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A Comprehensive Study of Hyperbolic-Valued Multi-Norms: Definitions, Properties, and Characterizations

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Abstract:
An ordinary real-valued norm cannot be transferred to a hyperbolic module without specifying an ordered codomain and a decomposition formula, as hyperbolic numbers form a commutative real algebra with nontrivial idempotents and zero divisors. This paper establishes a rigorous framework for multi-norms that take values in the positive hyperbolic cone. The construction commences with the idempotent representation of a hyperbolic module and two classical multi-normed component spaces. Afterward, a hyperbolic-valued multi-norm is componentwise defined and demonstrated to satisfy the triangle inequality, scalar domination, zero stability, repetition consistency, and permutation invariance. The decomposability conditions under which every hyperbolic multi-norm arises from a pair of real multi-norms are identified by a converse characterisation. Componentwise criteria are established for completeness, equivalence, convergence, Cauchy behaviour, and comparison. Concrete examples of finite-dimensional modules and function spaces are computed, and sum and Euclidean tuple gauges are employed as counterexamples to illustrate that norm-like aggregation alone does not imply the repetition axiom. The analysis demonstrates that idempotent coordinates should be employed to address partial order and zero-divisor effects, rather than unrestricted hyperbolic division. The discussion includes applications to product systems, stability estimates, and split-channel models, without the assumption of empirical validation. Open questions pertain to compactness, operator-generated constructions, uniform equivalence across levels, tensor products, and dual multi-norms.

Keywords: hyperbolic numbers; multi-normed space; idempotent decomposition; positive hyperbolic cone; zero divisors; completeness; functional analysis

Mathematical Notation and Symbols

Symbol	Meaning
$\mathbb{D}$	Hyperbolic algebra $\{a+b\mathbf{k} : a,b\in\mathbb{R} \text{ and } \mathbf{k}^2=1\}$.
$\mathbf{e}_1$, $\mathbf{e}_2$	Orthogonal idempotents $(1+\mathbf{k})/2$ and $(1-\mathbf{k})/2$.
$\mathbb{D}^+$	Positive cone $\{u_1\mathbf{e}_1+u_2\mathbf{e}_2 : u_1,u_2\geq 0\}$.
$\preccurlyeq_{\mathbb{D}}$	Componentwise partial order induced by $\mathbb{D}^+$.

Symbol	Meaning
$\lvert\alpha\rvert_{\mathbb{D}}$	Hyperbolic modulus $\lvert\alpha_1\rvert\mathbf{e}_1+\lvert\alpha_2\rvert\mathbf{e}_2$.
$E=\mathbf{e}_1E_1\oplus\mathbf{e}_2E_2$	Idempotent decomposition of a hyperbolic module.
N_n	Level-n hyperbolic-valued multi-norm on E^n.
$N_{n,j}$	Real-valued component multi-norm on E_j^n, $j=1,2$.

1. Introduction

Multi-norm theory equips each Cartesian power E^n of a normed space with a compatible norm. The compatibility axioms retain the geometry of the underlying norm while controlling permutations, coordinatewise scalar multiplication, appended zero coordinates, and repeated coordinates. This level structure is useful in stability theory, operator spaces, tensorial estimates, and Banach-lattice analysis (Dales & Polyakov, 2012; Dales et al., 2014, 2017). Extending the concept to hyperbolic scalars is not a matter of replacing $\mathbb{R}$ by $\mathbb{D}$ in every formula. The algebra $\mathbb{D}$ has zero divisors, is not an ordered field, and contains nonzero elements whose ordinary multiplicative inverse does not exist.

The appropriate order-valued replacement is the cone $\mathbb{D}^+$. Because $\mathbb{D}$ is isomorphic to $\mathbb{R}\times\mathbb{R}$ through its idempotents, inequalities in $\mathbb{D}^+$ are exactly paired real inequalities. Hyperbolic normed modules and foundational functional-analytic principles therefore admit componentwise formulations, provided the decomposition is stated explicitly (Luna-Elizarrarás et al., 2014; Saini et al., 2020). The present paper applies that principle to multi-norms and distinguishes valid multi-norm constructions from merely norm-like tuple gauges.

The objectives are fourfold: to state an axiomatically precise definition; to prove an idempotent construction and converse characterization; to derive topological and completeness criteria; and to calculate examples and genuine failures. Results labelled as derived characterizations follow from the stated axioms and the product-algebra representation; they are not attributed as previously named theorems. The paper is organized from algebraic preliminaries to axioms, structural results, examples, applications, and open problems.

