International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

On Hankel Determinant Inequalities

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Abstract: This article aims to obtain the second Hankel determinant inequalities for the inverse of the well-known classes of univalent functions, namely, starlike and convex functions.

Keywords: The Hankel determinant; starlike functions; convex functions; the inverse function.

1. Introduction

Let Δ denote the class of normalised analytic univalent functions $t(\xi)$ in the unit disk $d = \{\xi \in C, |\xi| < 1\}$,which have the form

$$t(\xi) = \xi + \sum_{n=2}^{\infty} \alpha_n \xi^n. \tag{1}$$

Let us recall the following:

Definition 1. Let t be given by (1). We say t is starlike $t \in \Delta^{\star}$ if and only if

$$\Re\left\{\frac{\xi t'(\xi)}{t(\xi)}\right\} > 0,\ \xi \in d.$$

Definition 1.2. Let t be given by (1). We say that t is convex $t \in C_v$ if and only if

$$\Re\left\{\frac{\xi t''(\xi)}{t'(\xi)} + 1\right\} > 0,\ \xi \in d.$$

The inverse t^{-1} of every function $t \in \Delta$, defined by $t^{-1}(t(\xi)) = \xi$, is analytic in $|w| < r(t)$, $(r(t) \geq \frac{1}{4})$ and has Maclaurin's series expansion

$$t^{-1}(w) = w + \sum_{n=2}^{\infty} \mu_n w^n\ ,\ (|w| < r(t)) \tag{2}$$

Early in 1923 Löwner invented the famous parametric method to find sharp bounds on all the coefficients for the inverse functions in Δ (or $\Delta^{\star}$). Thus if $t \in \Delta(or \Delta^{\star})$ is given by (2) then

$$|\mu_n| \leq \frac{1}{n+1}\binom{2n}{n};\ (n = 2,3,\dots)$$

with equality for every n for the inverse of the Koebe function

$$k(\xi) = \frac{\xi}{(1+\xi)^2} = \xi + \sum_{n=2}^{\infty} n\xi^n.$$

Further, if $t \in C_v$ then $|\mu_n| \leq 1, (n = 2,3,\dots,8)$, while $|\mu_{10}| > 1$

definition 3. the jth Hankel determinant for $j \geq 1$ and $n \geq 0$ is stated by Noonan and Thomas as

$$H_j(n) = \begin{vmatrix} \alpha_n & \alpha_{n+1} & \alpha_{n+j+1} \\ \alpha_{n+1} & \cdots & . \\ . & & \\ \alpha_{n+j-1} & \cdots & \alpha_{n+2j-2} \end{vmatrix}.$$

This determinant has been considered by several authors. For example, the Hankel determinant was considered in case $j = 2$ and $n = 2$,

$$H_2(2) = \begin{vmatrix} \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_4 \end{vmatrix}.$$

In addition, the upper bound for the functional $|\alpha_2\alpha_4 - \alpha_3^2|$ for functions f belongs to the class $\Delta^\star$ and C_v. The objective of this study is to obtain the upper bounds for the functional $|\mu_2\mu_4 - \mu_3^2|$ for the inverse function t^{-1} given by (2) if t belonging to $\Delta^\star$ and C_v, respectively.

2. Preliminaries

Let the function s given by the power series

$$s(z) = 1 + d_1 z + d_2 z^2 + \cdots$$

be analytic in a neighborhood of the origin. For a real number l define the function h by

$$h(z) = (s(z))^l = (1 + d_1 z + d_2 z^2 + \cdots)^l = 1 + \sum_{k=1}^{\infty} C_k^{(l)} z^k . \tag{3}$$

Thus $C_k^{(l)}$ denotes the k^{th} coefficient in Maclaurin's series expansion of the l^{th} of the function $s(z)$.

Lemma 1. Let the coefficients $C_k^{(l)}$ be defined as in (3), then

$$C_{k+1}^{(l)} = \sum_{j=0}^{k} \left[l - \frac{(l+1)j}{k+1} \right] d_{k+1-j} C_j^l \,;\ (k = 0,1,\ldots; C_0^l = 1).$$

Let P be the family of all functions p analytic in U for which $\Re p(z) > 0$ and

$$p(\xi) = 1 + c_1\xi + c_2\xi^2 + \cdots$$

for $\xi \in d$.

