A Small Sigma Asymptomatic Approach Generalized Quasi Minimax and Mock Minimax Estimators in Linear Regression Model

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ABSTRACT

In this paper concern with priority two generalized classes of estimators utilizing both types of information apriori and sample, for the estimation of coefficient vector in classical linear regression model in the presence of ellipsoidal Constraints on the parameter vector when the variance of the distribution are unknown. A vectors mean square matrices and weighted quadratic risk are found employing small sigma asymptotic. Some better estimators in the sense of having lesser risk than those of the estimators.

INTRODUCTION

The model and the estimators

Let the linear regression model be

$$y = x\beta + u$$

where y is $TX1$ vector of observations on the variable to be explained, $X\ is\ a$ non- stochastic TXP full Column rank matrix of observations on p-explanatory variable, β is a $px1$ vector of regression Coefficient and $u\ is\ TX1$ vector of disturbances. They found the minimax linear estimator of β to

$$\widehat{\beta_1} = X'X + \sigma^2 H^{-1}X'Y \qquad \text{........ (1)}$$

$$\widehat{\beta_2} = \left[1 - \frac{\sigma^2 + A(X'X)^{-1}}{\overline{Ch}\,(AH^{-1}) + \sigma^2 + A(X'X)^{-1}}\right].b \qquad \text{.......(2)}$$

Above estimators becomes generalized estimators when replacing σ^2 by σ_g^2 in eq 1 or 2 from SINGH and RIZVI (1999) then

$$b_1^* = \left[x'x + \sigma_g^2 H\right]^{-1} x'y \qquad \text{........(3)}$$

$$b_2^* = \left[1 - \frac{\sigma_g^2 + A(x'x)^{-1}}{\overline{ch}\,(AH^{-1}) + \sigma_g^2 + A(x'x)^{-1}}\right].b \qquad \text{........(4)}$$

b_1^* and b_2^* are quasi-minimax and mock minimax generalized classes of estimators respectively

A small sigma asymptomatic approach generalized quasi minimax and mock minimax estimators for b_1^*, b_2^* bais and risk.
We find the approximate bias and risk associated with the estimators b_1^* and b_2^* following small σ asymptotic due to Kadane (1971) when disturbances are small.

For u = v , we write

$$Y = X\beta + u$$

As

$$Y = X\beta + \sigma\, v$$

so that $v_1, v_2, v_3 \ldots\ldots\ldots\ldots\ldots\ v_t\ \ldots\ldots\ldots v_T$ are independently and identically distributed with $E(v_t) = 0,\ E(v_t^2) = 1,\ E(v_t^3) = v_1\ and\ E(v_t^4) = v_2 + 3$

We know that

$$(b + \beta) = \sigma(x'x)^1 x'v$$

And

$$s^2 = \frac{1}{n}(Y - Xb)'(Y - Xb)$$

$$= \frac{1}{n}\sigma\{(I - X(X'X)^{-1}X')v\}'\sigma\{(I - X(X'X)^{-1}X')v\}$$

$$= \frac{\sigma^2}{n}\, v'\overline{P_X}\, v$$

where

1. $\bar{P}_X = I - X(X'X)^{-1}X'$ and
2. $v'\bar{P}_X\, v$ follows x^2 with $n = T - p$ degree of freedom also

$$\hat{\sigma}_g^2 = s^2 + (\sigma_o^2 - s^2)g'(1) + \frac{s^2(u^* - 1)^2}{2!}g''(1) + \ldots\ldots$$

$$= \frac{\sigma^2}{n}(v'\bar{P}_X v) + \left\{\mu\sigma^2 - \frac{\sigma^2}{n}\left(v'\,\bar{P}_X v\right)\right\}g'(1) + \ldots\ldots$$

$$= \sigma^2\left\{\mu g'(1) + \frac{\left(1 - g'(1)\right)}{n}\,\left(v'\bar{P}_X v\right)\right\}$$

$$= \sigma^2 v \qquad \ldots\ldots\ldots\ldots(5)$$

where

$$\gamma = \mu g'(1) + \frac{\left(1 - g'(1)\right)}{n}(v'\bar{P}_X v), \mu = \frac{\sigma_o^2}{\sigma^2}$$

with

$$E[\gamma\,] = \mu g'(1) + \left(1 - g'(1)\right)$$

$$E[\gamma^2] = \mu^2\left(g'(1)\right)^2 + 2\mu g'(1)\left(1 - g'(1)\right) + \frac{\left(1 - g'(1)\right)^2}{n}(n + 2)$$