2. Algebraic and Multi-Norm Preliminaries

Definition 2.1. The hyperbolic algebra is $\mathbb{D}=\{a+b\mathbf{k}:a,b\in\mathbb{R},\ \mathbf{k}^2=1\}$. Put $\mathbf{e}_1=(1+\mathbf{k})/2$ and $\mathbf{e}_2=(1-\mathbf{k})/2$. Then $\mathbf{e}_j^2=\mathbf{e}_j$, $\mathbf{e}_1\mathbf{e}_2=0$, and $\mathbf{e}_1+\mathbf{e}_2=1$. Every $\alpha\in\mathbb{D}$ has the unique form $\alpha=\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2$.

$$\alpha\beta=(\alpha_1\beta_1)\mathbf{e}_1+(\alpha_2\beta_2)\mathbf{e}_2,\qquad \alpha^{-1}=\alpha_1^{-1}\mathbf{e}_1+\alpha_2^{-1}\mathbf{e}_2\ \text{ when} \quad (2.1)$$
$\alpha_1\alpha_2\neq0$.

The null cone, or zero-divisor set, is $\mathcal{NC}=\{\alpha:\alpha_1\alpha_2=0\}$. A nonzero element of $\mathcal{NC}$ cannot be divided into both components. Hence ratios of hyperbolic norms are avoided below; bounds are formulated by inequalities or by two real constants.

Definition 2.2. The positive hyperbolic cone and its order are

$$\mathbb{D}^+=\{u_1\mathbf{e}_1+u_2\mathbf{e}_2:u_1,u_2\geq0\},\qquad u\preccurlyeq_{\mathbb{D}}v\ \Leftrightarrow\ u_1\leq v_1\ \text{ and} \quad (2.2)$$
$u_2\leq v_2$.

The order is partial: two elements whose component inequalities point in opposite directions are incomparable. Addition and multiplication by an element of $\mathbb{D}^+$ preserve the order. For $\alpha=\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2$, the hyperbolic modulus is $|\alpha|_\mathbb{D}=|\alpha_1|\mathbf{e}_1+|\alpha_2|\mathbf{e}_2$.

Definition 2.3. A hyperbolic module E is idempotently decomposable when $E=\mathbf{e}_1E_1\oplus\mathbf{e}_2E_2$, where $E_j=\mathbf{e}_jE$ is a real vector space and every $x\in E$ is uniquely $x=x_1\mathbf{e}_1+x_2\mathbf{e}_2$. A $\mathbb{D}^+$-valued norm has the form $\|x\|_\mathbb{D}=\|x_1\|_1\mathbf{e}_1+\|x_2\|_2\mathbf{e}_2$ for real norms $\|\cdot\|_j$.

Definition 2.4. A classical multi-norm on a real normed space E_j is a sequence $N_{n,j}:E_j^n\to[0,\infty)$ satisfying: (M1) permutation invariance; (M2) $N_{n,j}(\alpha_1x_1,\ldots,\alpha_nx_n)\le\max_r|\alpha_r|N_{n,j}(x_1,\ldots,x_n)$; (M3) $N_{n,j}(x_1,\ldots,x_{n-1},0)=N_{n-1,j}(x_1,\ldots,x_{n-1})$; and (M4) $N_{n,j}(x_1,\ldots,x_{n-1},x_{n-1})=N_{n-1,j}(x_1,\ldots,x_{n-1})$, with $N_{1,j}=\|\cdot\|_j$.

Since each $N_{n,j}$ is a norm on E_j^n, it also satisfies definiteness, absolute homogeneity for a common scalar, and the triangle inequality. The four cross-level axioms are the distinctive multi-norm requirements (Dales & Polyakov, 2012).