Lemma 2. If $p \in P$ then $|c_k| \leq 2$ for each k.

Lemma 3. The power series for p given by (3) converges in d to a function in P if and only if the Toeplitz determinants

$$D_n = \begin{vmatrix} 2 & c_1 & c_2\cdots & c_n \\ c_{-1} & 2 & c_1\cdots & c_{n-1} \\ . & & & \\ . & & & \\ . & & & \\ c_{-n} & c_{-n+1} & c_{-n+2}\cdots & 2 \end{vmatrix}, n = 1,2,3,\ldots$$

and $c_{-k} = \bar{c_k}$, are all nonnegative. they are strictly positive except for $p(\xi) = \sum_{k=1}^{m} \rho_k \rho_o (e^{it_k}\xi)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$ for $k \neq j$; in this case $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \geq m$.

3. Main Result

Theorem 1. Let t^{-1} be defined in (2), if $f \in \Delta^\star$. Then

$$|\mu_2\mu_4 - \mu_3^2| \leq 3$$

.
The result is sharp.
Proof. We use the fact that

$$\mu_n = \frac{1}{2\pi i n}\int_{|\xi|=r}\frac{1}{(t(\xi))^n}d\xi.$$

Now for fixed n we write

$$h(\xi) = \left[\frac{\xi}{t(\xi)}\right]^n = \frac{1}{(1+\sum_{k=2}^{\infty}\alpha_k\xi^{k-1})^n} = 1 + \sum_{k=1}^{\infty} C_k^{(-n)}\xi^k. \tag{4}$$

Thus
Now, by using (Lemma 1) and (4) we can directly calculate the following:

$$\left(\frac{\xi}{t(\xi)}\right)^2 = \left(1+\sum_{k=2}^{\infty}\alpha_k\xi^{k-1}\right)^{-2} = 1+\sum_{k=1}^{\infty}C_k^{(-2)}\alpha^k.$$

$$\left(\frac{\xi}{t(\xi)}\right)^3 = \left(1+\sum_{k=2}^{\infty}\alpha_k\xi^{k-1}\right)^{-3} = 1+\sum_{k=1}^{\infty}C_k^{(-3)}\xi^k.$$

$$\left(\frac{\xi}{t(\xi)}\right)^4 = \left(1+\sum_{k=2}^{\infty}\alpha_k\xi^{k-1}\right)^{-4} = 1+\sum_{k=1}^{\infty}C_k^{(-4)}\xi^k.$$

which yields

$$\mu_2 = -\alpha_2$$
$$\mu_3 = -\alpha_3 + 2\alpha_2^2$$
$$\mu_4 = -\alpha_4 + 5\alpha_2\alpha_3 - 5\alpha_2^3$$

because $t \in \Delta^\star$, there exists $p \in P$ such that

$$\xi t'(\xi) = t(\xi)p(\xi),$$

we equating the coefficients and receive;

$$\alpha_2 = c_1$$
$$\alpha_3 = \frac{c_2}{2} + \frac{c_1^2}{2}$$
$$\alpha_4 = \frac{c_3}{3} + \frac{c_1c_2}{2} + \frac{c_1^3}{6}$$

we also have;

$$\mu_2 = -c_1$$
$$\mu_3 = -\frac{c_2}{2} + \frac{3c_1^2}{2}$$
$$\mu_4 = -\frac{c_3}{3} + 2c_1c_2 - \frac{8c_1^3}{3}$$

Thus, we find that

$$|\mu_2\mu_4 - \mu_3^2| = \left|\frac{1}{3}c_1c_3 - \frac{1}{4}c_2^2 + \frac{5}{12}c_1^4 - \frac{1}{2}c_1^2c_2\right|$$

Now, we consider Lemma 2 to obtain the upper bound. First, we assume that $c_1 \geq 0$. the cases $n = 2$ and $n = 3$, results in

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ c_1 & 2 & c_1 \\ \bar{c_2} & c_1 & 2 \end{vmatrix} = 8 + 2\Re\{c_1^2 c_2\} - 2|c_2|^2 - 4c_1^2 \geq 0,$$

which is equivalent to

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

for some x, $|x| \leq 1$.
Further, $D_3 \geq 0$ is equivalent to
$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|2c_2 - c_1^2|^2;$
and this provides the result;