Now

$$b_1^* = \{I + \sigma^2\gamma(X'X)^{-1}H\}^{-1}(X'X)^{-1}X'(X\beta + \sigma v)$$

$$= [I - \sigma^2\gamma(X'X)^{-1}H + \sigma^4\gamma^2(X'X)^{-1}H(X'X)^{-1}H + \ldots..].(X'X)^{-1}X'(X\beta + \sigma v)$$
$$= [I - \sigma^2\gamma(X'X)^{-1}H + \sigma^4 \quad {}^2(X'X)^{-1}H(X'X)^{-1}H + 0(\sigma^6) + \ldots\ldots][\beta + \sigma(X'X)^{-1}X'v]$$
$$= \beta + \sigma(X'X)^{-1}X'v - \sigma^2\gamma(X'X)^{-1}H\beta + \sigma^4\gamma^2.(X'X)^{-1}H(X'X)^{-1}H\beta - \sigma^3(X'X)^{-1}H(X'X)^{-1}X'v + 0(\sigma^5) + \ldots\ldots..$$
$$(b_1^* - \beta) = \sigma(X'X)^{-1}X'v - \sigma\gamma(X'X)^{-1}H\beta - \sigma^3\gamma(X'X)^{-1}H(X'X)^{-1}X'v + \sigma^4\gamma^2(X'X)^{-1}H(X'X)^{-1}H\beta$$
$$= \sigma\,\varphi_1 + \sigma^2\varphi_2 + \sigma^3\,\varphi_3 + \sigma^4\varphi_4 \qquad \ldots\ldots\ldots\ldots\ldots (6)$$

$\varphi_1 = (X'X)^{-1}X'v$
$\varphi_2 = -\gamma(X'X)^{-1}H\beta$
$\varphi_3 = -\gamma(X'X)^{-1}H(X'X)^{-1}X'V$
$\varphi_4 = \gamma^2(X'X)^{-1}H(X'X)^{-1}H\beta$

Taking expectation on both sides, the bias of b_1^* upto order $0(\sigma^2) is$
Bias $(b_1^*) = E(b_1^* - \beta)$
$$= -\sigma^2\{\mu g'(1) + (1 - g'(1))\}(X'X)^{-1}H\beta$$
$$= -\sigma^2\left\{\frac{\sigma_0^2}{\sigma^2}\,g'(1) + (1 - g'(1))^2\right\}(X'X)^{-1}H\beta$$
$$= -\{\sigma_0^2 g'(1) + \sigma^2(1 - g'(1))\}(X'X)^{-1}H\beta \qquad \ldots\ldots\ (7)$$

$V(b_1^*)\ upto\ order\ 0(\sigma^4)\ is$

$$V(b_1^*) = \sigma^2(X'X)^{-1} - 2\sigma^2\{\sigma_0^2\,g'(1) + \sigma^2(1 - g'(1))\}.$$
$$.(X'X)^{-1}H(X'X)^{-1} + \frac{2}{n}(1 - g'(1))^2\sigma^4(X'X)^{-1}$$
$$.H\beta\beta'H(X'X)^{-1} \qquad \ldots\ldots..(8)$$

MSE$(b_1^*)up\ to\ order\ 0(\sigma^4)$
$MSE\ \ (b_1^*) = E(b_1^* - \beta)(b_1^* - \beta)'$

$$= \sigma^2(X'X)^{-1} - 2\sigma^2\{\sigma_0^2 g'(1) + (1 - g'(1))\}.(X'X)^{-1}H(X'X)^{-1}$$
$$+\{\sigma_0^2 g'(1) + \sigma^2(1 - g'(1))\}^2 + \frac{2}{n}\sigma^4(1 - g'(1))^2(X'X)^{-1}H\beta\beta'H(X'X)^{-1} \qquad \ldots\ldots (9)$$

Weighted quadratic risk to order $0(\sigma^4)$

$$R(b_{g_1}, A) = \sigma^2\ tr\ A(X'X)^{-1} - 2\sigma^2\{\sigma_0^2 g'(1) + \sigma^2(1 - g'(1 - g'(1))\}.tr\ A(X'X)^{-1}H(X'X)^{-1}$$
$$+\{\sigma_0^2 g'(1) + \sigma^2(1 - g'(1))\}^2 + \frac{2}{n}\,\sigma^4(1 - g'(1))^2\,\beta'H(X'X)^{-1}A(X'X)^{-1}H\beta \qquad \ldots.(10)$$
$Minimax\ risk\ to\ order\ 0(\sigma^4)is$

$$\rho(b_1^*, A) = \sigma^2 tr\ A(X'X)^{-1} - 2\sigma^2\{\sigma_0^2 g'(1) + \sigma^2(1 - g'(1))\}\ tr\ A(X'X)^{-1}$$

$$.H(X'X)^{-1} + \{\sigma_0^2 g'(1) + \sigma^2(1-g'(1))\}^2 + \frac{2\sigma^4}{n}(1-g'(1))^2 \overline{ch}\,[H(X'X)^{-1}A(X'X)^{-1}] \quad \ldots\ldots(11)$$

