Closed Forms for the Apostol-Frobenius-Euler Polynomials

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Abstract
The Apostol-Frobenius-Euler polynomials are novel extension from the prolific studies on classical polynomials. In recent years, there has been great erudition focused on evaluating special functions. This paper obtained a simple and finite approach to evaluate the Apostol-Frobenius-Euler polynomials through their closed forms. Closed forms are mathematical expressions expressed using a definite number of standard operations. There can be constants, variables, some elementary operations, and functions, but conventionally, limits are excluded [1]. The researchers utilized some properties of the Bell polynomials of the second kind, the Faà di Bruno formula, and the determinantal representation of the quotient rule to obtain the closed forms. The closed forms for the Apostol-Frobenius-Euler number were also presented.

Keywords: closed forms, Apostol-Frobenius-Euler polynomial and numbers, Determinantal expression

Introduction
Special functions are labelled as "useful functions" in [2] as since they are substantially useful in mathematical physics and other branch of mathematics. They surfaced from the theories of a few mathematicians but persisted to be a compelling topic in modern mathematics (see [3], [4], [5], [6], [7]). One of many topics in special functions that emanated from the extended studies on classical polynomials are the Apostol-Frobenius-Euler polynomials denoted by $H_n(x,\lambda;u)$ which are defined by the exponential generating function

$$\left(\frac{1-u}{\lambda e^t - u}\right) e^{xt} = \sum_{n=0}^{\infty} H_n\,(x;\lambda;u)\frac{t^n}{n!}$$

where $\lambda, u \in \mathbb{C}$ with $\lambda \neq 1, u \neq 1, u \neq \lambda$ as cited in [8]. When $x = 0$ such that $H_n(0,\lambda;u) = H_n(\lambda;u)$, denotes the Apostol-Frobenius-Euler numbers. The Apostol-Frobenius-Euler polynomials are generalizations on λ -extension on Frobenius-Euler polynomials and these polynomials have been a subject for research for many researchers (see [9], [10], [11], [12], [13], [14], [23]).
There has been a great erudition in recent years focused on evaluating special functions (see [19], [20], [21], [22]). A simple way to evaluate special functions is to obtain its closed-form expression. A mathematical expression expressed using a definite number of standard operations pertain to closed-form

expressions. There can be constants, variables, some elementary operations. and functions, but conventionally, limits are excluded [1]. One particular study on evaluating special function was the study of Hu and Kim in 2018 [15] which they incorporate the generating function of the Apostol-Bernoulli polynomials to the Faà di Bruno formula for computing higher order derivatives of composite functions. Consequently, they obtained a closed form for the Apostol-Bernoulli polynomials.

This study will obtain two closed form expressions for Apostol-Frobenius-Euler polynomials. One of these closed forms involves the Stirling numbers of the second kind and the other will be presented in terms of the determinant of lower Hessenberg matrix. Closed forms for the Apostol-Frobenius-Euler numbers were also presented.

Result

Definition 2.1 [16]. The Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ are defined by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum \frac{n!}{\ell_1! \ell_2!, \dots, \ell_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{\ell_1} \left(\frac{x_2}{2!}\right)^{\ell_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{\ell_{n-k+1}}$$

where the sum is taken over all sequences $\ell_1, \ell_2, \dots, \ell_{n-k+1}$ of non-negative integers such that $\ell_1 + \cdots + \ell_{n-k+1} = k$ and $\ell_1 + 2\ell_2 + \cdots + (n-k+1)\ell_{n-k+1} = n$ for $n \geq k \geq 0$.

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind by

$$\frac{d^n}{dt^n}(f \circ g)(t) = \sum_{k=0}^{n} f^{(k)}\big(g(t)\big) B_{n,k}\left(g'(t), g''(t), \dots, g^{(n-k+1)}(t)\right) \qquad (1)$$

Some properties of (which are lemmas in this study) of the Bell polynomials of the second kind were as follows:

Lemma 2.2 [16]. For $n \geq k \geq 0$, the Bell polynomials of the second kind meets

$$B_{n,k}(x_1 + y_1, x_2 + y_2, \dots, x_{n-k+1} + y_{n-k+1}) = \sum_{r+s=k} \sum_{\ell+m=n} \binom{n}{\ell} B_{\ell,r}(x_1, x_2, \dots, x_{n-k+1}) B_{m,s}(y_1, y_2, \dots, y_{n-k+1}).$$

