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Conformal Curvature Tensor of Quasi-Para-Sasakian Manifolds Admitting Zamkovoy Connection

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Abstract:
The purpose of the present paper is to study various geometric properties of quasi-para-Sasakian manifold with respect to Zamkovoy connection. In the present article we have studied that quasi-conformally flat quasi-para-Sasakian manifold, conformally flat, ϕ-conformally flat quasi-para-Sasakian manifold and shows that the manifold is an η-Einstein manifold.

Keywords: Quasi-Para-Sasakian Manifold, Conformal Curvature Tensor, Zamkovoy Connection.
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1. INTRODUCTION

In [9] Kaneyuki and Konzai defined the almost paracontact structure on pseudo-Riemannian manifold M^n of dimension $(2n+1)$ and constructed the almost paracontact structure on $M^{(2n+1)} \times \Re$.
In 2008, S. Zamkovoy associated the almost paracontact structure [17] to a pseudo-Riemannian metric of signature $(n+1, n)$ and showed that any almost paracontact structure admits such a pseudo-Riemannian metric which is called compatible metric. He introduced the notion of Zamkovoy connection and it showed that its torsion is the obstruction of the paracontact manifold to be para-Sasakian. He defined a Zamkovoy connection on a paracontact metric manifold which seems to be the paracontact analogue of the (generalized) Tanaka-Webster connection [14].

In [2], A. M. Blaga introduced Zamkovoy connection on para Kenmotsu manifolds. This affine connection was further studied by [1]. For an ndimensional almost contact metric manifold M^n equipped with an almost contact metric structure (ϕ,ξ,η,g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g the Zamkovoy connection is defined by

$$\bar{\nabla}_X Y = \nabla_X Y + \nabla_X \eta)(Y)\xi - \eta(Y \nabla_X \xi + \eta(X)\phi Y. \tag{1.1}$$

A systematic study of almost paracontact metric manifolds was given in one of the Zamkovoy's papers [17]. Z. Olszak [11] studied normal almost contact metric manifolds of dimension 3. He obtained certain necessary and sufficient conditions for an almost contact metric structure to be normal and curvature properties of such structures were studied. Normal almost paracontact metric manifolds were studied by many other authors ([6], [7]).

The notion of quasi-Sasakian manifolds was introduced by D. E. Blair [3]. A quasi-Sasakian manifold is a normal almost contact metric manifold whose fundamental 2-form $\Omega := g(.,\phi.)$ is closed. Quasi-Sasakian manifolds can be viewed as an odd-dimensional counterpart of Kaehler structures. These manifolds have been studied by many authors ([8], [10], [13]). Although quasi-Sasakian manifolds were studied by several different authors and are considered a well-established topic in contact Riemannian geometry to the authors knowledge, there do not exists any comprehensive study about quasi-para-Sasakian manifold.

On the other hand ([15], [16]) constructed a generalized curvature tensor on Riemannian manifold which vanishes whenever the metric is (locally) conformally equivalent to a flat metric. The Weyl Conformal curvature tensor is defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{(2n-1)}\{S(Y,Z)X - S(X,Z)Y \\ + g(Y,Z)QX - g(X,Z)QY\} + \frac{r}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{1.2}$$

where *C, R, S,* and *r* denotes the conformal curvature tensor, Riemannian curvature tensor, Ricci tensor and scalar curvature tensor respectively.

Conformal curvature tensor on quasi-para-Sasakian manifold with respect to Zamkovoy connection is given by:

$$\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{(2n-1)}\{\bar{S}(Y,Z)X - \bar{S}(X,Z)Y \\ + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y\} + \frac{\bar{r}}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{1.3}$$

where $\bar{C}, \bar{R}, \bar{S}, \bar{Q}$ and $\bar{r}$denotes the conformal curvature tensor, Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature tensor of quasi-para-Sasakian manifold with respect to Zamkovoy connection respectively.

Motivated by these considerations, in this paper, we studied the conformal curvature tensor of quasi-para-Sasakian manifold admitting Zamkovoy connection and established some new results.

Definition 1.1. *An n-dimensional quasi-para Sasakian manifold M^n is said to be η-Einstein manifold if the Ricci tensor S is of the form*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

for all $X, Y \in \chi(M)$, where a and b are scalars.

