Somewhat mr-Continuous and Somewhat mr-Open Functions

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Abstract
The present article introduces and investigates new classes of functions, namely somewhat mr-continuous and somewhat mr-open functions, by utilizing minimal regular open sets and minimal regular closed sets. The paper establishes the relationship between these new categories and other classes of functions, such as somewhat continuous, somewhat r-continuous and completely continuous functions, while also providing examples, counter examples, and various properties. The study of these classes of functions represents a significant contribution to the field of mathematics and provides new insights into the properties and behaviour of functions in various contexts.

Keywords. somewhat mr – continuous, somewhat mr – open, mr – space, mr – seperable.

1. Introduction
Karl R Gentry and Hughes B Hoyle [4] studied the concept of Somewhat continuous functions and somewhat open functions. Anuradha N and Baby Chacko [1] introduced and discussed minimal regular open sets and maximal regular open sets. In this paper a new type of somewhat continuous and open functions namely somewhat mr-continuous and somewhat mr-open functions are introduced. These types of functions were discussed in sections (3) and (4). In section (2) basic definitions are introduced. In section (3) we define somewhat mr-continuous functions and study its properties. Characterizations of somewhat mr-continuous functions are given and its relation with some other types of functions is also studied. Section (4) deals with the definition of somewhat mr-open function and its properties.

2. Preliminaries
Definition 2.1. [4] A function \( g: (X, \gamma) \rightarrow (Z, \mu) \) is said to be Somewhat continuous if for \( V \in \mu \) and \( g^{-1}(V) \neq \emptyset \), there exists an open set \( W \) in \( X \) such that \( W \neq \emptyset \) and \( W \subset g^{-1}(V) \).
Definition 2.2. [2] A function \( g: (X, \gamma) \rightarrow (Z, \mu) \) is said to be Somewhat r-continuous if for \( V \in \mu \) and \( g^{-1}(V) \neq \emptyset \), there exists a regular open set \( W \) in \( X \) such that \( W \neq \emptyset \) and \( W \subset g^{-1}(V) \).
Definition 2.3. A function \( g: (X, \gamma) \rightarrow (Z, \mu) \) is said to be cl-super Continuous [8] (clopen continuous [7]) if for each \( x \in X \) and each open set \( V \) containing \( g(x) \) there exists clopen set \( W \) containing \( x \) such that \( g(W) \subset V \).
Definition 2.4. A function \( g: (X, \gamma) \rightarrow (Z, \mu) \) is said to be \( \delta \) – Continuous [7] if for each \( x \in X \) and for
each regular open set V containing \( g(x) \) there exists a regular open set W containing x such that \( g(W) \subset V \).

**Definition 2.5.** A function \( g: (X, \gamma) \to (Z, \mu) \) is said to be completely Continuous [3] if \( g^{-1}(W) \) is a regular open set in X, for every open set \( W \subset Z \).

**Definition 2.6.** A function \( g: (X, \gamma) \to (Z, \mu) \) is said to be almost completely Continuous [6] if \( g^{-1}(W) \) is a regular open set in X, for every regular open set \( W \subset Z \).

**Definition 2.7.** [9] A proper non-empty regular open subset U of a topological space \( (X, \tau) \) is said to be minimal regular open, if any regular open set which is contained in U is \( \emptyset \) or U. A proper non-empty regular closed subset F of a topological space \( (X, \tau) \) is said to be minimal regular closed, if any regular closed set which is contained in F is \( \emptyset \) or F.

**Definition 2.8.** [9] A proper non-empty regular open subset U of a topological space \( (X, \tau) \) is said to be maximal regular open, if any regular open set which contains U is X or U. A proper non-empty regular closed subset F of a topological space \( (X, \tau) \) is said to be maximal regular closed, if any regular closed set which contains F is X or F.

**Definition 2.9.** [5] If \( X \) is a set and \( \tau_1 \) and \( \tau_2 \) are topologies for \( X \). Then \( \tau_2 \) is said to be stronger than \( \tau_1 \) (or \( \tau_1 \) is weaker than \( \tau_2 \)) provided if \( U \in \tau_1 \) and \( U \neq \emptyset \), then there is an open set \( V \) in \( (X, \tau_2) \) such that \( V \subset U \).

**Definition 2.10.** [5] A is dense in \( X \) if and only if the only closed subset of \( X \) containing \( A \) is \( X \) itself.

### 3. Somewhat \( mr \)-Continuous functions

**Definition 3.1.** Let \( (X, \gamma) \) and \( (Z, \mu) \) be any two topological spaces. A function is said to be somewhat \( mr \)-continuous if for each \( V \in \mu \) and \( g^{-1}(V) \neq \emptyset \) there exists a minimal regular open set \( W \) in \( X \) such that \( W \neq \emptyset \) and \( W \subset g^{-1}(V) \).

**Example 3.1.** Let \( X = Z = \{1, 2, 3\} \)
\[ \gamma = \{X, \emptyset, \{2\}, \{1, 3\}\} \]
\[ \mu = \{X, \emptyset, \{1, 3\}\} \]

Define \( g: (X, \gamma) \to (Z, \mu) \) by \( f(1) = 3, f(2) = 2, f(3) = 1 \)

Here \( g^{-1}(\emptyset) = \emptyset, g^{-1}(X) = X, g^{-1}({1, 3}) = \{1, 3\} \) and \( \{1, 3\} \) is a minimal regular open set, then \( g \) is somewhat \( mr \)-continuous.