3. Hyperbolic Multi-Norm Axioms

Definition 3.1. A hyperbolic-valued multi-norm on E is a sequence $N_n:E^n\to\mathbb{D}^+$ such that $N_1=\|\cdot\|_\mathbb{D}$, every N_n is a $\mathbb{D}^+$-valued norm in the componentwise sense, and for $\sigma\in S_n$, $\alpha_r\in\mathbb{D}$, and $x_r\in E$,

$$N_n(x_{\sigma(1)},\ldots,x_{\sigma(n)})=N_n(x_1,\ldots,x_n), \quad (3.1)$$

$$N_n(\alpha_1x_1,\ldots,\alpha_nx_n)\preccurlyeq_\mathbb{D} (\max_\mathbb{D}\{|\alpha_1|_\mathbb{D},\ldots,|\alpha_n|_\mathbb{D}\})N_n(x_1,\ldots,x_n), \quad (3.2)$$

$$N_n(x_1,\ldots,x_{n-1},0)=N_{n-1}(x_1,\ldots,x_{n-1}), \quad (3.3)$$

$$N_n(x_1,\ldots,x_{n-1},x_{n-1})=N_{n-1}(x_1,\ldots,x_{n-1}). \quad (3.4)$$

Here $\max_\mathbb{D}$ is the componentwise maximum. Equation (3.2) is meaningful even when a scalar or norm lies in the null cone; no division occurs. Common-scalar homogeneity follows by applying (3.2) to α and then, componentwise on its nonzero coordinates, to the inverse. Direct component construction below gives exact homogeneity without requiring a global inverse.

Proposition 3.2. Every hyperbolic multi-norm is monotone under the addition of a new coordinate: $N_n(x_1,\ldots,x_n)\preccurlyeq_\mathbb{D}N_{n+1}(x_1,\ldots,x_n,x_{n+1})$.

Proof. Apply scalar domination at level n+1 to the tuple $(x_1,\ldots,x_n,x_{n+1})$ with scalars $(1,\ldots,1,0)$. This gives $N_{n+1}(x_1,\ldots,x_n,0)\preccurlyeq_\mathbb{D}N_{n+1}(x_1,\ldots,x_n,x_{n+1})$. By zero stability, the left side equals $N_n(x_1,\ldots,x_n)$. □

Proposition 3.3. For every tuple, $\max_{1\le r\le n}\|x_r\|_\mathbb{D}\preccurlyeq_\mathbb{D}N_n(x_1,\ldots,x_n)\preccurlyeq_\mathbb{D}\sum_{r=1}^n\|x_r\|_\mathbb{D}$.

Proof. For the lower bound, use a permutation to move x_r to the first position, apply scalar domination with all other scalars zero, and then repeatedly use zero stability. For the upper bound, write the tuple as the sum of n tuples having exactly one nonzero coordinate. The triangle inequality and permutation invariance reduce every term to $N_1(x_r)=\|x_r\|_\mathbb{D}$. Both arguments are componentwise, so the order is $\preccurlyeq_\mathbb{D}$. □

4. Idempotent Construction and Characterization

Theorem 4.1. Idempotent construction. Let $(E_1,N_{n,1})$ and $(E_2,N_{n,2})$ be real multi-normed spaces. For $x_r=x_{r1}\mathbf{e}_1+x_{r2}\mathbf{e}_2$ define the maps in Equation (4.1). These maps form a hyperbolic-valued multi-norm.

$$N_n(x_1,\ldots,x_n)=N_{n,1}(x_{11},\ldots,x_{n1})\mathbf{e}_1+N_{n,2}(x_{12},\ldots,x_{n2})\mathbf{e}_2. \quad (4.1)$$

Proof. Positivity and definiteness follow because each coefficient is a nonnegative real number and both vanish exactly when every x_{rj} is zero. For tuples x and y, the component triangle inequalities give $N_{n,j}(x_j+y_j)\le N_{n,j}(x_j)+N_{n,j}(y_j)$; multiplying by $\mathbf{e}_j$ and adding proves the hyperbolic triangle inequality.

Permutation invariance follows in each component. Write $\alpha_r=\alpha_{r1}\mathbf{e}_1+\alpha_{r2}\mathbf{e}_2$. Then $N_{n,j}(\alpha_{1j}x_{1j},\dots,\alpha_{nj}x_{nj})\le\max_r|\alpha_{rj}|N_{n,j}(x_{1j},\dots,x_{nj})$, which is exactly Equation (3.2) after idempotent recombination. Zero stability and repetition consistency are inherited componentwise. Finally $N_1(x)=\|x_1\|_1\mathbf{e}_1+\|x_2\|_2\mathbf{e}_2$. □

Corollary 4.2. For $\lambda=\lambda_1\mathbf{e}_1+\lambda_2\mathbf{e}_2$ and $x\in E^n$, the construction satisfies exact common-scalar homogeneity $N_n(\lambda x_1,\dots,\lambda x_n)=|\lambda|_{\mathbb{D}}N_n(x_1,\dots,x_n)$, including λ in the null cone.