2. PRELIMINARIES

An odd dimensional smooth manifold M^n, $(n = 2m + 1)$ has an almost paracontact structure (ϕ,ξ,η) if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the following conditions:

$$\phi(\xi) = 0, \tag{2.1}$$

$$\eta\circ\phi = 0, \tag{2.2}$$

$$\eta(\xi) = 1, \tag{2.3}$$

$$\phi^2 X = X - \eta(X)\xi. \tag{2.4}$$

Distribution $D : P \in M \to D_P \subseteq T_P M$; $D_P = Ker\eta = \{X \in T_P M : \eta(X) = 0\}$ is called paracontact distribution generated by η.

If a manifold M^n with (ϕ, ξ, η) structure admits a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2.5}$$

then we say that M^n has an almost paracontact metric structure and g is called compatible. Any compatible metric g with the given almost paracontact structure g is necesserialy of signature $(n + 1, n)$. Also if

$$\eta(X) = g(X,\xi) \tag{2.6}$$

and

$$g(X,\phi Y) = d\eta(X,Y), \tag{2.7}$$

where,

$$d\eta(X,Y) = \frac{1}{2}\{X\eta(Y) - Y\eta(X) - \eta[X,Y]\} \tag{2.8}$$

holds then η is paracontact form and the almost paracontact metric manifold $(M^n, \phi, \eta, \xi, g)$ is said to be paracontact metric manifold [5].

A paracontact metric manifold is para-Sasakian manifold ([4], [18]) if and only if

$$(\nabla_X\phi)(Y) = -g(X,Y)\xi + \eta(Y)X, \tag{2.9}$$

for all vector fields X and Y.
If

$$(\nabla_X\phi)(Y) = g(X,Y)\xi - \eta(Y)X, \tag{2.10}$$

then the manifold $(M^n, \phi, \eta, \xi, g)$ is said to be a quasi-para-Sasakian manifold.
Also

$$g(X,\varphi Y) = -g(\varphi X, Y), \tag{2.11}$$

$$(\nabla_X\eta)(Y) = -g(X,\varphi Y), \tag{2.12}$$

$$\nabla_X\xi = \varphi X, \tag{2.13}$$

$$d\eta(X,Y) = -g(X,\varphi Y), \tag{2.14}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.15}$$

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.16}$$

$$S(\varphi X,\varphi Y) = -(n-1)g(\varphi X,\varphi Y), \tag{2.17}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.18}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X). \tag{2.19}$$

In view of equation (2.12) and (2.13), equation (1.1) gives the expression for the Zamkovoy connection on quasi-para Sasakian manifold

$$(\bar\nabla_X Y) = (\nabla_X Y) + g(X,\phi Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y. \tag{2.20}$$

3. CURVATURE TENSOR OF QUASI-PARA-SASAKIAN MANIFOLD WITH RESPECT TO ZAMKOVOY CONNECTION

The curvature tensor $\bar R$ of Riemannian manifold M^n with respect to Zamkovoy connection $\bar\nabla$ is given by:

$$\bar R(X,Y)Z = \bar\nabla_X\bar\nabla_Y Z - \bar\nabla_Y\bar\nabla_X Z - \bar\nabla_{[X,Y]} Z.$$

In the view of equation (2.20), we have

$$\begin{aligned}\bar R(X,Y)Z &= R(X,Y)Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &\quad - 2g(X,\phi Y)\phi Z + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \\ &\quad + \{\eta(Y)X - \eta(X)Y\}\eta(Z),\end{aligned} \tag{3.1}$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z \tag{3.3}$$

is the Riemannian curvature tensor of Levi-Civita connection ∇.
Equation (3.2) is the relation between Riemannian curvature tensor with respect to Zamkovoy connection ∇ and Levita-Civita connection ∇.

Transvection of V in equation (3.2) give

$$\begin{aligned}\bar{R}(X,Y,Z,V) = {} & R(X,Y,Z,V) + g(X,\phi Z)g(V,\phi Y) \\ & - g(Y,\phi Z)g(V,\phi X) - 2g(X,\phi Y)g(V,\phi Z) + \{g(Y,Z)\eta(X) \\ & - g(X,Z)\eta(Y)\}\eta(V) + \{\eta(Y)g(X,V) - \eta(X)g(V,Y)\}\eta(Z),\end{aligned} \tag{3.4}$$

where

$$\bar{R}(X,Y,Z,V) = g(\bar{R}(X,Y)Z,V)$$

and

$$R(X,Y,Z,V) = g(R(X,Y)Z,V).$$

Putting $X = V = e_i$ $(1 \leq i \leq n)$ in equation (3.4) and taking summation over i, we get

$$\bar{S}(Y,Z) = S(Y,Z) + 2g(Y,Z) + (n-3)\eta(Y)\eta(Z), \tag{3.5}$$

where $\bar{S}$ and S denotes the Ricci tensor with respect to the connection $\bar{\nabla}$ and ∇ respectively.