Lemma 2.3 [16]. For $n \geq k \geq 0$, where and b are any complex numbers,

$$B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}).$$

Lemma 2.4 [16]. For $n \geq k \geq 0$,

$$B_{n,k}(\underbrace{1,1,\dots,1}_{n-k+1}) = S(n,k).$$

Definition 2.5. The formula for the nth derivative of the ratio $\frac{u(t)}{v(t)}$, where $u(t)$ and $v(t)$ are differentiable functions such that $v(t) \neq 0$, is given by $\frac{d^n}{dt^n}\left(\frac{u}{v}\right) = \frac{(-1)^n Wn}{v^{n+1}}$. The $\mathbf{W_n} = w_{ij}$, such that $1 \leq i \leq n+1$ and $1 \leq j \leq n+1$, is the determinant of the lower Hessenberg matrix which is given by,

$$\mathbf{W_n} = \begin{vmatrix} u & v & 0 & 0 & \dots & 0 \\ u' & v' & v & 0 & \dots & 0 \\ u'' & v'' & 2v' & v & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ u^{(n-1)} & v^{(n-1)} & \binom{n-1}{1} v^{(n-2)} & \binom{n-1}{2} v^{(n-3)} & \dots & v \\ u^{(n)} & v^{(n)} & \binom{n}{1} v^{(n-1)} & \binom{n}{2} v^{(n-2)} & \dots & \binom{n}{n-1} v' \end{vmatrix}$$

where $w_{ij} = 0$ when $j - i > 1$ [17]. In [1], this formula was described as

$$\frac{d^n}{dt^n}\left(\frac{u}{v}\right) = \frac{(-1)^n}{v^{n+1}} \left| A_{(n+1)x1} \quad B_{(n+1)\times n} \right|_{(n+1)x(n+1)} \tag{2}$$

where the matrices

$$A_{(n+1)\times 1} = \left(a_{\ell,1}\right)_{1\le \ell \le n+1} = \begin{vmatrix} u \\ u' \\ u'' \\ \vdots \\ u^{(n-1)} \\ u^{(n)} \end{vmatrix}$$

and

$$B_{(n+1)\times n} = \left(b_{\ell,m}\right)_{1\le \ell\le n+1, 1\le m\le n} = \begin{vmatrix} v & 0 & 0 & \dots & 0 \\ v' & v & 0 & \dots & 0 \\ v'' & 2v' & v & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v^{(n-1)} & \binom{n-1}{1} v^{(n-2)} & \binom{n-1}{2} v^{(n-3)} & \dots & v \\ v^{(n)} & \binom{n}{1} v^{(n-1)} & \binom{n}{2} v^{(n-2)} & \dots & \binom{n}{n-1} v' \end{vmatrix}$$

The first theorem is the closed form for the Apostol-Frobenius-Euler polynomials in terms of the Stirling numbers of the second kind.

Theorem 2.6. The Apostol-Frobenius-Euler polynomials $H_n(x, \lambda; u)$ for $n \in \mathbb{N}$ can be expressed as,

$$H_n(x; \lambda; u) \quad = (1-u) \sum_{k=0}^{n} \frac{(-1)^k k!}{(\lambda - u)^{k+1}} \sum_{r+s=k} \sum_{\ell+m=n} \binom{n}{\ell} \lambda^r (1-x)^\ell S(\ell, r)(-u)^s (-x)^m S(m, s)$$

where λ ,u $\in \mathbb{C}, \lambda \neq 1, u \neq 1, u \neq \lambda$.

Proof. Set

$$m(t) = 1 - u, \quad y = g(t) = \frac{\lambda e^t - u}{e^{xt}}, \quad and \quad f(y) = \frac{1}{y}$$

then say a function $h(t)$

$$h(t) = m(t)(f \circ g)(t) = \left(\frac{1-u}{\lambda e^t - u} e^{xt}\right)$$

which corresponds to the generating function of the Apostol-Frobenius-Euler polynomials. Note that as t $\to 0$,

$$
\begin{aligned}
g(t) &= \frac{\lambda e^{t} - u}{e^{xt}} = \lambda e^{t-xt} - u e^{-xt} = \lambda e^{(1-x)t} - u e^{-xt} \\
&= \lambda \sum_{m=0}^{\infty} \frac{(1-x)^m}{m!} t^m - u \sum_{m=0}^{\infty} \frac{(-x)^m}{m!} t^m \\
&\to \lambda - u\,. \qquad (3)
\end{aligned}
$$