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\xi,$$

for some ξ, $|\xi| \leq 1$.
Suppose that $c_1 = c$ and $0 \leq c \leq 2$. we conclude the following;

$$\left|\frac{1}{3}c_1c_3 - \frac{1}{4}c_2^2 + \frac{5}{12}c_1^4 - \frac{1}{2}c_1^2c_2\right|$$
$$= \left|\frac{3c^4}{16} + \frac{(4 - c^2)(1 - |x|^2)c\xi}{6} - \frac{(4 - c^2)(c^2 + 12)x^2)}{48} - \frac{5(4 - c^2)c^2 x}{24}\right|$$

The triangle inequality gives

$$\left|\frac{1}{3}c_1c_3 - \frac{1}{4}c_2^2 + \frac{5}{12}c_1^4 - \frac{1}{2}c_1^2c_2\right| \leq \frac{3c^4}{16} + \frac{c(4 - c^2)}{6} + \frac{(4 - c^2)(c - 2)(c - 6)y^2}{48} + \frac{5(4 - c^2)c^2 y}{24}$$

with $y = |x| \leq 1$. By elementary calculus, we compute the first derivative;

$$F'(y) = \frac{5c^2(4 - c^2)}{24} + \frac{(4 - c^2)(c - 2)(c - 6)y}{24}$$

it can be shown that $F'(y) > 0$ and thus $F(y)$ is increasing function which implies that it attained its maximum at $y = 1$ and $c = 2$,in which case

$$\left|\frac{1}{3}c_1c_3 - \frac{1}{4}c_2^2 + \frac{5}{12}c_1^4 - \frac{1}{2}c_1^2c_2\right| \leq 3$$

Equality is attained for $k^{-1}(\xi)$, the inverse of the Koebe function. □

Theorem 2. Let t^{-1} be defined in (2) if $t \in C_v$. Then

$$|\mu_2\mu_4 - \mu_3^2| \leq \frac{1}{8}$$

.
The result is sharp.
Proof. Since $t \in C_v$,it follows from (3) that there exists $p \in P$ such that

$$(\xi t'(\xi))' = t'(\xi)p(\xi)$$

Equating coefficients, we get

$$\alpha_2 = \frac{c_1}{2}$$
$$\alpha_3 = \frac{c_2}{6} + \frac{c_1^2}{6}$$
$$\alpha_4 = \frac{c_3}{12} + \frac{c_1c_2}{8} + \frac{c_1^3}{24}$$

these yields

$$\mu_2 = \frac{-c_1}{2}$$
$$\mu_3 = \frac{-c_2}{6} + \frac{c_1^2}{3}$$
$$\mu_4 = \frac{-c_3}{12} + \frac{7c_1c_2}{24} - \frac{c_1^3}{4}$$

So, we have

$$|\mu_2\mu_4 - \mu_3^2| = \frac{1}{432}|18c_1c_3 - 15c_1^2c_2 - 12c_2^2 + 6c_1^4|$$

Now, assume that $c_1 = c$, $(0 \le c \le 2)$, we have:

$|18c_1c_3 - 15c_1^2c_2 - 12c_2^2 + 6c_1^4|$

$$= \left|-\frac{9c^2(4-c^2)x}{2} - \frac{3(4-c^2)(c^2+8)x^2}{2} + 9c(4-c^2)(1-|x|^2)\xi\right|$$

and application of triangle inequality shows that

$$|18c_1c_3 - 15c_1^2c_2 - 12c_2^2 + 6c_1^4| \le 9c(4-c^2) + \frac{9c^2(4-c^2)y}{2} + \frac{3(4-c^2)(c-2)(c-4)y^2}{2}$$

with $y = |x| \le 1$. And

$$F'(y) = \frac{9c^2(4-c^2)}{2} + 3(4-c^2)(c-2)(c-4)y > 0$$

Thus $F(\mathrm{y})$ is an increasing function that attained its maximum at $\mathrm{y} = 1$. The upper bound for the above corresponds to $\mathrm{y} = 1$ and $c = 1$,in which case

$|18c_1c_3 - 15c_1^2c_2 - 12c_2^2 + 6c_1^4| \le 54.$

Letting $c_1 = 1$, $c_2 = 2$ and $c_3 = 1$ shows that the result is sharp. □

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