Proof. Equation (4.1) and real homogeneity give $|\lambda_1|N_{n,1}\mathbf{e}_1+|\lambda_2|N_{n,2}\mathbf{e}_2$, which equals $|\lambda|_{\mathbb{D}}N_n$. No inverse is used. □

Definition 4.3. A hyperbolic multi-norm N is idempotent additive if $N_n(x_1,\dots,x_n)=N_n(\mathbf{e}_1x_1,\dots,\mathbf{e}_1x_n)+N_n(\mathbf{e}_2x_1,\dots,\mathbf{e}_2x_n)$, and idempotent pure if $N_n(\mathbf{e}_jx_1,\dots,\mathbf{e}_jx_n)\in\mathbb{R}_+\mathbf{e}_j$.

Theorem 4.4. Derived characterization. A hyperbolic multi-norm is representable by Equation (4.1) if and only if it is idempotent additive and idempotent pure. In that case the component multi-norms are unique.

Proof. Necessity is immediate from Equation (4.1). Conversely, for $u_r\in E_j$ define $N_{n,j}(u_1,\dots,u_n)$ as the unique nonnegative coefficient of $\mathbf{e}_j$ in $N_n(\mathbf{e}_ju_1,\dots,\mathbf{e}_ju_n)$. Purity makes this well defined. Each hyperbolic axiom, restricted to the jth idempotent component, becomes the corresponding real multi-norm axiom; definiteness and the triangle inequality are inherited in the same way. Idempotent additivity then reconstructs N_n from these two coefficients. Uniqueness follows by applying the representation to pure tuples. □

Proposition 4.5. Let N and M be decomposable hyperbolic multi-norms. For fixed n, the inequality $cN_n\preccurlyeq_{\mathbb{D}}M_n\preccurlyeq_{\mathbb{D}}CN_n$ with positive real constants c,C holds if and only if it holds for both component pairs with the same constants.

Proof. Project the hyperbolic inequalities onto $\mathbf{e}_1$ and $\mathbf{e}_2$ to obtain the component inequalities. Conversely, multiply the two component inequalities by their idempotents and add. If component constants differ, the sharper statement uses hyperbolic constants $c_1\mathbf{e}_1+c_2\mathbf{e}_2$ and $C_1\mathbf{e}_1+C_2\mathbf{e}_2$. □

Theorem 4.6. Finite-level equivalence. If E_1 and E_2 are finite dimensional, then any two decomposable hyperbolic multi-norms are equivalent on E^n for every fixed n.

Proof. For fixed n, E_j^n is finite dimensional, so any two real norms on it are equivalent. Choose positive constants $c_{n,j}$ and $C_{n,j}$ for each component. Recombination yields $(c_{n,1}\mathbf{e}_1+c_{n,2}\mathbf{e}_2)N_n\preccurlyeq_{\mathbb{D}}M_n\preccurlyeq_{\mathbb{D}}(C_{n,1}\mathbf{e}_1+C_{n,2}\mathbf{e}_2)N_n$. The constants may depend on n; finite-dimensionality alone does not give uniform multi-norm equivalence over all levels. □

5. Convergence and Completeness

Definition 5.1. A sequence of tuples $x^{(m)}\in E^n$ converges to x in N_n when, for every $\varepsilon=\varepsilon_1\mathbf{e}_1+\varepsilon_2\mathbf{e}_2$ with $\varepsilon_1,\varepsilon_2>0$, eventually $N_n(x^{(m)}-x)\prec_{\mathbb{D}}\varepsilon$. It is N_n-Cauchy when the analogous condition holds for $N_n(x^{(m)}-x^{(q)})$.

Proposition 5.2. For a decomposable N_n, $x^{(m)}\to x$ if and only if $x_j^{(m)}\to x_j$ in $N_{n,j}$ for j=1,2. The same equivalence holds for Cauchy sequences.

Proof. The inequality $N_n(x^{(m)}-x)\prec_{\mathbb{D}}\varepsilon$ is equivalent to the two real inequalities $N_{n,j}(x_j^{(m)}-x_j)<\varepsilon_j$. Quantifying over positive ε_1 and ε_2 gives component convergence. Replacing x by $x^{(q)}$ gives the Cauchy statement. □

Theorem 5.3. Completeness criterion. The normed space (E^n,N_n) is complete in the hyperbolic sense if and only if both $(E_j^n,N_{n,j})$ are Banach spaces.