Next for estimator b_2^* we observe that

$$b_2^* = \left[1 - \frac{\sigma_g^2\, tr\, A(X'X)^{-1}}{\overline{Ch}\,(AH^{-1}) + \sigma_g^2 tr\, A(X'X)^{-1}}\right] b$$

$$= \left[\frac{\overline{Ch}\,(AH^{-1})}{\overline{Ch}\,(AH^{-1}) + \sigma_g^2 tr\, A(X'X)^{-1}}\right] b$$

$$= \left[\frac{\overline{Ch}\,(AH^{-1})}{\overline{Ch}\,(AH^{-1}) + \sigma_g^2 \frac{tr\, A(X'X)^{-1}}{\overline{Ch}(AH^{-1})}}\right] b$$

$$= \left[1 + \frac{\sigma^2\, g'(1) + \frac{(1-g'(1))}{n}\,(v'\bar{P}_X V\,)\, tr\, A(X'X)^{-1}}{\overline{Ch}(AH^{-1})}\right]^{-1} .(X'X)^{-1}X'Y$$

$$= \left[1 + \sigma^2 \mu g'(1) + \frac{(1-g'(1))}{n}(v'\bar{P}_X v)\frac{\gamma tr A(X'X)^{-1}}{\overline{Ch}(AH^{-1})}\right]^{-1} .(X'X)^{-1}X'(X\beta + \sigma v)$$

$$= \left[I + \sigma\mu g'(1) + \frac{(1-g'(1))}{n}(v'\bar{P}_X v)\gamma\, tr \frac{A(X'X)^{-1}}{\overline{Ch}(AH^{-1})}\right]^{-1} .(\beta + (X'X)^{-1}X'\sigma v)$$

$$= \left[1 - \sigma^2\mu g'(1) + \frac{(1-g'(1))}{n}\,(v'\bar{P}_X v)\,\gamma \frac{tr A(X'X)^{-1}}{\overline{Ch}(AH^{-1})} + \left\{\sigma^2\mu g'(1) + \frac{(1-g'(1))}{n}\,(v'\bar{P}_X v)\right\}^2 . \left\{\frac{tr\, A(X'X)^{-1}}{\overline{Ch}(AH^{-1})}\right\}^2 \ldots.\right](\beta + (X'X)^{-1}X'v\sigma)$$

$$= \beta + \sigma(X'X)^{-1}X'v - \left[\sigma^2\, g'(1) + \frac{(1-g'(1))}{n}(v'\bar{P}_X v).\frac{tr\, A(X'X)^{-1}}{Ch\,(AH^{-1})} + \sigma^4\left\{\mu g'(1) + \frac{(1-g'(1))}{n}\,(v'\bar{P}_X v)\right\}^2 .\left\{\frac{tr\, A(X'X)^{-1}}{Ch\,(AH^{-1})}\right\}^2 \ldots..\right](\beta + (X'X)^{-1}X'v)$$

$$(b_2^* - \beta) = \sigma(X'X)^{-1}X'v - \gamma\frac{tr\, A(X'X)^{-1}}{Ch\,(AH^{-1})}\beta - \sigma^3\gamma\left(\frac{tr\, A(X'X)^{-1}}{Ch\,(AH^{-1})}\right)(X'X)^{-1}X'v + \sigma^4\gamma^2\left(\frac{tr\, A(X'X)^{-1}}{Ch\,(AH^{-1})}\right)^2\beta$$

$$= \sigma\varphi_1 + \sigma^2\varphi_2^* + \sigma^3\varphi_3^* + \sigma^4\varphi_4^* \quad \ldots\ldots\ldots(12)$$

Where

φ_1

$$\varphi_1 = (X'X)^{-1}X'v$$

$$\varphi_2^* = -\gamma\left(\frac{tr\ A(X'X)^{-1}}{\overline{Ch}\ (AH^{-1})}\right)\beta$$

$$\varphi_3^* = -\gamma\left(\frac{tr\ A(X'X)^{-1}}{\overline{Ch}\ (AH^{-1})}\right)(X'X)^{-1}X'V$$

$$\varphi_4^* = \gamma^2\left(\frac{tr\ A(X'X)^{-1}}{\overline{Ch}\ (AH^{-1})}\right)^2\beta$$

Bias $(b_2^*) = -\{g'(1)\sigma_0^2 + (1-g'(1))\sigma^2\}\left(\frac{tr\ A(X'X)^{-1}}{\overline{Ch}\ (AH^{-1})}\right)\beta$