Again putting $Y = Z = e_i$ in equation (3.5) and taking summation over i, $(1 \leq i \leq n)$, we get

$$\bar{r} = r + 3n - 3, \tag{3.6}$$

where $\bar{r}$ and r denotes the scalar curvature with respect to the connection $\bar{\nabla}$ and ∇ respectively.

From equation (3.5), we have

$$\bar{Q}Y = QY + 2Y + (n-3)\eta(Y)\xi, \tag{3.7}$$

where $\bar{Q}$ and Q denotes the Ricci operator with respect to the connection $\bar{\nabla}$ and ∇ respectively and

$$\bar{S}(Y,\xi) = 0 = \bar{S}(\xi,Z). \tag{3.8}$$

Now from equation (3.2), we have

$$\bar{R}(X,Y)\xi = 0,\ \bar{R}(Y,\xi)Z) = 0,\ \bar{R}(\xi,Y)Z = 0. \tag{3.9}$$

4. QUASI-CONFORMALLY FLAT QUASI-PARA-SASAKIAN MANIFOLD WITH RESPECT TO ZAMKOVOY CONNECTION

Definition 4.1. *A quasi-para-Sasakian manifold M^n is said to be quasi-conformally flat with respect to Zamkovoy connection* [12] *if*

$$g(\bar{C}(\phi X,Y)Z,\phi V) = 0, \tag{4.1}$$

where $\bar{C}$ is the conformal curvature tensor with respect to Zamkovoy connection $\bar{\nabla}$.

Theorem 4.2. *A quasi-conformally flat quasi-para-Sasakian manifold M^n with respect to Zamkovoy connection is an η-Einstein manifold.*

Proof. In the view of equation (1.3), we have

$$\begin{aligned}g(\bar{C}(\phi X,Y)Z,\phi V) = {} & g(\bar{R}(\phi X,Y)Z,\phi V) - \frac{1}{(2n-1)}\{\bar{S}(Y,Z)g(\phi X,\phi V) \\ & - \bar{S}(\phi X,Z)g(Y,\phi V) + g(Y,Z)\bar{S}(\phi X,\phi V) - g(\phi X,Z)\bar{S}(Y,\phi V) \\ & + \frac{\bar{r}}{2n(2n-1)}\{g(Y,Z)g(\phi X,\phi V) - g(\phi X,Z)g(\phi Y,\phi V)\}.\end{aligned} \tag{4.2}$$

Now, let us suppose that M^n is quasi-conformally flat with respect to Zamkovoy connection. Then from equation (4.1) and (4.3), we have

$$\begin{aligned}g(\bar{R}(\phi X,Y)Z,\phi V) = {} & \frac{1}{(2n-1)}\{\bar{S}(Y,Z)g(\phi X,\phi V) - \bar{S}(\phi X,Z)g(Y,\phi V)\} \\ & +g(Y,Z)\bar{S}(\phi X,\phi V) - g(\phi X,Z)\bar{S}(Y,\phi V)\} \\ & -\frac{\bar{r}}{2n(2n-1)}\{g(Y,Z)g(\phi X,\phi V) - g(\phi X,Z)(\phi Y,\phi V)\end{aligned} \tag{4.3}$$

Using equation (3.2), (3.5) and (3.6) in above equation, we have

$$\begin{aligned} g(R(\phi X,Y)Z,\phi V) &= -g(\phi X,\phi Z)g(\phi V,\phi Y) + g(Y,\phi Z)g(\phi V,X) \\ &+ 2g(\phi X,\phi Y)g(\phi V,\phi Z) - g(\phi X,\phi V)\eta(Y)\eta(Z) \\ &+ \frac{1}{(2n-1)}\{S(Y,Z)g(\phi X,\phi V) + 2g(Y,Z)g(\phi X,\phi V) \\ &- S(\phi X,Z)g(Y,\phi V) - 2g(\phi X,Z)g(Y,\phi V) + S(\phi X,\phi V)g(Y,Z) \\ &+ 2g(Y,Z)g(\phi X,\phi V) - S(Y,\phi V)g(\phi X,Z) - 2g(Y,\phi V)g(\phi X,Z)\} \\ &- \frac{r+3n-3}{2n(2n-1)}\{g(Y,Z)g(\phi X,\phi V) - g(\phi X,Z)g(Y,\phi V)\}. \end{aligned} \tag{4.4}$$