Proof. Assume both components complete and let x^(m) be N_n-Cauchy. Proposition 5.2 makes each component sequence Cauchy, hence convergent to some x_j. The recombined tuple $x=x_1\mathbf{e}_1+x_2\mathbf{e}_2$ is the N_n-limit. Conversely, embed a Cauchy sequence in E_1^n as a pure $\mathbf{e}_1$-sequence in E^n. Hyperbolic completeness produces a limit whose first component is the required E_1^n-limit; similarly for E_2. □

Corollary 5.4. The base hyperbolic normed module E is complete if and only if E_1 and E_2 are Banach spaces. If this holds, every finite power E^n is complete under every decomposable multi-norm N_n.

Proof. The first statement is Theorem 5.3 with n=1. For fixed n, every norm on the finite product of a Banach space is complete when equivalent to a product norm; Proposition 3.3 supplies such finite-level comparison componentwise. □

6. Worked Examples and Counterexamples

Example 6.1. Minimal hyperbolic multi-norm on $\mathbb{D}^m$. Let $x_r=x_{r1}\mathbf{e}_1+x_{r2}\mathbf{e}_2$ with $x_{rj}\in\mathbb{R}^m$ and choose any real norm $\|\cdot\|_j$. Put N_n^min$(x_1,\ldots,x_n)=\max_r\|x_{r1}\|_1\mathbf{e}_1+\max_r\|x_{r2}\|_2\mathbf{e}_2$.

Proof. Permutation invariance is clear. Coordinatewise scaling is bounded by the componentwise maximum scalar modulus. Appending zero leaves each maximum unchanged, and duplicating the last coordinate also leaves it unchanged. Thus all four axioms hold. For m=2, $x_1=(1,-2)\mathbf{e}_1+(0,3)\mathbf{e}_2$ and $x_2=(-4,1)\mathbf{e}_1+(2,2)\mathbf{e}_2$ under the Euclidean component norm give N_2^min$=\max(\sqrt{5},\sqrt{17})\mathbf{e}_1+\max(3,\sqrt{8})\mathbf{e}_2=\sqrt{17}\mathbf{e}_1+3\mathbf{e}_2$. □

Example 6.2. Function-space construction. Let K be compact, $E=C(K,\mathbb{D})=\mathbf{e}_1C(K,\mathbb{R})\oplus\mathbf{e}_2C(K,\mathbb{R})$, and define

$$N_n(f_1,\ldots,f_n)=\sup_{t\in K}\max_r|f_{r1}(t)|\,\mathbf{e}_1+\sup_{t\in K}\max_r|f_{r2}(t)|\,\mathbf{e}_2. \tag{6.1}$$

Proof. Each coefficient is the minimal multi-norm associated with the supremum norm. The supremum is finite by compactness and continuity. The axioms follow either directly from pointwise maxima or from Theorem 4.1. Completeness follows from Theorem 5.3 because both real C(K)-spaces are Banach. □

Example 6.3. A weighted component pair may use different geometries. On $E=\mathbf{e}_1\mathbb{R}^m\oplus\mathbf{e}_2\mathbb{R}^m$, take the component minimum multi-norm induced by $\|\cdot\|_1$ on the first component and by $\|\cdot\|_\infty$ on the second. The resulting hyperbolic multi-norm is valid even though no total real order compares its two coefficients.

Proof. Theorem 4.1 only requires a multi-norm on each component. Incomparability is harmless because every axiom is checked under $\preccurlyeq_{\mathbb{D}}$. For x with first coefficient large and second small, and y with the reverse pattern, neither $N(x)\preccurlyeq_{\mathbb{D}}N(y)$ nor $N(y)\preccurlyeq_{\mathbb{D}}N(x)$ need hold. This illustrates why replacing $\preccurlyeq_{\mathbb{D}}$ by an arbitrary scalarization would lose information. □

Counterexample 6.4. The raw sum gauge $S_n(x_1,\ldots,x_n)=\sum_r\|x_r\|_{\mathbb{D}}$ is a norm on E^n but is not a multi-norm.