And $v(b_2^*) = \sigma^2(X'X)^{-1} - 2\sigma^2\{g'(1)\sigma_0^2 + (1-g'(1))\}.\left(\frac{tr\ A(X'X)^{-1}}{\overline{Ch}\ (AH^{-1})}\right)(X'X)^{-1} + \frac{2}{h}(1-g'(1))^2\sigma^2\left(\frac{tr\ A(X'X)^{-1}}{\overline{Ch}\ (AH^{-1})}\right)^2\beta\beta'$ ....(13)

$MSE(b_2^*) upto\ order\ 0(\sigma^4)$

$MSE(b_2^*) = E(b_2^* - \beta)(b_2^* - \beta)'$

$$= \sigma^2(X'X)^{-1} - 2\sigma^2\{g'(a)\sigma_0^2 + (1-g'(1))\sigma^2\}.\left(\frac{tr\ A(X'X)^{-1}}{\overline{Ch}\ (AH^{-1})}\right)(X'X)^{-1}$$
$$+\{g'(1)\sigma_0^2 + (1-g'(1))\sigma^2\}^2 + \frac{2}{n}(1-g'(1))^2\sigma^4\left(\frac{tr\ A(X'X)^{-1}}{\overline{Ch}\ (AH^{-1})}\right)^2\beta\beta'$$

$weighted\ quadratic\ risk\ to\ order\ 0(\sigma^4)$ ....(14)

$R(b_2^*, A) = \sigma^2 tr\ A(X'X)^{-1} - 2\sigma^2\{g'(1)\sigma_0^2 + (1-g'(1))\sigma^2\}.\left(\frac{tr\ A(X'X)^{-1}}{\overline{Ch}\ (AH^{-1})}\right)^2 + \{g'(1)\sigma_0^2 + (1-g'(1))\sigma^2\}^2 + \frac{2}{n}(1-g'(1))^2\sigma^4\left(\frac{tr\ A(X'X)^{-1}}{C\ h(AH^{-1})}\right)^2(\beta'A\beta)$ ....(15)

$minimax\ risk\ to\ order\ 0(\sigma^4) is$

From dealing with weighted quadratic risk of the generalized quasi minimax estimator b_1^* under small sigma asymptotic theory, weting that b_1^* is superior to the OLS estimator b in the sense of havily smaller risk if and only if.

$$\frac{\beta'H(X'X)^{-1}A(X'X)^{-1}H\beta}{tr\ A(X'X)^{-1}H(X'X)^{-1}} < \frac{2\sigma^2[g'(1)\sigma_0^2 + (1-g'(1))\sigma^2]}{[g'(1)\sigma_0^2 + (1-g'(1))\sigma^2]^2 + \frac{2}{n}(1-g'(1))^2\sigma^4}$$

$which\ hold\ true\ at\ least\ as\ long\ as$

$\left(\frac{\sigma_0^2}{\sigma^2} - 1\right)^2 < \frac{1}{(g'(1))^2}\ 1 - \frac{2}{n}(1-g'(1))^2$ ......(16)

Further, from the small sigma asymptotic expression (15) conarning risk of the generalized mock-minimax estimator b_2^* , we see that b_2^* is superior to the OLS estimator b if

$$\frac{\beta' A\beta}{\overline{Ch}\,(AH^{-1})} < \frac{2\left[g'(1)\sigma_0^2 + \left(1 - g'(1)\right)\sigma^2\right]}{\left[g'(1)\sigma_0^2 + \left(1 - g'(1)\right)\sigma^2\right]^2 + \frac{2}{n}\left(1 - g'(1)\right)^2\sigma^4}$$

which holds true as long as the condition (2.4.11) is satisfied. Thus, we get the same superiority condition for $b_1^* and\ b_2^*$ over the OLS estimator b. In particular, for the estimators b_{10}^* in (2.4.5) and b_{20}^* in (2.4.6), the sufficient superiority condition (2.4.11) under small asymptotic reduces to

$$\left(\frac{\sigma_0^2}{\sigma} - 1\right)^2 < \frac{1}{(\delta k)^2}\left[1 - \frac{2}{n}(1 - \delta k)^2\right] \qquad \ldots\ldots(17)$$

From Shukla (1993) , the sufficient superiority condition for estimators b_1 and b_2 over the OLS estimator b under small sigma asymptotic theory is given by

$$\left(\frac{\sigma_0^2}{\sigma} - 1\right)^2 < \frac{1}{(\delta)^2}\left[1 - \frac{2}{n}(1 - \delta)^2\right] \qquad \ldots\ldots\ldots\ldots(18)$$

If we choose k in (16) so that the range of the superiority condition (17) becomes wider than that of the superiority condition (18), the estimators b_{10}^* and b_{20}^* become superior to b_1 ,b_2 and b in the extended range of the superiority condition. It is to be mentioned here that minimax risk comparison provides the same results regarding the superiority conditions for the estimators.

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