**Theorem 3.1.** Every somewhat \( mr \)-continuous functions are somewhat \( r \) – continuous.

**Remark:** The opposite proposition does not holds.

**Example 3.2.** Let \( X = Z = \mathcal{R}, \gamma = \mu \) = usual topology on \( \mathcal{R} \).

Define \( g: (X, \gamma) \to (Z, \mu) \) by \( f(x) = x, x \in \mathcal{R} \), then \( g \) is somewhat \( r \) – continuous function but \( g \) is not somewhat \( mr \) – continuous function, since \( g^{-1}(a, b) = (a, b) \), there does not exist minimal regular open set contained in \( (a, b) \).

**Remark:** Minimal regular open set need not be a minimal open set and Minimal open set need not be a minimal regular open set.

Let \( X = \{ p, q, r, s \} \) with \( \gamma = \{ X, \emptyset, \{ p, q \}, \{ r, s \}\} \).
Here \( \{p, q\} \) is a minimal regular open but not a minimal open.

**Theorem 3.2.** Any function from a discrete space to any other space is somewhat \( mr \) – continuous.

**Proof.** Let \( g: (X, \gamma) \to (Z, \mu) \) with \( \gamma = \) discrete topology and \( \mu \) be any topology. For each \( V \in \mu \) with \( g^{-1}(V) \neq \emptyset \). There exist some \( x \in g^{-1}(V) \), then \( \{x\} \) is a minimal regular open set contained in \( g^{-1}(V) \).
Implies that $g$ is a somewhat $mr$ – continuous function.

**Definition 3.2.** A space $X$ is said to be $mr$ – space if for each $x \in X$ and every $r$ – neighbourhood (regular open set) $V$ of $x$, there exist a minimal regular open set $W$ such that $W \subset V$ and $x \in W$.

**Example 3.3.**
1. Every discrete space is $mr$ – space.
2. $X = \{p, q, r\}$ with $\gamma = \{X, \emptyset, \{q\}, \{p, q\}\}$, $(X, \gamma)$ is $mr$ – space.

**Theorem 3.3.** Every finite space is $mr$ – space.

**Theorem 3.4.** Every $cl$-super continuous function in $mr$-space is somewhat $mr$ – continuous.

*Proof.* Let $g: (X, \gamma) \rightarrow (Z, \mu)$ be $cl$ – super continuous and $X$ is $mr$ – space. For each $x \in X$ and open set $V$ containing $g(x)$, there exists a clopen set $U$ containing $x$ such that $g(U) \subset V$. Since $X$ is $mr$ – space and $U$ is regular open, there exist a minimal regular open set $W$ containing $x$ such that $W \subset U$. Then, there exists a minimal regular open set $W$ such that $x \in W$ and $g(W) \subset V$. Then $W \subset g^{-1}(V)$, since $W \subset g^{-1}(g(W)) \subset g^{-1}(V)$. Implies that $g$ is somewhat $mr$ – continuous.

**Example 3.4.** $X = \{p, q, r, s\}$, $Z = \{p, q, r\}$
$\gamma = \{X, \emptyset, \{q\}, \{p\}, \{q\}, \{p, q, s\}\}$
$\mu = \{Z, \emptyset, \{p\}, \{q\}\}$
Let $g: (X, \gamma) \rightarrow (Z, \mu)$ be the identity function. Then $g$ is somewhat $mr$ – continuous, but $g$ is not $cl$ – super continuous.

**Definition 3.3.** [10] A topological space is locally indiscrete if every open set is closed.

**Theorem 3.5.** Let $g: (X, \gamma) \rightarrow (Z, \mu)$ be somewhat $mr$ – continuous and $X$ is locally in-discrete. Then $f$ is $cl$ – super continuous.

**Theorem 3.6.** Every completely continuous function in $mr$-space is somewhat $mr$ – continuous.

*Proof.* Let $g: (X, \gamma) \rightarrow (Z, \mu)$ be a completely continuous function. For each $W \in \mu$ with $g^{-1}(W) \neq \emptyset$ is regular open. Since $X$ is $mr$ – space, there exist a minimal regular open set $W$ such that $V \subset g^{-1}(W)$. Which implies $g$ is somewhat $mr$ – continuous.

**Remark:** The opposite proposition does not hold.

**Example 3.5.** Let $X = \{p, q, r\} = Z$ and $\gamma = \{X, \emptyset, \{p\}, \{p, q\}\}, \mu = \{Z, \emptyset, \{p, q\}\}$.
Let $g: (X, \gamma) \rightarrow (Z, \mu)$ be the identity function. Then $g$ is somewhat $mr$ – continuous. Since int $cl\{\{p, q\}\} = int cl \{\{p, q\}\} = intX = X$ and $\{q, p\}$ is not regular. Then $g$ is not completely continuous.

**Theorem 3.7.** If $X$ is a discrete space and $g: (X, \gamma) \rightarrow (Z, \mu)$ is somewhat $mr$-continuous, then $g$ is completely continuous.

**Corollary 3.7.1** Let $g: (X, \gamma) \rightarrow (Z, \mu)$ is somewhat $mr$ – continuous, where $X$ is finite and $T_1$, then $g$ is completely continuous.

