International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

Representations of Lie Algebras of Vector Fields on Pairing Between Gauge Modules and Rudakov Modules

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Abstract
For an irreducible affine variety X over an algebraically closed field of characteristic zero, we define two new classes of modules over the Lie algebra of vector fields on X: gauge modules and Rudakov modules. These modules admit a compatible action of the algebra of functions on X. We prove general simplicity theorems for these two types of modules, demonstrating their irreducibility under specific conditions. Additionally, we establish a pairing between gauge modules and Rudakov modules, highlighting the connections and interactions between these two classes of modules. We have established that gauge modules and Rudakov modules, corresponding to simple gl_N -modules, remain irreducible as modules over the Lie algebra of vector fields unless they appear in the de Rham complex. Additionally, we have studied the irreducibility of tensor products of Rudakov modules, providing a comprehensive understanding of these module structures and their applications.

Keywords: Vector Field, Lie Algebra, Gauge Module, Rudakov Module

Introduction
The topic of representations of Lie algebras of vector fields on affine varieties involves studying modules over these Lie algebras, often in the context of algebraic geometry and representation theory. These Lie algebras are associated with the algebraic structure of polynomial rings or coordinate rings on affine varieties, and their modules provide insights into the geometric and algebraic properties of these varieties.

Key aspects of this study include:

Lie Algebra of Vector Fields: Understanding the Lie algebra structure formed by vector fields on affine varieties, typically represented as derivations of the coordinate ring of the variety.

Modules over Lie Algebras: Investigating different types of modules over these Lie algebras, which can include modules of differential operators, modules of differential forms, or more specialized constructions such as gauge modules and Rudakov modules.

Representation Theory: Applying tools from representation theory to analyze these modules, such as studying their irreducibility, simplicity, and constructing specific types of modules related to geometric or physical considerations.

Applications: These studies often have applications in areas like gauge theory, mathematical physics, and algebraic geometry, providing a bridge between geometric intuition and algebraic structures associated with vector fields on varieties.

Historical Context and Classification of Lie Algebras

The classification of complex simple finite-dimensional Lie algebras by Wilhelm Killing in 1889 and Élie Cartan in 1894 was pivotal in the development of Lie theory during the first half of the 20th century. Their work laid the foundation for understanding the structure and classification of Lie algebras, which are closely related to the symmetries of geometric structures.

Connection to Symmetries and Infinite-Dimensional Lie Algebras

Since the time of Sophus Lie, Lie groups and their corresponding Lie algebras (viewed as infinitesimal transformations) have been essential in describing the symmetries of geometric structures. While initially focusing on finite-dimensional structures, the theory later expanded to include infinite-dimensional Lie groups and algebras. These are particularly relevant in the context of systems with an infinite number of independent degrees of freedom, such as those found in Conformal Field Theory.

Sophus Lie and Cartan: Introduction of Simple Infinite-Dimensional Lie Algebras

The exploration of infinite-dimensional Lie algebras began with Sophus Lie, who introduced certain pseudogroups of transformations in low dimensions. This foundational work was later completed by Élie Cartan, who identified four corresponding classes of simple infinite-dimensional Lie algebras:

1. **W_n (Witt algebras)**: These algebras are related to the algebra of derivations of the polynomial ring in nnn variables.
2. **S_n (Special algebras)**: These are the algebras of divergence-free vector fields.
3. **H_n (Hamiltonian algebras)**: These are related to the Hamiltonian vector fields on symplectic manifolds.
4. **K_n (Contact algebras)**: These are associated with contact vector fields on contact manifolds.

These four classes, known as Cartan type algebras, were the first examples of simple infinite-dimensional Lie algebras. Their discovery marked a significant milestone in the study of Lie algebras.

Current State of Infinite-Dimensional Lie Algebra Theory

Despite these early advancements, the general theory of simple infinite-dimensional Lie algebras remains underdeveloped, particularly regarding their representation theory. The complexity and vastness of infinite-dimensional structures pose significant challenges, making this an active area of ongoing research. Understanding the representations of these algebras is crucial for their application in various fields, including mathematical physics and geometry.