Let $(e_1, e_2, \ldots\ldots e_{n-1}, \xi)$ be a local orthonormal basis of vector fields in M^n. Then $(\phi e_1, \phi e_2, \ldots\ldots \phi e_{n-1}, \xi)$ is also local orthonormal basis of M^n. Putting $X = V = e_i$ in equation (4.4) and taking summation over i, $1 \le i \le n-1$, we get

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z), \tag{4.5}$$

which shows that M^n is an η-Einstein manifold, where

$$a = [-\frac{(r-3) + n(6n^2 + 8n - 15}{2n(n+2)}]$$

and

$$[b = \frac{n(2n-1)}{(n+2)}].$$

□

5. CONFORMALLY FLAT QUASI-PARA-SASAKIAN MANIFOLD WITH RESPECT TO ZAMKOVOY CONNECTION

An n-dimensional quasi-para-Sasakian manifold M^n is said to be conformally flat if the conformal curvature tensor vanishes.
In this section, we assume that $\bar{C}(X,Y)Z = 0$, where $\bar{C}$ denotes the conformal curvature tensor with respect to the Zamkovoy connection $\bar{\nabla}$.

Theorem 5.1. *A conformally flat quasi-para-Sasakian manifold M^n $(n \geq 2)$ admitting Zamkovoy connection $\bar{\nabla}$ is an η-Einstein manifold.*
Proof. Let M^n be an n-dimensional conformally flat quasi-para-Sasakian manifold with respect to the Zamkovoy connection, *i.e.* $\bar{C} = 0$, then from equation (1.3), we have

$$\begin{aligned} \bar{R}(X,Y)Z &= \frac{1}{(2n-1)}\{\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X \\ &- g(X,Z)\bar{Q}Y\} - \frac{\bar{r}}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{5.1}$$

Transvection of V in equation (5.1), gives

$$\begin{aligned} g(\bar{R}(X,Y)Z,V) &= \frac{1}{(2n-1)}\{\bar{S}(Y,Z)g(X,V) - \bar{S}(X,Z)g(Y,V) \\ &+ g(Y,Z)\bar{S}(X,V) - g(X,Z)\bar{S}(Y,V)\} - \frac{\bar{r}}{2n(2n-1)}\{g(Y,Z)g(X,V) \\ &- g(X,Z)g(Y,V)\}. \end{aligned} \tag{5.2}$$

Let $e_i, (1 \leq i \leq n)$ be an orthonormal basis. Taking summation over $X = V = e_i$ $(1 \leq i \leq n)$ in above equation, we get

$$\bar{S}(Y,Z) = \frac{\bar{r}(n-2) - 2n^2}{2n(n+2)} g(Y,Z) \tag{5.3}$$

Using equations (3.5) and (3.6) in equation (5.3), we get

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z), \tag{5.4}$$

which shows that M^n is an η-Einstein manifold,
where
$a = \frac{(\, r(n-2) + (n^2 - n + 6)\,)}{(\, 2n(n+2)\,)}$ and $b = -(n-3)$.

□

Theorem 5.2. *A* $\xi-$*conformally flat quasi-para-Sasakian manifold* M^n $(n \geq 2)$ *admitting Zamkovoy connection* $\bar{\nabla}$ *is an* η*-Einstein manifold.*

Proof. If M^n be $\xi-$conformally flat quasi-para-Sasakian manifold with respect to the Zamkovoy connection, *i.e.* $\bar{C}(X,Y)\xi = 0$, then from equation (1.3), we have

$$\eta(X)\bar{Q}Y + g(X,V)\bar{Q}\xi = \frac{1}{2n}\{\eta(Y)X - \eta(X)Y\}. \tag{5.5}$$

Transvection of V in equation (5.5), we get

$$\eta(X)\bar{S}(Y,V) = 0. \tag{5.6}$$

Putting $X = \xi$ and using equation (3.5) in equation, we get

$$S(Y,V) = -2g(Y,V) - (n-3)\eta(Y)\eta(V), \tag{5.7}$$

which shows that the manifold M^n is an $\eta-$Einstein manifold.