**Theorem 3.8.** Every somewhat $mr$-continuous function is $\delta$ – continuous.

*Proof.* Let $g: (X, \gamma) \rightarrow (Z, \mu)$ be somewhat $mr$ – continuous. Let $V$ be a non-empty regular open set in $Z$, then it is open. Since $g$ is somewhat $mr$ – continuous, there exists a minimal regular open set $U$ such that $U \subset g^{-1}(V)$. Since every minimal regular open set is regular open, then $g$ is $\delta$ – continuous.

**Remark:** The opposite proposition does not hold.

**Example:** Let $X = Z = \{p, q, r\}, \gamma = \{X, \emptyset, \{q\}, \{r\}\}, \mu = \{Z, \emptyset, \{p\}, \{q\}, \{q, r\}\}$.
Define $g: (X, \gamma) \rightarrow (Z, \mu)$ by $g(p) = q, g(q) = r, g(r) = p$. Then $g$ is $\delta$ – continuous, but not somewhat $mr$ – continuous.
Theorem 3.9. Let $W$ be any non-empty finite regular open set. Then there exists at least one (finite) minimal regular open set such that $V \in W$.

Theorem 3.10. Let $X$ be finite, $g: (X, \gamma) \to (Z, \mu)$ is somewhat $r -$ continuous if and only if $g$ is somewhat $mr -$ continuous.

Proof. Suppose $g: (X, \gamma) \to (Z, \mu)$ is somewhat $r -$ continuous and $X$ is finite. Let $U \in \mu$, and $g^{-1}(U) \neq \emptyset$. Then there exist a regular open set $V$ such that $V \subseteq f^{-1}(U)$. Since $X$ is finite, $V$ is a finite regular open set. Then by the Theorem 3.9, there exist a minimal regular open set $W$ such that $W \subseteq V \subseteq g^{-1}(U)$. That implies $g$ is somewhat $mr -$ continuous. Converse part is follows from the definition.

Theorem 3.11. If $g: (X, \gamma) \to (Z, \mu)$ is almost completely continuous and $Y$ is locally indiscrete and $X$ is $mr$-space, then $g$ is somewhat $mr -$ continuous.

Proof. Let $V$ be open in $Z$ and $g^{-1}(V) \neq \emptyset$. Since $Z$ is locally in-discrete, $V$ is clopen and so is regular open. Since $g$ is almost completely continuous $g^{-1}(V)$ is regular open. Since $X$ is $mr$ - space there exist a minimal regular open set $W$ such that $W \subseteq g^{-1}(V)$. There for $g$ is somewhat $mr -$ continuous.

Theorem 3.12. If $g: (X, \gamma) \to (Z, \mu)$ is somewhat $mr$-continuous and $X$ is a discrete space, then $g$ is almost completely continuous.

Proof. Let $V$ be a regular open set in $Z$ and $g^{-1}(V) \neq \emptyset$. Since $g$ is somewhat $mr$ - continuous, there exist a minimal regular open set $W$ such that $W \subseteq g^{-1}(V)$. Since $X$ is $mr$ - space $g^{-1}(V)$ is clopen, then it is regular open. Therefore $g$ is almost completely continuous. It is follows from the definition, any function from discrete space is almost completely continuous.

Corollary 3.12.1. If $g: (X, \gamma) \to (Z, \mu)$ is somewhat $mr$ - continuous and $X$ is finite and $T_1$, then $f$ is almost completely continuous.

Theorem 3.13. $g: (X, \gamma) \to (Z, \mu)$ somewhat continuous and $X$ be locally indiscrete and finite, then $g$ is somewhat $mr$ - continuous function.

Proof. For each $U \in \mu$ with $g^{-1}(U) \neq \emptyset$ there exists open set $V \neq \emptyset$ and $V \subseteq g^{-1}(U)$. Since $X$ is locally in-discrete, $V$ is closed. Then, $V$ is regular open. Since $X$ is finite, then $V$ is a nonempty finite regular open set. By the Theorem 3.9 there exist a minimal regular open set $W$ such that $W \subseteq V$. Implies $W \subseteq V \subseteq g^{-1}(U)$, then $W \subseteq g^{-1}(U)$. Since $U \in \mu$ is arbitrary, therefore $g$ is somewhat $mr$ - continuous function.

Definition 3.4. A topological space is minimal space ($m$ - space) if every open set in $X$ is minimal open.

Theorem 3.14 $g: (X, \gamma) \to (Z, \mu)$ is $\delta$ - continuous and open function and $Z$ is $m$-space, then $g$ is somewhat $mr$ - continuous.

Proof. Let $U \in \mu$ and $g^{-1}(U) \neq \emptyset$, since $Z$ is $m$ - space, $U$ is minimal open. Since $g$ is $\delta$ - continuous, for each $x \in g^{-1}(U)$, there exist a regular open set $V$ containing $x$, such that $g(V) \subseteq U$, implies $V \subseteq g^{-1}(U), V \neq \emptyset$. If $V$ is minimal regular open set, then $g$ is somewhat $mr$ - continuous. On contrary, suppose that $V$ is not a minimal regular open set, then there exists a nonempty regular open $W$ such that $W \subseteq g^{-1}(U)$. Then $g(W) \subseteq U$ and $g(W)$ is open, since $g$ is an open function. Implies $g(W) \neq \emptyset, g(W) \subseteq U$. Which is a contradiction, since $U$ is a minimal open set.

Theorem 3.15. Every $m$ - space contains at most two proper open set.