The classification work by Killing and Cartan provided a solid foundation for Lie theory, leading to the identification of both finite-dimensional and infinite-dimensional Lie algebras. While significant progress has been made in the finite-dimensional case, the study of infinite-dimensional Lie algebras continues to be a rich and evolving field, with many open questions and potential applications in theoretical physics and beyond.

Rudakov Modules and Their Generalization

Definition and Origin

Rudakov modules $R_p(U)$ are generalizations of certain induced modules over the Lie algebra of derivations of a polynomial ring, initially studied by Rudakov in . These modules are constructed in the following context:

- **Affine Space**: The setting involves an affine space X.
- **Point p**: The modules are associated with a specific point p∈X.
- **Representation U**: U is a finite-dimensional representation of the Lie algebra L^+, which is the Lie algebra of vector fields of non-negative degree on the affine space.

Specific Case Studied by Rudakov

In Rudakov's original work [24], the focus was on the following specific scenario:

- **Affine Space $X=A^n$**: The affine space considered is A^n.
- **Point p=0**: The point associated with the modules is the origin, p=0.
- **Simple Modules U**: The representation U is assumed to be simple.

Generalization

Rudakov modules $R_p(U)$ extend the concept of induced modules by incorporating the following elements:

1. **Representation of L^+:** These modules take into account finite-dimensional representations U of the Lie algebra L^+.
2. **Association with Point ppp**: The modules are linked to a point p in the affine space X, allowing for a broader and more flexible construction than the specific case of the origin p=0.

Importance and Applications

Rudakov modules are significant in the study of the representation theory of Lie algebras, especially in understanding the modules over the Lie algebra of derivations of polynomial rings. Their generalization allows for a deeper exploration of the structure and properties of these modules in various geometric and algebraic settings.

Rudakov modules $R_p(U)$ are a powerful generalization of induced modules studied by Rudakov, associated with a point ppp in an affine space and a finite-dimensional representation UUU of the Lie algebra L^+. This generalization broadens the scope of the original modules, providing a rich framework for further investigation in the representation theory of Lie algebras.

Gauge Modules and Tensor Modules

Definition and Inspiration

Gauge modules are a generalization of tensor modules, which themselves were inspired by concepts from non-abelian gauge theory. In the context of Lie algebras of vector fields on affine varieties, particularly W_n and W_1, these modules play a crucial role in classification theory and representation theory.

Tensor Modules and Their Role

Tensor modules were originally central to the classification theory for W_n, particularly as modules of intermediate series in the case of W_1. They encompass various fundamental structures such as:

- **Algebra of Functions A(X)**: This includes the space of functions defined on XXX, which naturally form a tensor module under suitable actions.

- **Algebra of Vector Fields V(X)**: These are vector fields defined on XXX, which also act on tensor modules.
- **Space of 1-forms Ω1(X)**: The differential forms on XXX form another example of tensor modules.

Construction of Gauge Modules

To construct gauge AV-modules, we localize A(X) by a standard minor hhh of the Jacobian matrix of the defining ideal of X, denoted as A(h). These modules are defined as submodules of UA(h)⊗U, where U is a finite-dimensional L^+ module. Here:

- L^+ represents the Lie algebra of vector fields of non-negative degree on X.
- V(X) involves gauge fields {Bi}} which act on U.

Relationship to Tensor Modules

Gauge modules encompass tensor modules as a special case where the gauge fields BiB_iBi are all zero. This includes classical modules of tensor densities, particularly relevant when XXX is a torus and WnW_nWn is the derivation algebra of Laurent polynomials.

Gauge modules extend the concept of tensor modules by introducing gauge fields and their associated actions into the module structure. They are crucial in the study of Lie algebras of vector fields on affine varieties, providing a framework for understanding representation theory and classification theory in these contexts. This broader perspective enriches the mathematical understanding of symmetries and geometric structures in algebraic and differential geometry.

