) \cap S_H)| \\
&= |(N_G(u) \cap S_G) \cup S_H|, \text{ since } u \in V(G) \setminus S \\
&= |N_G(u) \cap S_G| + |S_H| \\
&\neq |N_G(v) \cap S_G| + |S_H| \\
&= |(N_G(v) \cap S_G) \cup S_H| \\
&= |(N_{G+H}(v) \cap S_G) \cup (N_{G+H}(v) \cap S_H)|, \text{ since } v \in V(G) \setminus S \\
&= |N_{G+H}(v) \cap (S_G \cup S_H)| \\
&= |N_{G+H}(v) \cap S|
\end{aligned}$$

This contradict to our assumption that S is a fair dominating set of $G + H$. Therefore, S_G must be a fair dominating set of G. Similarly, S_H is a fair dominating set of H. Thus, for every vertex $u \in V(G) \setminus S_G$,

$$|N_G(u) \cap S_G| = r, \text{ where } r = 1 \text{ or } r = 2 \text{ since } G = P_n \text{ is a path,}$$

and for every vertex $v \in V(H) \setminus S_H$,

$|N_H(v) \cap S_H| = s$, where $s = 1$ or $s = 2$ since $H = P_m$ is a path.

This implies that S_G is an r-fair dominating set of G and S_H is an s-fair dominating set of H.

Now, let $u \in V(G) \setminus S_G$ and $v \in (H) \setminus S_H$. Then,

$$\begin{aligned} |N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap (S_G \cup S_H)| \\ &= |(N_{G+H}(u) \cap S_G) \cup (N_{G+H}(u) \cap S_H)| \\ &= |(N_G(u) \cap S_G) \cup S_H| \\ &= |(N_G(u) \cap S_G| + |S_H| \\ &= r + |S_H| \text{ and} \end{aligned}$$

$$\begin{aligned} |N_{G+H}(v) \cap S| &= |N_{G+H}(v) \cap (S_G \cup S_H)| \\ &= |(N_{G+H}(v) \cap S_G) \cup (N_{G+H}(v) \cap S_H)| \\ &= |S_G \cup (N_H(v) \cap S_H)| \\ &= |S_G| + |N_H(v) \cap S_H| \\ &= |S_G| + s. \end{aligned}$$

This proves statement (iv).

For the converse, suppose that statement (i) is satisfied. Since $S = V(G)$, S is a fair dominating set of $G + H$ by Remark 2.1. Since $\gamma(H) = 1$ and $H = P_m$, it follows that $m = 2$ or $m = 3$. Let $v \in S$. Then for every $u \in V(G+H) \setminus S = V(H)$, there exists $z \in V(H)$ where $(z \neq u)$ such that $uz, uv \in E(G+H)$. That is, S is a restrained dominating set of $G+H$. Now, let $S' = \{y\}$ be the dominating set of H since $\gamma(H) = 1$. Then S' is a minimum fair dominating set of $G + H$. Since $n \geq 2$, for every $u \in V(G + H) \setminus S'$, there exists $z \in V(G + H) \setminus S'$ where $(z \neq u)$ such that $uz, uy \in E(G + H)$. Hence, S' is a restrained dominating set of $G + H$, that is, S' is a minimum fair restrained dominating set of $G + H$. This implies that $S \subset V(G + H) \setminus S'$ is an inverse fair restrained dominating set of $G + H$ with respect to S'.

Suppose that statement (ii) is satisfied. If $S = V(H)$, then S is a fair dominating set of $G+H$ by Remark 2.1. Since $\gamma(G) = 1$ and $G = P_n$, it follows that $n = 2$ or $n = 3$. Let $v \in S$. Then for every $u \in V(G + H) \setminus S = V(G)$, there exists $z \in V(G)$ where $(z \neq u)$ such that $uz, uv \in E(G + H)$. That is, S is a restrained dominating set of $G + H$. Now, let $S'' = \{x\}$ be the dominating set of G since $\gamma(G) = 1$. Then S'' is a minimum fair dominating set of $G + H$. Since $m \geq 2$, for every $u \in V(G + H) \setminus S''$, there exists $z \in V(G + H) \setminus S''$ where $(z \neq u)$ such that $uz, ux \in E(G+H)$. Hence, S'' is a restrained dominating set of $G + H$, that is, S'' is a minimum fair restrained dominating set of $G+H$. This implies that $S \subset V(G+H) \setminus S''$ is an inverse fair restrained dominating set of $G+H$ with respect to S''.