Therefore, the following higher-order derivatives of $g(t)$ as t → 0 were obtained as follows

$$
\begin{aligned}
g'(t) &= \lambda \sum_{m=1}^{\infty} \frac{(1-x)^m}{(m-1)!} t^{m-1} - u \sum_{m=1}^{\infty} \frac{(-x)^m}{(m-1)!} t^{m-1} \\
&= \lambda(1-x) - u(-x). \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
g''(t) &= \lambda \sum_{m=2}^{\infty} \frac{(1-x)^m}{(m-2)!} t^{m-2} - u \sum_{m=2}^{\infty} \frac{(1-x)^m}{(m-2)!} t^{m-2} \\
&= \lambda(1-x)^2 - u(-x)^2. \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
g'''(t) &= \lambda \sum_{m=3}^{\infty} \frac{(1-x)^m}{(m-3)!} t^{m-3} - u \sum_{m=3}^{\infty} \frac{(-x)^m}{(m-3)!} t^{m-3} \\
&= \lambda(1-x)^3 - u(-x)^3. \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
&\vdots \\
g^{(n-k+1)}(t) &= \lambda \sum_{m=n-k+1}^{\infty} \frac{(1-x)^m}{m-(n-k+1)!} t^{m-(n-k+1)} \\
&\quad - u \sum_{m=n-k+1}^{\infty} \frac{(-x)^m}{m-(n-k+1)} t^{m-(n-k+1)} \\
&= \lambda(1-x)^{n-k+1} - u(-x)^{n-k+1}. \qquad (7)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
f(y) &= \frac{1}{y} \\
f'(y) &= (-1)(y)^{-2} \\
f''(y) &= (-1)(-2)(y)^{-3} \\
f'''(y) &= (-1)(-2)(-3)(y)^{-4} \\
&\vdots \\
f^{(k)}(y) &= (-1)^k (k!)(y)^{-(k+1)} \\
&= \frac{(-1)^k k!}{(\lambda - u)^{k+1}} \qquad (8)
\end{aligned}
$$

Substituting (3), (4), (5), (6), (7), (8) to (1), it follows that

$$\begin{aligned}\frac{d^n}{dt^n}(f\circ g)(t) &= \sum_{k=0}^{n} f^{(k)}\left(g(t)\right)B_{n,k}\left(g'(t),g''(t),\dots,g^{(n-k+1)}(t)\right)\\ &= \sum_{k=0}^{n}\frac{(-1^k)k!}{(\lambda-u)^{k+1}}\\ &\times B_{n,k}(\lambda(1-x)-u(-x),\lambda(1-x)^2-u(-x)^2,\dots,x(1-x)^{n-k+1}-u(x)^{n-k+1})\end{aligned}$$

as t → 0. Furthermore,

$$\begin{aligned}\frac{d^n}{dt^n}(f\circ g)(t)|_{t=0} &= \sum_{k=0}^{n}\frac{(-1)^k k!}{(\lambda-u)^{k+1}}\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}B_{\ell,r}(\lambda(1-x),\lambda(1-x)^2,\dots\lambda(1-x)^{n-k+1})\\ &\times B_{m,s}(-u(-x),-u(-x)^2,\dots,-u(-x)^{n-k+1})\quad (By\quad Lemma\quad 2.2)\\ &= \sum_{k=0}^{n}\frac{(-1)^k k!}{(\lambda-u)^{k+1}}\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\lambda^r(1-x)^\ell B_{\ell,r}(1,1,1,\dots,1)\\ &\times(-u)^s(-x)^m B_{m,s}(1,1,1,\dots,1)\quad (By\quad Lemma\quad 2.3)\\ &= \sum_{k=0}^{n}\frac{(-1)^k k!}{(\lambda-u)^{k+1}}\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\lambda^r(1-x)^\ell S(\ell,r)(-u)^s(-x)^m S(m,s).\\ &(By\quad Lemma\quad 2.4)\end{aligned}$$

By Leibnitz's formula for the nth derivative of the product of two functions,

$$\begin{aligned}\frac{d^n}{dt^n}h(t) &= \frac{d^n}{dt^n}m(t)(f\circ g)(t)\\ &= \sum_{\iota=0}^{n}\binom{n}{\iota}(f\circ g)(t)^{(n-\iota)}m(t)^{(\iota)}\\ &= (1-u)\frac{d^n}{dt^n}(f\circ g)(t)\\ &= (1-u)\sum_{k=0}^{n}\frac{(-1)^k k!}{(\lambda-u)^{k+1}}\\ &\times\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\lambda^r(1-\lambda)^\ell S(\ell,r)(-u)^s(-x)^m S(m,s)\end{aligned}$$

as t → 0. ■

The next result is an immediate consequence of Theorem 2.6.