Conjecture and Main Theorem in Lie Algebra Theory

Conjecture 1: Every AVX-module that is finitely generated over AX is a gauge module.

This conjecture proposes that any *AV X*-module which is finitely generated over *AX* (the algebra of functions on *X*) is actually a gauge module. In other words, the structure and properties of finitely generated modules over *AX* are inherently related to the framework of gauge modules.

Main Theorem

The main theorem asserts the following significant results:

1. **Rudakov Modules Rp(U)**:

- For any non-singular point p∈X , the Rudakov module Rp(U), associated with a finite-dimensional simple $gl_s(k)$-module U, is a simple *AV*-module. This indicates that Rudakov modules provide simple and robust structures within the AVAVAV-module framework.

2. **Gauge AV-modules**:

- If *X* is smooth, then any gauge *AV*-module associated with U (where U is a finite-dimensional simple gls(k)gl_s(k)gls(k)-module) is also simple. This result extends the simplicity property to a broader class of modules that incorporate gauge fields and their associated actions.

Implications and New Families of Simple AV-modules

These results have several implications:

- They provide a method to construct new families of simple AVAVAV-modules using Rudakov modules and gauge modules associated with U.
- The simplicity of these modules underscores their fundamental nature and utility in the study of Lie algebras of vector fields on affine varieties.

The theorem acknowledges that the question of the simplicity of restrictions of Rudakov and gauge modules to the Lie algebra VX (the algebra of vector fields on X) remains open. This suggests that while the main theorem provides foundational results, there are still avenues for further exploration and refinement in the study of these modules.

The conjecture and main theorem contribute significantly to the understanding of AV-modules and their simplicity properties in the context of Lie algebras of vector fields on affine varieties. They highlight the role of Rudakov modules and gauge modules as key constructs in constructing and analyzing simple modules over these algebras, paving the way for further developments in representation theory and algebraic geometry.

Algebraic Variety and Ideal

- **Variety X**: $X \subset A^n$ is an irreducible affine algebraic variety defined over k.
- **Ideal IX**: IX=⟨g1,…,gm⟩ is the ideal in k[x1,…,xn] consisting of all polynomials that vanish on X.

Algebras and Modules

- **Algebra of Functions A=AX**: A=k[x1,…,xn]/ IX is the algebra of polynomial functions on X. It represents the quotient of the polynomial ring by the ideal IXIXIX.
- **Lie Algebra of Vector Fields V=VX: V** $Der_k(A)$ denotes the Lie algebra of polynomial vector fields on X, consisting of k-derivations of A.

Explicit Description of V

- **Lie Algebra Wn**: Wn=Der_k(k[x1,…,xn]) represents the Lie algebra of vector fields on A^n, the ambient space of X.
- **Isomorphism**: There exists an isomorphism of Lie algebras:
  V≅{μ∈Wn|μ(IX)⊆IX}/{μ∈Wn|μ(k[x1,…,xn])⊆IX}.
- This isomorphism indicates that V can be understood as a quotient of certain derivations on A^n that respect the ideal IX.

Module Structure

- **Module Structure**: V acts naturally on A, making A a left V-module. This module structure is fundamental in studying the interactions between functions on X and vector fields on X.

The passage describes an alternative perspective on the Lie algebra V=VX of polynomial vector fields on an affine variety $X \subset A^n$X, particularly focusing on its relationship with the ideal IX and the structure of X.

Matrix Representation and Charts

- **Subalgebra in A^n Derivations**: V can be viewed as a subalgebra of $L_{i=1}^n$ A∂∂xi. This means V consists of all linear combinations of derivatives with respect to xi, where A is the algebra of polynomial functions on X.
- **Matrix J**: Defined by J=$(∂gi∂xj)_{i,j}$ where gi are generators of the ideal IX. This matrix J maps A^n to A^m, and $P_{i=1}^n$fi∂∂xi∈V if and only if (f1,...,fn)∈KerJ.
- **Rank and Minors**: Let r:=rankF where F is the field of fractions of A. hi denotes the nonzero r×r -minors of J. The charts N(hi):=p∈X|hi(p)≠0 cover X, particularly in the case where X is smooth.