Suppose that statement (iii) is satisfied. Consider that $n = 2$ that is, $G = P_2$ and let $V(G) = \{x, y\}$. The S = {x} is a 1-fair dominating set of G, that is, S is a 1-fair dominating set of $G + H$. Since for every $u \in V(G + H) \setminus S$ there exists $v \in V(G + H) \setminus S$ such that $uv, ux \in E(G + H)$. Thus, S is a restrained dominating set of $G + H$, that is, S is a fair restrained dominating set of $G + H$. Similarly, $S' = \{y\}$ is a fair restrained dominating set of $G + H$. This implies that $S \subset V(G + H) \setminus S'$ is an inverse fair restrained dominating set of $G+H$ with respect to S'. Consider that $n=3$ and let $G=P_3=[x_1,x_2,x_3]$. Then $S=\{x_1,x_3\}$ is a 2-fair dominating set of G, that is, S is a 2-fair dominating set of $G+H$. Since for every $u\in V(G+H)\setminus S$ there exists $v\in V(G+H)\setminus S$ such that $uv, ux_1\in E(G+H)$ or $uv, ux_3\in E(G+H)$, S is a restrained dominating set of $G+H$, that is, S is a fair restrained dominating set of $G+H$. Similarly, $S'=\{x_2\}$ is a minimum fair restrained dominating set of $G+H$. This implies that $S\subset V(G+H)\setminus S'$ is an inverse fair restrained dominating set of $G+H$ with respect to S'.

Suppose that statement (iv) is satisfied. Consider that $m=2$, that is, $H=P_2$ and let $V(H)=\{x,y\}$. Then $S=\{x\}$ is a 1-fair dominating set of H, that is, S is a 1-fair dominating set of $G+H$. Since for every $u\in V(G+H)\setminus S$ there exists $v\in V(G+H)\setminus S$ such that $uv, ux\in E(G+H)$. Thus, S is a restrained dominating set of $G+H$, that is, S is a fair restrained dominating set of $G+H$. Similarly, $S'=\{y\}$ is a fair restrained dominating set of $G+H$. This implies that $S\subset V(G+H)\setminus S'$ is an inverse fair restrained dominating set of $G+H$. Consider that $m=3$ and let $H=P_3=[x_1,x_2,x_3]$. Then $S=\{x_1,x_3\}$ is a 2-fair dominating set of H, that is, S is a 2-fair dominating set of $G+H$. Since for every $u\in V(G+H)\setminus S$ there exists $v\in V(G+H)\setminus S$ such that $uv, ux_1\in E(G+H)$ or $uv, ux_3\in E(G+H)$. Thus, S is a restrained dominating set of $G+H$, that is, S is a fair restrained dominating set of $G+H$. Similarly, $S'=\{x_2\}$ is a minimum fair restrained dominating set of $G+H$. This implies that $S\subset V(G+H)\setminus S'$ is an inverse fair restrained dominating set of $G+H$.

Finally, suppose that statement (v) is satisfied. Then $S=S_G\cup S_H$ where $S_G\subset V(G)$ is a r-dominating set of G, and $S_H\subset V(H)$ is a s-fair dominating set of H, and $|S_G|+s=r+|S_H|$. By Lemma 2.3, S is a fair restrained dominating set of $G+H$. Consider that $\gamma(G+H)=1$. Then $\gamma(G)=1$ or $\gamma(H)=1$. Supposed that $\gamma(G)=1$. Then $G=P_2=[x,y]$ $(or\ G=P_3=[x,y,z])$. Set $D=V(G)\setminus S_G=\{y\}$. Then D is a minimum fair restrained dominating set of $G+H$, where $S_G=\{x\}$ (or $S_G=\{x,z\}$) and $S_H=\{u\}\subset V(H)$. Since S_H is a dominating set of $H=P_m, m=2$ or $m=3$, that is, $P_2=\{u,v\}$ (or $P_3=\{t,u,v\}$). Hence, $S=S_G\cup S_H=\{x,u\}$ (or $S=\{x,z,u\}$) is an inverse fair dominating set of $G+H$ with respect to D. Suppose that $\gamma(H)=1$. Then S is an inverse fair restrained dominating set of $G+H$ by similar arguments above. Now, consider that $\gamma(G+H)\neq 1$. Then $n\geq 4$ and $m\geq 4$ for $G=P_n=[x_1,x_2,x_3,\ldots,x_n]$ and $H=P_m=[y_1,y_2,y_3,\ldots,x_m]$. If $n=m$, then by Lemma 2.4, D is a minimum fair restrained dominating set of $G+H$ and S is an inverse fair restrained dominating set of $G+H$. If $n\neq m$, say $m=n+1$, then by Lemma 2.5, D is a minimum fair restrained dominating set of $G+H$ and S is an inverse fair restrained dominating set of $G+H$. This completes the proof. ■

The following result is an immediate consequence of Theorem 2.6.