Proof. Suppose that $X$ contains more than two open sets. Without loss of generality $X$ contains three open sets $A, B, C$.  


Let $D = A \cup B$, open set.

Case I: If $D \neq X$.

Since $D = A \cup B$, then $A \subset D$ and $B \subset D$. Since $D$ is open, then it is minimal open. Which is contradiction.

Case II: If $D = X$

$C \subset A \cup B \Rightarrow C \cap A \neq \emptyset$ and $C \cap B \neq \emptyset$. $C \cap A$ is open set and $C \cap A \subset C$, $C \cap B$ is open set and $C \cap B \subset C$. Which is a contradiction, since $C$ is minimal open set.

If possible $C \cap B = \emptyset \Rightarrow C \subset A$. Which is a contradiction, since $A$ is a minimal open. Similarly we get a contradiction if $C \cap A = \emptyset$. Therefore we can conclude that every $m$–space contains at most two proper open set.

**Theorem 3.16.** Let $g: (X,\gamma) \to (Y, \mu), g: (Y, \mu) \to (Z, \eta)$ be any two functions. If $f$ is somewhat $mr$–continuous function, and $g$ is continuous then $g \circ f$ is somewhat $mr$–continuous.

**Proof.** Let $U \in \eta$ and $(g \circ f)^{-1}(U) \neq \emptyset$. That is $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Since $g$ is continuous $(g^{-1}(U)) \in \mu$ and $(g^{-1}(U)) \neq \emptyset$. Since $f$ is somewhat $mr$–continuous, there exist a minimal regular open set in $X$, $V \neq \emptyset$ such that $V \subset f^{-1}(g^{-1}(U))$. That is $V \subset (g \circ f)^{-1}(U), V \neq \emptyset$ and $V \in \gamma$. Therefore $g \circ f$ is a somewhat $mr$–continuous function.

**Remark:** Somewhat $mr$–continuous function need not be continuous.

$X = Z = \{a, b, c, d\}, \gamma = \{X, \emptyset, \{b\}, \{a, c\}, \{a, b, c\}\}, \mu = \{X, \emptyset, \{a, b\}, \{a, b, c\}\}$.

Define $g: (X, \gamma) \to (Y, \mu)$ by $g(a) = a, g(b) = b, g(c) = d, g(d) = c$.

Here $g^{-1}\{(a, b, c)\} = \{a, b, d\}$, $g^{-1}\{(a, b)\} = \{a, b\}$ and $\{b\}$ is minimal regular open set in $(X, \gamma)$. Also $g^{-1}\{(a, b, c)\} = \{a, b, d\}$ and $g^{-1}\{a, b\} = \{a, b\}$ not open in $(X, \gamma)$. Therefore $g$ is somewhat $mr$–continuous function, but not continuous.

**Theorem 3.17.** If $f: (X, \gamma) \to (Y, \mu)$ is continuous $g: Y \to Z$ is somewhat $mr$–continuous, then $g \circ f: X \to Z$ is somewhat continuous.

**Proof.** Let $W \subset Z$ be open and $(g \circ f)^{-1}(W) \neq \emptyset, then g^{-1}(W) \neq \emptyset$. Since $g$ is somewhat $mr$–continuous function, there exist minimal regular open set $V$ such that $V \subset g^{-1}(W)$. Since $V \neq \emptyset$ is open and $f$ is continuous, implies $f^{-1}(V) \subset f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ and $f^{-1}(V) \neq \emptyset$ is an open set. Therefore $g \circ f: X \to Z$ is somewhat continuous.

**Theorem 3.18.** Let $(X, \gamma)$ be a topological space and $B$ be a regular open set of $X$ and $W$ be a minimal regular open subset of $(B, \gamma / B)$, then $W$ is a minimal regular open subset of $X$.

**Proof.** Let $W$ is a regular open subset of $(B, \gamma / B)$ and $W = T \cap B, T$ is regular open subset of $X$ and $B$ is a regular open subset of $X$. Implies $W$ is regular open subset of $X$.

If possible $W$ is not minimal regular open subset of $X$, there exist a regular open set $S$ of $X$ such that $S \subset W$. $(S \cap B) \subset (W \cap B) = T \cap B = W, S \cap B \subset W$ and $S \cap B$ is a regular open subset of $(B, \gamma / B)$. But $W$ is a minimal regular open subset of $(B, \gamma / B)$, which is a contradiction. Implies that $W$ is a minimal regular open subset of $X$.

**Theorem 3.19.** Let $(X, \gamma)$ and $(Z, \mu)$ be any two topological spaces. Let $B$ be a regular open set of $(X, \gamma)$ and $g: (X, \gamma / B) \to (Z, \mu)$ be somewhat $mr$–continuous such that $g(B)$ is dense in $Z$. Then any extension $G$ of $g$ is somewhat $mr$–continuous.

**Proof.** Let $U$ be any open set in $(Z, \mu)$ such that $G^{-1}(U) \neq \emptyset$. Since $g(B) \subset Z$ is dense in $Z, U \cap f(B) \neq \emptyset$. So $g^{-1}(U) \cap B \neq \emptyset$. Since $g$ is somewhat $mr$–continuous, there exist a minimal regular open set $V$ with respect to $\gamma / B$ such that $V \subset g^{-1}(U)$. Since $V$ is minimal regular open set with respect
to $\gamma/B$ and $B$ is regular open subset of $X$, then $V$ is a minimal regular open subset of $X$ with respect to $\gamma$ and $V \subset G^{-1}(U)$. Implies that $G$ is somewhat $mr$-continuous.