Chart Parameters and Derivations

- **Chart Parameters t1,...,ts**: These are algebraically independent over k, forming k[t1,...,ts]⊂ is algebraic over k[t1,...,ts]k[t1, . . . , ts]k[t1,...,ts]. The derivation ∂∂ti of k[t1,...,ts] extends uniquely to a derivation of the localized algebra A(h).
- **Dimensionality and Derivations**: s=dim⁡X and Der(A(h))=Σ$s_{i=1}$A(h)∂∂. Since V=Der(A)⊂Der(A(h)), every vector field ηηη has a unique representation η=Σ$_{si=1}$fi∂∂ for some fi∈A(h).

This setup elucidates how V, the Lie algebra of vector fields on X, interacts with A, the algebra of polynomial functions on X, through its derivation structure and the chart parameters t1,...,. It provides a framework to understand the algebraic and geometric relationships inherent in X and its associated Lie algebra V.

Lemma 2.

Let t1,…,t be standard chart parameters in the chart N(h). Then h∂/∂ti∈V for all i.

Lemma 3.

Let t1,…,ts be standard chart parameters in the chart N(h), and let p∈N(h). Define $t\tilde{}_i=t_i-t_i$. Then t1,…, are local parameters at p

Proof: We need to show that {t1,…,ts} forms a basis for $\frac{mp}{mp2}$. Clearly, ti∈m_p.

Since s=dim⁡X, it suffices to prove linear independence. Suppose $\sum_{i=1}^{s} c_i t_i \in m\frac{2}{p}$ for some c_i∈k. Then $\sum_{i=1}^{s} c_i t\sim i \in m\frac{2}{p}$.

Consider d$\sum_{i=1}^{s}$cit˜i∈m_p for all derivations d∈Der_A. Taking d=h∂/∂_{tk}, we get h(p)ck=0h(p) c_k = 0h(p)ck =0 for all kkk, which implies ck=0 for all k. Thus, {t1,…,ts } is linearly independent in $\frac{mp}{mp2}$ proving that it is a basis.

Rudakov module

The Rudakov module Rp(U) is defined as an induced module: Rp(U):=A#U(V)⊗A#U(V+)U..

Here, A#U(V) denotes the skew group algebra of U(V) over A, and V^+ denotes the vector space V with the opposite action of U. The module U is considered as a module over A#U(V^+) via the action induced by U on V^+.

Theorem 8, we begin with some preliminary results involving the Rudakov module Rp(U). We define a chain of subspaces within Rp(U) as follows: R_0:=1⊗U R_{i+1}:=Ri+V·R_i where R_i:=(0) for i<0. This establishes a filtration: R0⊂R1⊂R2 with . $\sum_{i=1}$R1=Rp(U).

The understand the pairing between gauge modules and Rudakov modules, let's clarify the steps involved:

Lemma 25: Let MMM be an AVAVAV-module that is finitely generated over AAA, and let ppp be a non-singular point of XXX. Define U:=M/mpMU := M / mpMU:=M/mpM.

Proof:

1. **Verification of mpMmpMmpM as a V^+-submodule:** Since M is an AV-module, for μ∈V^+, f∈mp, and m∈M in M, we have: μ·(f·m)=μ(f)·m+f·(μ·m). Here, μ(f)∈mp because μ∈V^+ and f∈mp. Therefore, the right-hand side belongs to mpM, confirming that mpM is closed under the action of V^+.

2. **mpM as an A#U(V+)A -submodule:** Since mpM is closed under the action of V^+, it is also an A-submodule of M. Thus, mpMmp inherits a module structure over A#U(V^+) through the action induced by U, where U=M mpMU=M.
3. **Evaluation module property of U over A:** By definition, U=M mpMU=M The action of A on U satisfies f·u=f(p)·u for f∈A in u∈Uu , indicating that U behaves like an evaluation module over A.

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