Corollary 2.7 *Let $G=P_n$ and $H=P_m$ where $n,m\geq 2$, and S is an inverse fair restrained dominating set of $G+H$. Then*

$$\gamma_{frd}^{-1}(G+H)=\begin{cases}1, & if\ S\ is\ a\ 1-fair\ \ dominating\ set\ of\ G\ and\ n=2\\ & or\ S\ is\ a\ 1-fair\ dominating\ set\ of\ H\ and\ m=2.\\ 2, & if\ S\ is\ a\ 2-fair\ dominating\ set\ of\ G\ and\ n=3,(m\geq 4)\\ & or\ S\ is\ a\ 2-fair\ dominating\ set\ of\ H\ and\ m=3,(n\geq 4\\ |S|, & if\ S=S_G\cup S_H,\ S_G\ is\ a\ min\ fair\ dominating\ set\ of\ G,\\ & S_H\ is\ a\ min\ fair\ dominating\ set\ of\ H, and\ m,n\geq 4\end{cases}$$

Proof: Suppose that S is a 1-fair dominating set of G and $n=2$, say $V(G)=\{x_1,x_2\}$. Let $S=\{x_1\}$ and $D=\{x_2\}$. Since for every $u\in V(G+H)\setminus D$ there exists $u'\in V(G+H)\setminus D$ $(u\neq u')$ such that

$uu' \in E(G + H)$ and $ux_1 \in E(G+H)$, D is a restrained dominating set of G. Since D is a 1-fair dominating set of G, D is a minimum fair restrained dominating set of $G + H$. Similarly, S is a fair restrained dominating set of $G + H$, that is, S is a minimum inverse fair restrained dominating set of $G + H$ with respect to D. Hence, $\gamma_{frd}^{-1}(G + H) = |S| = 1$. If S is a 1-fair dominating set of H and $m = 2$, then $\gamma_{frd}^{-1}(G + H) = |S| = 1$ by using the same arguments above. Next, if S is a 2-fair dominating set of G (or H) and $n = 3$ (or $m = 3$). *Let* $G = [x_1, x_2, x_3]$ (or $H = [x_1, x_2, x_3]$). Then $D = \{x_2\}$ is a minimum fair restrained dominating set of $G + H$. The $S = \{x_1, x_3\} \subset V(G)$ $(or\ S \subset V(H))$ is a minimum inverse fair restrained dominating set of $G + H$ since $m \geq 4$ (or $n \geq 4$) with respect to D. Thus, $\gamma_{frd}^{-1}(G + H) = |S| = 2$ Finally, suppose that $S = S_G \cup S_H, S_G$ is a minimum fair dominating set of G, S_H is a minimum fair dominating set of H and $m, n \geq 4$. Let $G = P_n = [x_1, x_2, x_3, \ldots, x_n]$ and $H = P_m = [y_1, y_2, y_3, \ldots, x_m]$. Consider that $n = m$. By Lemma 2.4, $D = \left\{x_{3i-2}, y_{3i-2} : i = 1, 2, 3, \ldots, \frac{n+2}{3}\right\}$ is a minimum fair dominating set of $G + H$ and

$$\begin{aligned} S &= S_G \cup S_H \\ &= \left\{x_2, x_{3i}: i = 1,2,3, \ldots, \frac{n-1}{3}\right\} \bigcup \left\{y_2, y_{3i}: i = 1,2,3, \ldots, \frac{n-1}{3}\right\} \\ &= \left\{x_2, y_2, x_{3i}, y_{3i} : i = 1,2,3, \ldots, \frac{n-1}{3}\right\} \end{aligned}$$

is an inverse fair dominating set of $G + H$ with respect to D if $n = m = 3k + 1$ for all positive integer k. Since

$$\begin{aligned} |S| &= \left|\left\{x_2, y_2, x_{3i}, y_{3i} : i = 1,2,3, \ldots, \frac{n-1}{3}\right\}\right| \\ &= 1 + 1 + \frac{n-1}{3} + \frac{n-1}{3} \\ &= \frac{3 + 3 + (n-1) + (n-1)}{3} \\ &= \frac{6 + 2n - 2}{3} \\ &= \frac{2n + 4}{3} \\ &= \frac{n+2}{3} + \frac{n+2}{3} \\ &= \left|\left\{x_{3i-2}, y_{3i-2} : i = 1,2,3, \ldots, \frac{n+2}{3}\right\}\right| \\ &= |D| \end{aligned}$$

where D is a minimum fair restrained dominating set of $G + H$, it follows that S is also a minimum inverse fair restrained dominating set of $G + H$ with respect to D. Therefore, $\gamma_{frd}^{-1}(G + H) = \frac{2n+4}{3} = |S|$. This complete the proofs. ■

3. Conclusion and Recommendations

In this work, the fair restrained domination in the join of two paths of order $n \geq 2$ were characterized and the exact fair restrained domination number resulting from this binary operation of two paths were computed. This study will result to new research such as bounds and other binary operations of two graphs.

Other parameters involving the inverse fair restrained domination in graphs may also be explored. Finally, the characterization of a fair restrained domination in graphs and its bounds is a promising extension of this study.

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