**Theorem 3.20.** Let $(X, \gamma)$ be a topological space and $B$ be a regular open set of $X$ and $U$ be a minimal regular open subset of $(B, \gamma/B)$, then $U$ is a minimal regular open set of $X$.

**Proof.** Let $U$ is a regular open subset of $(B, \gamma/B)$ and $U = W \cap B$, $W$ is regular open subset of $X$ and $B$ is a regular open subset of $X$. That is, $U$ is regular open subset of $X$. If possible $U$ is not minimal regular open set of $X$, there exist a regular open set $W_1$ of $X$ such that $W_1 \neq \emptyset$ and $W_1 \subset U$. Then $W_1 \cap B \subset U \cap B = U$. Since $W_1$ is regular open set in $X$, $W_1 \cap B \neq \emptyset$ is regular open set in $B$. That implies $U$ is not minimal regular open subset of $B$, which is a contradiction. Therefore $U$ is a minimal regular open subset of $X$.

**Definition 3.5.** Let $N$ be a subset of a topological space $(X, \gamma)$. Then $N$ is said to be $mr$-dense in $X$ if there does not exist a maximal regular closed set $D$ in $X$ such that $N \subset D \subset X$.

**Theorem 3.21.** Let $X$ be a topological space and $E \subset X$. $E$ is a minimal regular closed set if and only if $X - E$ is a maximal regular open.

**Theorem 3.22.** Let $X$ be a topological space and $T \subset X$. $T$ is a maximal regular open set if and only if $X - T$ is a maximal regular closed set.

**Theorem 3.23.** Let $(X_1, \tau_1)$ and $(X_2, \tau_2)$ be an injective function. Then the following are equivalent.
1. $g$ is somewhat $mr$-continuous.
2. If $B$ is a closed subset of $X_2$ such that $g^{-1}(B) \neq X_1$, then there is a minimal regular closed subset $E$ of $X_1$ such that $E \supset g^{-1}(B)$.
3. If $N$ is an $mr$-dense in $X_1$, then $g(N)$ is a dense subset of $X_2$.

**Proof.** (1) $\Rightarrow$ (2)

Let $B$ be a closed subset of $X_2$ such that $g^{-1}(B) \neq X_1$, then $X_2 - B$ is open in $X_2$ such that $g^{-1}(X_2 - B) = X_1 - g^{-1}(B) \neq \emptyset$. Since $g$ is somewhat $mr$-continuous, there exist a minimal regular open set $W$ such that $W \subset X_1 - g^{-1}(B) \Rightarrow g^{-1}(B) \subset X_1 - W$. Since $W$ is a minimal regular open, then $E = X_1 - W$ is maximal regular closed. That is there exists maximal regular closed set $E$ such that $E \supset g^{-1}(B)$.

(2) $\Rightarrow$ (3)

Let $N$ be a $mr$-dense subset of $X_1$. Suppose $g(N)$ is not dense in $X_2$. Then there exists a proper closed set $B$ in $X_2$ such that $g(N) \subset B \subset X_2$. Clearly $g^{-1}(B) \neq X_1$. Hence by ii), there exists a maximal regular closed set $E$ such that $E \supset g^{-1}(B)$. That is $N \subset g^{-1}(B) \subset E \subset X_1$. This contradicts the fact that $N$ is $mr$-dense in $X_1$. So $g(N)$ is dense in $X_2$.

(3) $\Rightarrow$ (2)

Suppose ii) is not true, this means that there exists a closed set $B$ in $X_2$ such that $g^{-1}(B) \neq X_1$. But there is no maximal regular closed set in $E$ in $X_1$ such that $g^{-1}(B) \subset E$. This means $g^{-1}(B)$ is $mr$-dense in $X_1$. But by iii) $g(g^{-1}(B)) = B$ must be dense in $X_2$. Which is contradiction to the choice of $B$. So ii) is true.

(2) $\Rightarrow$ (1)

Let $T \subset E$ and $g^{-1}(T) \neq \emptyset$. Then $X_2 - T$ is closed in $X_2$ and $g^{-1}(X_2 - T) = X_1 - g^{-1}(T) \neq X_1$. So by ii) there exists a maximal regular closed subset $E$ of $X_1$ such that $E \supset g^{-1}(X_2 - T) = X_1 - g^{-1}(T)$. That is $X_1 - E \subset g^{-1}(T)$ and $X_1 - E$ is a minimal regular open subset. So $g$ is somewhat $mr$-continuous function.
Theorem 3.24. Every dense subset in topological space \((X, \gamma)\) is m\(r\) – dense.

Proof. Let \((X, \gamma)\) be any topological space and \(C\) be any dense subset of \(X\). Then by the definition, there does not exist a proper closed set \(V\) such that \(C \subset V \subset X\). That implies there does not exist any maximal regular closed set \(F\) satisfying this property \(C \subset F \subset X\) since every maximal regular closed set is a closed set. Which implies \(C\) is m\(r\) – dense in \(X\).

Remark: m\(r\) – denseness of a set in a space \(X\) does not imply the denseness of that set.

Example 3.6. \(X = \{a, b, c, d\}, \gamma = \{\emptyset, \{a\}, \{a, b\}, X\}, \{a, b\} = X, \{a, b\} \) is dense.