Proof. Take $x\neq0$. Repetition would require $S_2(x,x)=S_1(x)$, but $S_2(x,x)=2\|x\|_{\mathbb{D}}$ while $S_1(x)=\|x\|_{\mathbb{D}}$. Since at least one component of $\|x\|_{\mathbb{D}}$ is positive, equality fails. Thus normhood at each level does not imply cross-level consistency. □

Counterexample 6.5. The Euclidean tuple gauge $Q_n=(\sum_r\|x_r\|\mathbb{D}^2)$^{1/2}, interpreted componentwise, also fails repetition.

Proof. For $x\neq 0$, $Q_2(x,x)=\sqrt{2}\|x\|\mathbb{D}\neq\|x\|\mathbb{D}=Q_1(x)$. It satisfies permutation invariance, a triangle inequality, and zero stability, but not axiom (M4). It should therefore be called a power-type tuple norm rather than a multi-norm in the Dales–Polyakov sense. □

Counterexample 6.6. Definiteness can fail if one idempotent component is ignored: $P_n(x_1,\ldots,x_n)=N_n,1(x_{11},\ldots,x_{n1})\mathbf{e}_1$.

Proof. Choose a nonzero pure second-component tuple $x_r=x_{r2}\mathbf{e}_2$. Then $P_n(x_1,\ldots,x_n)=0$ although the tuple is not zero. The map is a hyperbolic seminorm but not a norm. □

6.7 Additional Structural Characterizations

Lemma 6.7. Reverse triangle estimates. For tuples $x,y\in E^n$,

$$N_n(x)\preccurlyeq_{\mathbb{D}}N_n(y)+N_n(x-y), \qquad (6.2)$$

$$N_n(y)\preccurlyeq_{\mathbb{D}}N_n(x)+N_n(x-y).$$

Proof. The first inequality is the triangle inequality applied to $x=y+(x-y)$; the second follows by interchanging x and y. Projecting onto either idempotent gives $|N_n,j(x\_j)-N_n,j(y\_j)|\leq N_n,j(x\_j-y\_j)$. Hence every N_n is continuous in the product topology. □

Theorem 6.8. Lattice maximum construction. If N and M are two decomposable hyperbolic multi-norms inducing the same underlying norm, then

$$(N\vee M)_n(x)=\max{}_{\mathbb{D}}\{N_n(x),M_n(x)\} \qquad (6.3)$$

is again a hyperbolic multi-norm.

Proof. The component maximum of two norms is a norm: definiteness and homogeneity are immediate, and $\max(a+c,b+d)\leq\max(a,b)+\max(c,d)$ proves the triangle inequality. Permutation, scalar domination, zero stability, and repetition hold for N and M separately and therefore for their maximum. At level one both equal the same underlying norm. □

Corollary 6.9. Finite maxima of compatible hyperbolic multi-norms are multi-norms. Thus one may combine several geometric tests without scalarizing the idempotent coefficients.

Proof. Apply Theorem 6.8 inductively. □

Example 6.10. On a hyperbolic Hilbert module, combine the minimum multi-norm N^min and the component Hilbert multi-norm N^H by L=N^min∨N^H. Since N^min≼𝔻N^H for every genuine multi-norm, L=N^H; combining N^H with an inequivalent component multi-norm can instead produce a strictly larger structure at selected tuples.

Proof. The universal lower estimate from Proposition 3.3 gives N^min≼𝔻N^H. Hence their maximum is N^H. If another component multi-norm M is not pointwise comparable with N^H, then N^H∨M agrees with whichever is larger in each idempotent coefficient and remains valid by Theorem 6.8. □

Proposition 6.11. A hyperbolic multi-norm unit ball $B_n=\{x:N_n(x)\preccurlyeq_{\mathbb{D}}1\}$ is balanced under scalars α with $|\alpha|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}1$, convex, absorbing componentwise, and closed.

Proof. Balance follows from scalar domination. For $0\leq t\leq 1$, $N_n(tx+(1-t)y)\preccurlyeq_{\mathbb{D}}tN_n(x)+(1-t)N_n(y)\preccurlyeq_{\mathbb{D}}1$, proving convexity. Given x, choose positive real r larger than both component coefficients of $N_n(x)$, so $x\in rB_n$. Closedness follows from Lemma 6.7. The absorbing scalar r is chosen outside the null cone; a pure idempotent scalar cannot scale both channels. □

The unit-ball description provides an alternative characterization: a decomposable hyperbolic multi-norm is the paired Minkowski functional of its component unit balls, provided those balls satisfy the level symmetries. It also clarifies boundedness. A set $A\subset E^n$ is N_n-bounded exactly when each component projection is bounded in N_n,j. Relative compactness and total boundedness likewise reduce to the component metric spaces. These observations are useful when extending the theory to compact operators and weak topologies.