Corollary 2.7. Suppose $x=0$. The Apostol-Frobenius-Euler numbers $H_n(\lambda;u)$ $(n\in\mathbb{N})$ can be expressed as

$$H_n(\lambda;u)=(1-u)\sum_{k=0}^{n}\frac{(-1)^k k!}{(\lambda-u)^{k+1}}\lambda^k S(n,k).$$

where λ ,u $\in \mathbb{C}, \lambda \neq 1, u \neq 1, u \neq \lambda$.
The next theorem is the determinantal expression for the Apostol-Frobenius-Euler polynomials.

Theorem 2.8. The Apostol-Frobenius-Euler polynomials $H_n(x;\lambda;u)$ for $n \in \mathbb{N}$ where λ ,u $\in \mathbb{C}, \lambda \neq 1, u \neq 1, u \neq \lambda$ can be expressed as

$$(x,\lambda;u) = \frac{(1-u)(-1)^n}{(\lambda-u)^{n+1}} \begin{vmatrix} 1 & \lambda-u & 0 & 0 & \dots & 0 \\ x & \lambda & \lambda-u & 0 & \dots & 0 \\ x^2 & \lambda & 2\lambda & \lambda-u & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x^{n-1} & \lambda & \binom{n-1}{1}\lambda & \binom{n-1}{2}\lambda & \dots & \lambda-u \\ x^n & \lambda & \binom{n}{1}\lambda & \binom{n}{2}\lambda & \dots & \binom{n}{n-1}\lambda \end{vmatrix}$$

with $|\cdot|$ denotes the $(n+1)\times(n+1)$ determinant of the lower Hessenberg matrix.
Proof. Set

$$m(t) = 1-u, \quad u(t) = e^{xt}, \quad v(t) = \lambda e^t - u.$$

Then say a function $h(t)$

$$h(t) = m(t)\left(\frac{u(t)}{v(t)}\right) = \left(\frac{1-u}{\lambda e^t - u}\right)e^{xt}$$

which corresponds to the generating function of the Apostol-Frobenius-Euler polynomials. Now as t → 0,

$$\begin{aligned} v(t) &= \lambda e^t - u \\ &= \lambda \sum_{m=0}^{\infty} \frac{t^m}{m!} - u \\ &\to \lambda - u \end{aligned}$$

$$\begin{aligned} v'(t) &= \lambda \sum_{m=1}^{\infty} \frac{t^{m-1}}{(m-1)!} \\ &= \lambda \end{aligned}$$

$$\begin{aligned} v''(t) &= \lambda \sum_{m=2}^{\infty} \frac{t^{m-2}}{(m-2)!} \\ &= \lambda \end{aligned}$$

$$\vdots$$

$$\begin{aligned} v^{(n)}(t) &= \lambda \sum_{m=n}^{\infty} \frac{t^{m-n}}{(m-n)!} \\ &= \lambda \end{aligned}$$

Furthermore,

$$u(t) = e^{xt} \quad = \sum_{m=0}^{\infty} \frac{x^m}{m!} t^m \; = \; 1$$

$$u'(t) \quad = \sum_{m=1}^{\infty} \frac{x^m}{(m-1)!} t^{m-1}$$
$$= x$$

$$u''(t) \quad = \sum_{m=2}^{\infty} \frac{x^m}{(m-2!} t^{m-2}$$
$$= x^2$$

$$\vdots$$

$$u^{(n)}(t) \quad = \sum_{m=n}^{\infty} \frac{x^m}{(m-n)!} t^{m-n}$$
$$= x^n$$

as t → 0. Therefore the matrix $A_{(n+1)\times 1} = \left(a_{\ell,1}\right)_{1\leq\ell\leq n+1}$ and $B_{(n+1)\times n} = \left(b_{\ell,m}\right)_{1\leq\ell\leq n+1, 1\leq m\leq n}$ in (2.) can be represented as

$$A_{(n+1)\times 1} = \left(a_{\ell,1}\right)_{1\leq\ell\leq n+1} = \begin{vmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{n-1} \\ x^n \end{vmatrix}$$

and

$$B_{(n+1)\times n} = \left(b_{\ell,m}\right)_{1\leq\ell\leq n+1, 1\leq m\leq n} = \begin{vmatrix} \lambda - u & 0 & 0 & \dots & 0 \\ \lambda & \lambda - u & 0 & \dots & 0 \\ \lambda & 2\lambda & \lambda - u & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda & \binom{n-1}{1}\lambda & \binom{n-1}{2}\lambda & \dots & \lambda - u \\ \lambda & \binom{n}{1}\lambda & \binom{n}{2}\lambda & \dots & \binom{n}{n-1}\lambda \end{vmatrix}$$