□

Corollary 5.3. *If a quasi-para-sasakian manifold is* ξ*-conformally flat with respect to Zamkovoy connection then its scalar curvature is constant.*

Proof. Suppose that the quasi-para-Sasakian manifold is ξ-conformally flat with respect to Zamkovoy connection then taking summation over $Y = V = e_i, (1 \leq i \leq n)$ in equation (5.7), we get

$$r = -3(n-1). \tag{5.8}$$

Theorem 5.4. *On an n-dimensional quasi-para-Sasakian manifold* M^n*,* ξ*-conformal curvature tensor of Zamkovoy connection and Levi-Civita connection are identical provided that the vector fields on* M^n *are horizontal vector fields.*

Proof. From equations (1.2), (1.3), (3.2), (3.5), and (3.7), we have

$$\begin{aligned}\bar{C}(X,Y)Z = C(X,Y)Z &- \frac{1}{(2n-1)}\{2g(Y,Z)X + (n-3)\eta(Y)\eta(Z)X \\ &- 2g(X,Z)Y - (n-3)\eta(X)\eta(Z)Y + 2g(Y,Z)X - (n-3)\eta(Y)g(X,Z)\xi \\ &- 2g(X,Z)Y + (n-3)\eta(Y)g(X,Z)\xi\} + \frac{3n-3}{2n(2n-1)}\{g(Y,Z)X \\ &- g(X,Z)Y\} + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X - 2g(X,\phi Y)\phi Z \\ &+ \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi + \eta(Y)X - \eta(X)Y\}\eta(Z).\end{aligned} \tag{5.9}$$

Substitute $Z = \xi$ in above equation (5.9), we get

$$\bar{C}(X,Y)\xi = C(X,Y)\xi + \frac{2n^2 - n - 3}{2n(2n \quad 1)}\{\eta(Y)X - \eta(X)Y\} \tag{5.10}$$

If X and Y are horizontal vector fields then from equation (5.10), it follows that $\bar{C}(X,Y)\xi = C(X,Y)\xi$. □

6. ϕ-CONFORMALLY FLAT QUASI-PARA-SASAKIAN MANIFOLD WITH RESPECT TO THE ZAMKOVOY CONNECTION $\bar{\nabla}$

Definition 6.1. *Conformal curvature tensor with respect to Zamkovoy connection is said to be ϕ-conformally flat if* $\bar{C}(\phi X,\phi Y,\phi Z,\phi W) = 0$.

In this section we consider ϕ-conformally flat quasi-para-Sasakian manifold admitting Zamkovoy connection and showed that it is an η-Einstein manifold.

Theorem 6.2. *A ϕ-conformally flat quasi-para-Sasakian manifold M^n admitting Zamkovoy connection $\bar{\nabla}$ is an $\eta-$Einstein manifold.*

Proof. We assume that quasi-para-Sasakian manifold M^n be ϕ-conformally flat with respect to the Zamkovoy connection, *i.e.* $\bar{V}(\phi X,\phi Y,\phi Z,\phi W) = 0$, for all $X,Y,Z,W \in \chi(M)$. Then from equation (1.3), we have

$$\begin{aligned} g(\bar{R}(\phi X,\phi Y)\phi Z,\phi W) &= \frac{1}{(2n-1)}\{\bar{S}(\phi Y,\phi Z)g(\phi X,\phi W) \\ &- \bar{S}(\phi X,\phi Z)g(\phi Y,\phi W) + g(\phi Y,\phi Z)\bar{S}(\phi X,\phi W) - g(\phi X,\phi Z)\bar{S}(\phi Y,\phi W)\} \\ &- \frac{r}{2n(2n-1)}\{g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W). \end{aligned} \tag{6.1}$$