\([a] = X, \{a\} \) is dense and \([b, c]\) = \(\{b, c, d\}\), \(\{b, c\}\) is not dense. But \(\{b, c\}\) is m\(r\) – dense. Because each closed set is not regular closed, there is no maximal regular closed set containing \(\{b, c\}\) and property contained in \(X\).

Theorem 3.25. Let \((X_1, \gamma_1)\) and \((X_2, \gamma_2)\) be any two topological spaces. \(X_1 = D_1 \cup D_2\) where \(D_1\) and \(D_2\) are regular open subsets of \(X_1\). Let \(g: (X_1, \gamma_1) \rightarrow g: (X_2, \gamma_2)\) be a function such that \(g|D_1\) and \(g|D_2\) are somewhat m\(r\)–continuous. Then \(g\) is a somewhat m\(r\)– continuous function

\[\text{Proof.} \] Let \(V\) be any open set in \((X_2, \gamma_2)\) such that \(g^{-1}(V) \neq \emptyset\). Then either \((g|D_1)^{-1}(V) \neq \emptyset\) or \((g|D_2)^{-1}(V) \neq \emptyset\).

Case-1: \((g|D_1)^{-1}(V) \neq \emptyset\)

Since \(g|D_1\) is somewhat m\(r\)–continuous, there exists minimal regular open set \(W\) in \(D_1\) such that \(W \neq \emptyset\) and \(W \subset (g|D_1)^{-1}(V) \subset g^{-1}(V)\). Since \(W\) is a minimal regular open set in \(D_1\) and \(D_1\) is a regular open in \(X_1\), then \(W\) is a minimal regular set. So \(g\) is somewhat m\(r\)–continuous.

Case-2: \((g|D_2)^{-1}(V) \neq \emptyset\)

This can be proved by the same argument as in case-1

Case-3: \((g|D_2)^{-1}(V) \neq \emptyset\) and \((g|D_2)^{-1}(V) \neq \emptyset\)

The proof follows from the proofs of case-1 and case-2.

Definition 3.6. A topological space \((Z, \mu)\) is said to be m\(r\)–seperable if there exists a countable subset \(D\) of \(Z\) which is m\(r\)–dense in \(Z\) (or if there exists a countable m\(r\)–dense subset \(D\) of \(Z\)).

Example 3.7. Let \(Z = \{a, b, c\}, \mu = \{Z, \emptyset, \{a\}, \{a, b\}\}\).

Then \(\{a, b\}\) is m\(r\)–dense in \(Z\) and it is countable. So \(Z\) is m\(r\)–seperable.

Theorem 3.26. If \(g\) is somewhat m\(r\)–continuous function from \(Z_1\) onto \(Z_2\) and if \(Z_1\) is m\(r\)–seperable, then \(Z_2\) is seperable.

\[\text{Proof.} \] Let \(g: Z_1 \rightarrow Z_2\) be somewhat m\(r\)–continuous function such that \(Z_1\) is m\(r\)–seperable. Then there exists a countable set \(B\) of \(Z_1\) which is m\(r\)–dense in \(Z_1\). Then by theorem 3.23 \(g(B)\) is dense in \(Z_2\). Since \(B\) is countable and \(g\) is onto then \(g(B)\) is countable, so \(Z_2\) is seperable.

Definition 3.7. If \(X\) is a set and \(\tau_1\) and \(\tau_2\) are topologies for \(X\). Then \(\tau_2\) is said to be m\(r\) – weakly stronger than \(\tau_1\) (or \(\tau_1\) is m\(r\) – weakly weaker than \(\tau_2\)) provided if \(U \in \tau_1\) and \(U \neq \emptyset\), then there is a minimal regular open set \(V\) in \((X, \tau_1)\) such that \(V \subset C\).

Example 3.8. \(X = \{a, b, c\}, \tau_1 = \{X, \emptyset, \{b, c\}, \{a\}\}\).

\(\text{int} \{a\} = \{a\}, \{a\} \) is minimal regular open
\(\text{int} \{b, c\} = \{b, c\}, \{b, c\} \) is a minimal regular open.
\(\tau_2 = \{X, \emptyset, \{a, b\}\}\)
\(\text{int} \{a, b\} = \text{int} X = X, \{a, b\} \) is not regular open.

Here \(\tau_1\) is m\(r\) – weakly stronger than \(\tau_2\).
Theorem 3.27. Let $\tau$ and $\tau^*$ be two topologies in $X_1$ and $\tau^*$ is $mr$-weakly stronger than $\tau$, If $g: (X_1, \tau) \to (X_2, \sigma)$ be a somewhat continuous function. Then the function $g: (X_1, \tau^*) \to (X_2, \sigma)$ is somewhat $mr$ – continuous.

**Proof.** Let $V$ be any open set in $X_2$ such that $g^{-1}(V) \neq \emptyset$. Since $g$ is somewhat continuous, there exists an open set $W$ in $X_1$ such that $W \neq \emptyset$ and $W \subseteq g^{-1}(V)$

Since $\tau^*$ is $mr$ – stronger than $\tau$, there exists a minimal regular open $W_1$ in $(X_1, \tau^*)$ such that $W_1 \subseteq W$. That is $W_1 \subseteq W \subseteq g^{-1}(V)$. So $g$ is somewhat $mr$ – continuous.