Therefore,

$$\frac{d^n}{dt^n}\left(\frac{u}{v}\right) = \frac{(-1)^n}{(v)^{n+1}}\left|A_{(n+1)x1} \quad B_{(n+1)\times n}\right|$$

$$= \frac{(-1)^n}{(\lambda-u)^{n+1}}\begin{vmatrix} 1 & \lambda-u & 0 & 0 & \dots & 0 \\ x & \lambda & \lambda-u & 0 & \dots & 0 \\ x^2 & \lambda & 2\lambda & \lambda-u & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x^{n-1} & \lambda & \binom{n-1}{1}\lambda & \binom{n-1}{2}\lambda & \dots & \lambda-u \\ x^n & \lambda & \binom{n}{1}\lambda & \binom{n}{2}\lambda & \dots & \binom{n}{n-1}\lambda \end{vmatrix}$$

By Leibnitz's formula for the nth derivative of the product of two functions,

$$\frac{d^n}{dt^n}h(t) = \frac{d^n}{dt^n}\left(m(t)\left(\frac{u}{v}\right)\right)$$

$$= \sum_{\iota=0}^{n}\binom{n}{\iota}\left(\frac{u}{v}\right)^{(n-\iota)} m(t)^{(\iota)}$$

$$= (1-u)\left(\frac{u}{v}\right)^n$$

$$= (1-u)\frac{d^n}{dt^n}\left(\frac{u}{v}\right)$$

$$\rightarrow \frac{(1-u)(-1)^n}{(\lambda-u)^{n+1}}\begin{vmatrix} 1 & \lambda-u & 0 & 0 & \dots & 0 \\ x & \lambda & \lambda-u & 0 & \dots & 0 \\ x^2 & \lambda & 2\lambda & \lambda-u & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x^{n-1} & \lambda & \binom{n-1}{1}\lambda & \binom{n-1}{2}\lambda & \dots & \lambda-u \\ x^n & \lambda & \binom{n}{1}\lambda & \binom{n}{2}\lambda & \dots & \binom{n}{n-1}\lambda \end{vmatrix}$$

as t → 0 and $|.|_{(n+1)\times(n+1)}$ denotes the $(n+1)\times(n+1)$ determinant of the lower Hessenberg matrix. ■

The next result is an immediate consequence of Theorem 2.8.

Corollary 2.9. Suppose $x = 0$. The Apostol-Frobenius-Euler numbers $H_n(\lambda;u)$ $(n \in \mathbb{N})$ where λ ,u $\in \mathbb{C}, \lambda \neq 1, u \neq 1, u \neq \lambda$ can be expressed as

$$H_n(\lambda;u) = \frac{(1-u)(-1)^n}{(\lambda-u)^{n+1}}\begin{vmatrix} 1 & \lambda-u & 0 & 0 & \dots & 0 \\ 0 & \lambda & \lambda-u & 0 & \dots & 0 \\ 0 & \lambda & 2\lambda & \lambda-u & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \lambda & \binom{n-1}{1}\lambda & \binom{n-1}{2}\lambda & \dots & \lambda-u \\ 0 & \lambda & \binom{n}{1}\lambda & \binom{n}{2}\lambda & \dots & \binom{n}{n-1}\lambda \end{vmatrix}$$

with $|\cdot|$ denotes the $(n+1)\times(n+1)$ determinant of the lower Hessenberg matrix.

Conclusion

With the proofs presented from the previous section, the closed forms for the Apostol-Frobenius-Euler polynomials were established. The generating function of the Apostol-Frobenius-Euler polynomials was crucial in determining its closed forms because it paved way for the researchers to make use with the following: (1) some properties of the Bell polynomials of the second kind, (2) the Faà di Bruno formula for computing higher order derivatives of composite functions, (3) by Leibnitz formula for the nth derivative of the product of two function, (4) and the determinantal representation of the quotient rule. These allow the closed forms be obtained.

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