Which on using equations (3.2) and (3.5) equation (6.1) reduced to

$$\begin{aligned} &g(R(\phi X,\phi Y)\phi Z,\phi W) + g(\phi X,Z)g(\phi W,Y) - g(\phi Y,Z)g(\phi W,X) \\ &- 2g(\phi X,Y)g(\phi W,Z) = \frac{1}{(2n-1)}\{S(\phi Y,\phi Z)g(\phi X,\phi W) \\ &+ 2g(\phi Y,\phi Z)g(\phi X,\phi W) - S(\phi X,\phi Z)g(\phi Y,\phi W) - 2g(\phi X,\phi Z)g(\phi Y,\phi W) \\ &+ g(\phi Y,\phi Z)S(\phi X,\phi W) + 2g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)S(\phi Y,\phi W) \\ &- 2g(\phi X,\phi Z)g(\phi Y,\phi W)\} - \frac{r+3n-3}{2n(2n-1)}\{g(\phi Y,\phi Z)g(\phi, X\phi W) \\ &- g(\phi X,\phi Z)g(\phi Y,\phi W)\}. \end{aligned} \tag{6.2}$$

Let $\{e_1,\ e_2,\dots,e_{n-1},\xi\}$ are the local orthonormal basis of the vector field in M^n. Using the fact that $\{\phi e_1,\ \phi e_2,\dots,\phi e_{n-1},\xi\}$ is also a local orthonormal basis. Putting $X = W = e_i, (1 \le i \le n-1)$ in equation (6.2), we have

$$S(\phi Y,\phi Z) = \frac{(2n-1)}{(n+2)}g(Y,Z) - \frac{r(n-2)+2n^3-3n^2+7n+6}{2n(n+2)}\ g(\phi Y,\phi Z). \tag{6.3}$$

Replacing Y by ϕY and Z by ϕZ in equation (6.3), we get

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z), \tag{6.4}$$

where $a = \frac{r(n-2)+ 2n^3- 5n^2- 5n + 6}{2n(n+2)}$ and $b = \frac{r(n-2)+2n^3-3n^2+7n+6}{2n(n+2)}$. □

7. QUASI-PARA-SASAKIAN MANIFOLD ADMITTING ZAMKOVOY CONNECTION $\bar{\nabla}$ SATISFYING $\bar{C}(\xi,U).\bar{S} = 0$

In this section, we define a relation on quasi-para-Sasakian manifold $\bar{C}.\bar{S} = 0$, where $\bar{C}$ and $\bar{S}$ are the conformal curvature tensor and Ricci tensor with respect to the Zamkovoy connection respectively.

Theorem 7.1. *On an n-dimensional quasi-para-Sasakian manifold* M^n *admitting Zamkovoy connection* $\bar{\nabla}$, *if the condition* $\bar{C}(\xi, U).\bar{S} = 0$ *holds, then the manifold is an* η*-Einstein manifold.*

Proof. Assume that a quasi-para-Sasakian manifold M^n admitting Zamkovoy connection satisfying the condition

$$(\bar{C}(\xi, U).\bar{S})(X, Y) = 0,$$

where $\bar{C}$ and $\bar{S}$ are the conformal curvature tensor and Ricci tensor with respect to Zamkovoy connection respectively and $X, Y, U \in \chi(M)$, then we have

$$\bar{S}(\bar{C}(\xi, U)X, Y) + \bar{S}(X, \bar{C}(\xi, U)Y) = 0. \tag{7.1}$$

From equation (1.3), we have

$$\bar{C}(\xi, U)Z = -\frac{1}{(2n-1)}\{\bar{S}(U, Z)\xi + g(U, Z)\bar{Q}\xi - \eta(X)\bar{Q}U\} + \frac{\bar{r}}{2n(2n-1)}\{g(U, Z)\xi - \eta(X)U\}, \tag{7.2}$$

Using equation (7.2) in equation (7.1), we have

$$\frac{1}{(n-2)}\{\eta(X)\bar{S}(\bar{Q}U, Y) - \eta(Y)\bar{S}(\bar{Q}U, X)\} + \frac{\bar{r}}{2n(2n\ \ 1)}\{\eta(X)\bar{S}(U, Y) - \eta(Y)\bar{S}(U, X)\} = 0 \tag{7.3}$$

Putting $X = \xi$ in above equation, we get

$$\frac{1}{(n-2)}\bar{S}(\bar{Q}U, Y) + \frac{\bar{r}}{2n(2n-1)}\bar{S}(U, Y) = 0. \tag{7.4}$$

Using equations (3.5), (3.6) and (3.7) in equation (7.4), we get

$$S(U, Y) = -2g(U, Y) - (n-3)\eta(U)\eta(Y). \tag{7.5}$$

□

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