**Theorem 3.28.** Let $g: (X_1, \tau) \to (X_2, \sigma)$ be a somewhat $mr$-continuous and onto function. Let $\sigma^*$ be another topology for $X_2$ and $\sigma$ is stronger than $\sigma^*$, then $g: (X_1, \tau) \to (X_2, \sigma^*)$ is somewhat $mr$ – continuous.

**Proof.** Let $V_1$ be an open set in $(Y, \sigma^*)$ with $g^{-1}(V_1) \neq \emptyset$, then $V_1 \neq \emptyset$. Since $\sigma$ is stronger than $\sigma^*$, there exists an open set $V_2$ in $(Y, \sigma)$ such that $V_2 \subseteq V_1$ and $V_2 \neq \emptyset$. Since $g$ is onto, then $g^{-1}(V_2) \neq \emptyset$.

Since $g: (X_1, \tau) \to (X_2, \sigma)$ is a somewhat $mr$ – continuous function, then there exists a minimal regular open set $W$ such that $W \subseteq g^{-1}(V_2)$ and $g^{-1}(V_2) \subseteq g^{-1}(V_1)$. That is $W \subseteq g^{-1}(V_1)$ and $W$ is a minimal regular open set. Therefore $g: (X_1, \tau) \to (X_2, \sigma^*)$ is a somewhat $mr$ – continuous.

4. Somewhat $mr$-open functions

**Definition 4.1.** A function $g: (X_1, \tau_1) \to (X_2, \tau_2)$ is said to be somewhat $mr$ – open provided for $V_1 \in \tau_1$ and $V_1 \neq \emptyset$, there exists a minimal range open set $V_2$ in $X_2$ such that $V_2 \subseteq g(V_1)$.

**Example 4.1.** Let $X_1 = X_2 = \{a, b, c\}, \tau_1 = \{X_1, \emptyset, \{a\}\}, \tau_2 = \{X_2, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. Define a function $g: (X_1, \tau_1) \to (X_2, \tau_2)$ by $g(a) = b, g(b) = c, g(c) = a$. Then $g$ is somewhat $mr$ – open function.

**Definition 4.2.** (1). A function $g: (X_1, \tau_1) \to (X_2, \tau_2)$ is said to be somewhat clopen provided for $V_1 \in \tau_1$ and $V_1 \neq \emptyset$, there exists a clopen set $V_2$ in $X_2$ and $V_2 \neq \emptyset$ such that $V_2 \subseteq g(V_1)$.

**Theorem 4.1.** Let $g: (X_1, \tau_1) \to (X_2, \tau_2)$ be a somewhat clopen function and $X_2$ is finite then $g$ is somewhat $mr$ – open.

**Proof.** Let $g: (X_1, \tau_1) \to (X_2, \tau_2)$ be a somewhat clopen function and let $V_1 \in \tau_1$ and $V_1 \neq \emptyset$. Since $g$ is somewhat clopen, then there exist a clopen set $V_2$ such that $V_2 \subseteq g(V_1)$. Then $V_2$ is regular open set. Since $X_2$ is finite, then $V_2$ is finite regular open set. Then by Theorem 3.9, there exist a minimal regular open set $W$ such that $W \subseteq V_2 \subseteq g(V_1)$. Implies $W \subseteq g(V_1)$ and $W \neq \emptyset$. Therefore $g$ is somewhat $mr$ – open function.

**Theorem 4.2.** If $g: X_1 \to X_2$ is somewhat $mr$-open, where $X_1$ is locally indiscrete, then $g$ is somewhat clopen.

**Proof.** Let $V_1 \neq \emptyset$ be open in $X_1$. Since $g$ is somewhat $mr$ – open, there exists a minimal regular open set $V_2$ in $X_1$ such that $V_2 \subseteq g(V_1)$. But minimal regular open set is open and open set in a locally indiscrete space is a clopen. So $g$ is somewhat clopen.

**Definition 4.3.** (1) A function $g: (X_1, \tau_1) \to (X_2, \tau_2)$ is said to be somewhat open provided for $V_1 \in \tau_1$ and $V_1 \neq \emptyset$, there exists an open set $V_2$ in $X_2$ such that $V_2 \neq \emptyset$ and $V_2 \subseteq g(V_1)$.

**Theorem 4.3.** If $g: X_1 \to X_2$ is somewhat $mr$-open, then $g$ is somewhat open.

**Remark:** The opposite proposition does not hold.

**Example 4.2.** $X_1 = X_2 = \{a, b, c\}, \tau_1 = \{X_1, \emptyset, \{a\}, \{a, b\}\}, \tau_2 = \{X_2, \emptyset, \{b\}\}.$
Define $g: (X_1, \tau_1) \to (X_2, \tau_2)$ by $g(a) = b$, $g(b) = c$, $g(c) = a$ and $g(\{a\}) = \{b\}$, $g(\{a, b\}) = \{b, c\}$. Therefore $g$ is somewhat open. Also intcl(\{b\}) = intX_1 = X_1$, then \{b\} is not regular open. But g is not somewhat mr-open function.

**Theorem 4.4.** If $g: (X_1, \tau_1) \to (X_2, \tau_2)$ is an open map and $h: (X_2, \tau_2) \to (X_3, \tau_3)$ is a somewhat mr-open map, then $h \circ g: (X_1, \tau_1) \to (X_3, \tau_3)$ is a somewhat mr-open map.