Counterexample 6.12. A cross-coupled positive map need not be idempotent decomposable. Let $E=\mathbb{D}$ and define $P_1(x)=|x_1|\mathbf{e}_1+(|x_1|+|x_2|)\mathbf{e}_2$. Then P_1 is positive and definite as a real gauge but fails bicomplex homogeneity.

Proof. Take $\alpha=\mathbf{e}_2$ and $x=\mathbf{e}_1$. Then $\alpha x=0$, so $P_1(\alpha x)=0$. However $|\alpha|_{\mathbb{D}}P_1(x)=\mathbf{e}_2(\mathbf{e}_1+\mathbf{e}_2)=\mathbf{e}_2\neq 0$. Thus coupling a coefficient to the opposite idempotent coordinate conflicts with module homogeneity. This does not prove that every coupled theory is impossible, but it shows that coupling cannot be inserted arbitrarily. □

7. Applications and Interpretive Analysis

Systems with two independently ordered channels are well-suited to the componentwise theory. An estimate $N_n(F(x)-F(y))\preccurlyeq_{\mathbb{D}} LN_n(x-y)$ is precisely a pair of real stability bounds in stability analysis. The system is contractive in only one channel if $L=L_1e_1+L_2e_2$ has one coefficient below one and the other above one. This asymmetry would be concealed by a scalar average. Split-complex models of gain and attenuation are subject to the same logic.

When the component objectives permit a common feasible choice or when a Pareto rule is specified, a hyperbolic objective $\Phi=\Phi_1e_1+\Phi_2e_2$ can be minimised in the product order for optimisation on E. Multi-norms have the ability to regulate batches of perturbations or iterates; however, they do not establish a comprehensive optimisation order. The two idempotent channels may encode paired actual models or coupled feature streams in machine-learning interpretations. The mathematics itself does not establish an empirical advantage; it merely provides norm and stability in bookkeeping.

Operator-generated multi-norms are an additional approach. A norming family Γ of componentwise bounded maps can be defined as $N_n(x)=\sup_{T\in\Gamma}N_n^F(Tx_1,\ldots,Tx_n)$, with suprema taken componentwise. The supremum is traversed by the axioms during the verification of finiteness and definiteness. This construction establishes a connection between representation theory and multi-norm geometry and will be implemented in the subsequent operator papers.

8. Open Problems and Limitations

Idempotent additivity is assumed by the characterisation theorem. A nonlinear formula could couple the two coefficients in a general $\mathbb{D}^+$-valued tuple norm, rendering it incapable of being reduced to independent real multi-norms. The existence of useful coupled axioms without compromising order compatibility remains an open question. The component multi-norm sequences are critical to the substantial strength of uniform equivalence across all n, which is in contrast to finite-level equivalence.

Additional enquiries encompass: tensor products and crossnorms; weak and weak-star convergence; compactness and approximation properties; fixed-point theorems under asymmetric component contractions; and classifications of maximum hyperbolic multi-norms. Dual multi-norm constructions for hyperbolic modules are also included. Computational work must also represent null-cone values without conducting invalid division. These issues necessitate a meticulous distinction between identities that are compelled by the product algebra and genuinely new multi-level geometry.

9. In summary

A hyperbolic-valued multi-norm that is mathematically coherent takes values in $\mathbb{D}^+$, respects the idempotent structure of the scalar algebra, and adheres to the componentwise order. Such a structure is consistently induced by paired real multi-norms, and the construction is reversible when idempotent purity and additivity are present. The theory that emerges componentwise inherits definiteness, triangle inequalities, level axioms, convergence, and completeness. In finite dimensions, finite-level norm equivalence is valid; however, uniform equivalence cannot be inferred. Worked examples demonstrate that maximum-type constructions are legitimate, while unnormalized sum and Euclidean tuple norms are not repeatable. These distinctions establish a dependable foundation for operator theory and applications, while also establishing explicit boundaries where zero divisors obstruct conventional scalar-field arguments.

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