**Proof.** Let $V_1 \in \tau_1$ and $V_1 \neq \emptyset$, since $g$ is an open map, $g(V_1)$ is open. Also $g(V_1) \neq \emptyset$. Since $h$ is somewhat mr-open and $g(V_1) \in \tau_2$ with $g(V_1) \neq \emptyset$, there exists a minimal regular open set $V_2$ in $\tau_3$ such that $V_2 \subset h(g(V_1))$, ie, $V_2 \subset (h \circ g)(V_1)$. So $h \circ g$ is somewhat mr-open.

**Theorem 4.5.** If $g: (X_1, \tau_1) \to (X_2, \tau_2)$ is a one-one and onto mapping, then the following are equivalent.

1. $g$ is somewhat mr-open map.
2. If $E$ is a closed subset of $X_1$ such that $g(E) \neq X_2$, then there exists a maximal regular closed subset $F$ of $X_2$ such that $F \neq X_2$ and $F \supset g(E)$.

**Proof.** (1) $\implies$ (2)

Let $E$ be a closed subset of $X_1$ such that $g(E) \neq X_2$. Then $X_1 - E$ is open in $X_1$ and $X_1 - E \neq \emptyset$. Since $g$ is somewhat mr-open there exist a minimal regular open set $U \neq \emptyset$ such that $U \subset g(X_1 - E)$. Let $F = X_2 - U$, clearly $F$ is maximal regular closed set in $X_2$.

We claim that $F \neq X_2$. If possible $F = X_2$, then $U = \emptyset$, a contradiction.

$$U \subset g(X_1 - E) \implies X_2 - U \supset X_2 - g(X_1 - E) \implies F = X_2 - U \supset g(E) \implies F \supset g(E)$$

(2) $\implies$ (1)

Let $U \in \tau_1$ and $U \neq \emptyset$ and let $E = X_1 - U$. Then $E$ is a closed subset of $X_1$ and $g(E) = g(X_1 - U) = X_2 - g(U)$. Also $g(E) \neq X_2$. So by (2), there exists a maximal regular closed subset $F$ of $X_2$ such that $F \neq X_2$ and $F \supset g(E)$. Let $W = X_2 - F$, then $W$ is minimal regular open and $W \neq \emptyset$. Also $W = (X_2 - F) \subset (X_2 - g(E)) = X_2 - (X_2 - g(U))$. Implies $W \subset g(U)$. So $g$ is somewhat mr-open.

**Theorem 4.6.** Let $g: (X_1, \tau_1) \to (X_2, \tau_2)$ is a somewhat mr-open function. Let $B$ be any open subset $X_1$. Then $g/B: (B, \tau_{1/B}) \to (X_2, \tau_2)$ is also Somewhat mr-open.

**Proof.** Let $U_1 \in \tau_{1/B}$ and $U_1 \neq \emptyset$. Since $U_1$ is open in $B$ and $B$ is open in $X_1$, $U_1$ is open in $X_1$. Since $g: (X_1, \tau_1) \to (X_2, \tau_2)$ is somewhat mr-open, there exists a minimal regular open set $U_2$ in $X_2$ such that $U_2 \subset g(U_1)$. Thus for any open set $U_1$ in $\tau_{1/B}$ with $U_1 \neq \emptyset$, there exists minimal regular open set $U_2$ in $X_2$ such that $U_2 \subset (g/B)(U_1))$. So $g/B$ is somewhat mr-open.

**Theorem 4.7.** Let $(X_1, \tau_1)$ and $(X_2, \tau_2)$ be any two topological space and $X_1 = B_1 \cup B_2$, where $B_1$ and $B_2$ are open subsets of $X_1$. Let $g: (X_1, \tau_1) \to (X_2, \tau_2)$ be a function such that $g/B_1$ and $g/B_2$ are somewhat mr-open. Then $g$ is also somewhat mr-open.

**Proof.** Let $U_1 \in \tau_1$ and $U_1 \neq \emptyset$. Then either $(g/B_1)(U_1) \neq \emptyset$ or $(g/B_2)(U_1) \neq \emptyset$ or both $(g/B_1)(U_1) \neq \emptyset$ and $(g/B_2)(U_1) \neq \emptyset$.

**Case-1:** $(g/B_1)(U_1) \neq \emptyset$

Since $g/B_1$ is a somewhat mr-open, there exists a minimal regular open set $U_2$ in $B_1$ such that $U_2 \subset (g/B_1)(U_1) \subset g(U_1)$. Since $U_2$ is a minimal regular open in $B_1$ and $B_1$ is a regular open in $X_1$, then $U_2$ is a minimal regular open set in $X_1$. So $g$ is somewhat mr-open function.

**Case-2:** $(g/B_2)(U_1) \neq \emptyset$

This can be proved by the same argument as in case-1.

**Case-3:** $(g/B_1)(U_1) \neq \emptyset$ and $(g/B_2)(U_1) \neq \emptyset$

$B_1 \subseteq X_1$, $U_2$ is minimal regular open in $X_1$, then $U_2$ is minimal regular open in $X_1$.
Suppose $U_2$ is not minimal regular open set in $X_1$, then there exist a regular open set $U_1$ in $X_1$ such that $U_1 \subset U_2, U_1 \cap B_1 \subset U_2 \cap B_1 = U_2 \implies U_1 \cap B_1 \subset U_2$.

The proofs follow from the proofs of case-1 and case-2.

References