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A Reviewed Version of Lagrangian Mechanics: An Introduction of Citician Dynamics in Classical Mechanics

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Abstract
Citician is an advanced concept in classical dynamics, serving as a reviewed version of the Lagrangian formalism. It is defined as the total time derivative of a system's Lagrangian. While its functional structure mirrors that of the Lagrangian, the behaviour of Citician aligns more closely with the Hamiltonian. Fundamentally, Citician represents twice the instantaneous power of a holonomic system where the Lagrangian is not an explicit function of time.
A defining feature of Citician is its dependence on both the first and second derivatives of generalized coordinates, despite having only a single independent variable, similar to the Lagrangian. This dependency leads to the derivation of three canonical equations using conjugate coordinates (q, p) within phase space. In this regard, Citician's action parallels that of the Hamiltonian while offering a broader spectrum of transformations, making it particularly valuable for simplifying complex solutions.
Citician mechanics is characterized by its succinctness and adaptability, especially in scenarios involving moving or disconnecting fluid boundaries. It is applicable to simple systems such as a bouncing ball, pendulum, or oscillating spring—where energy oscillates between kinetic and potential forms. However, its true strength lies in modelling more intricate dynamic systems, such as planetary orbits in celestial mechanics.

Keywords: Canonical, Generalised Coordinates, Hamiltonian, Lagrangian

The development of classical mechanics has progressed through multiple generations, each marked by significant contributions from prominent physicists.
The first generation, Newtonian mechanics, was established by Sir Isaac Newton, who focused on the concepts of effective force and momentum to describe the dynamics of classical systems.
The second generation emerged with Lagrangian mechanics, introduced by the Italian theoretician Joseph-Louis Lagrange. Building on the principles of the calculus of variations, Lagrange shifted the focus to the kinetic and potential energies of a system, using these as the primary tools for analysing the behaviour of classical systems.
The third generation was pioneered by the Irish physicist Sir William Rowan Hamilton, who extended Lagrangian mechanics by introducing two canonical equations based on generalized coordinates (q, p). This development marked the transition to Hamiltonian mechanics, emphasizing the phase-space representation of dynamical systems.

Citician mechanics represents a significant enhancement of Lagrangian mechanics. It introduces three canonical equations in a Hamiltonian style while offering a broader range of possible transformations than Hamiltonian mechanics. This greater versatility makes Citician mechanics particularly valuable for simplifying and analysing a variety of classical systems, expanding the scope of classical mechanics into new domains.

1. Formation of Citician

We consider that for a holonomic[1] conservative system, $L = L(q_j, \dot{q}_j)$ and we also consider that Lagrangian[2] of the system not function of time explicitly i.e. $\frac{\partial L}{\partial t} = 0$. So we have $\frac{dH}{dt} = 0$ for the given conservative[3] system.

Thus we have for $L = L(q_j, \dot{q}_j)$, $dL = \sum \frac{\partial L}{\partial q_j} dq_j + \sum \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j = \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) dq_j + \sum \left(\frac{\partial L}{\partial \dot{q}_j}\right) d\dot{q}_j \Rightarrow dL = \sum \dot{p}_j dq_j + \sum p_j d\dot{q}_j$ and we get $\frac{dL}{dt} = \sum \dot{p}_j\, \dot{q}_j + \sum p_j\, \ddot{q}_j$

We now take another parameter C which is the total time rate of change of Lagrangian of the system then $C = \frac{dL}{dt}$. Here by considering $C = C(q_j, \dot{q}_j, \ddot{q}_j)$ we get

$$dC = \sum \frac{\partial C}{\partial q_j} dq_j + \sum \frac{\partial C}{\partial \dot{q}_j} d\dot{q}_j + \sum \frac{\partial C}{\partial \ddot{q}_j} d\ddot{q}_j \ldots\ldots\ldots\ldots (1)$$

$$\text{As} \quad C = \frac{dL}{dt} \Rightarrow dC = \frac{d}{dt}(dL) \Rightarrow dC = \frac{d}{dt}\left(\sum \dot{p}_j dq_j + \sum p_j d\dot{q}_j\right)$$

$$\Rightarrow dC = \frac{d}{dt}\left(\sum \dot{p}_j\, dq_j\right) + \frac{d}{dt}\left(\sum p_j\, d\dot{q}_j\right) = \sum \ddot{p}_j\, dq_j + \sum \dot{p}_j\, d\dot{q}_j + \sum \dot{p}_j\, d\dot{q}_j + \sum p_j\, d\ddot{q}_j$$

So we get $dC = \sum \ddot{p}_j\, dq_j + 2\sum \dot{p}_j\, d\dot{q}_j + \sum p_j\, d\ddot{q}_j \ldots\ldots\ldots\ldots\ldots (2)$

Finally from equations (1) and (2) $\frac{\partial C}{\partial q_j} = \ddot{p}_j$, $\frac{\partial C}{\partial \dot{q}_j} = 2\dot{p}_j$, $\frac{\partial C}{\partial \ddot{q}_j} = p_j$ These are Citician canonical equations.

2. Application of Citician Dynamics

a) Linear Harmonic Oscillator (One Dimension):

We have Lagrangian for one dimensional harmonic oscillator[4], kinetic energy $T = \frac{1}{2}m\dot{x}^2$ and potential energy $V = \frac{1}{2}kx^2$. So we get Lagrangian of that oscillator $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

The generalized momentum[5] of this oscillator is $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$.

As in this case $C = \frac{dL}{dt} = m\dot{x}\ddot{x} - kx\dot{x}$ we get from Citician canonical equations

$$\frac{\partial C}{\partial q_j} = \ddot{p}_j \Rightarrow \frac{\partial C}{\partial x} = \ddot{p}_x \Rightarrow \frac{\partial C}{\partial x} = \frac{d}{dt}(m\ddot{x}) \Rightarrow -k\dot{x} = \frac{d}{dt}(m\ddot{x}) \Rightarrow \frac{d}{dt}(m\ddot{x}) = -k\frac{dx}{dt}$$

$$\Rightarrow \quad m\ddot{x} + kx = c_1 \qquad (\text{where } c_1 = \text{constant of integration})$$

But in this case, at $x = 0$, acceleration $\ddot{x} = 0$, we get $c_1 = 0$

Finally we get $m\ddot{x} + kx = 0 \Rightarrow \ddot{x} + \frac{k}{m}x = 0 \Rightarrow \ddot{x} + \omega_0^2 x = 0$. This is general equation of motion for one dimensional harmonic oscillator.

Again from another canonical equation $\frac{\partial C}{\partial \dot{q}_j} = 2\dot{p}_j$

We get $\frac{\partial C}{\partial \dot{x}} = 2\dot{p}_x \Rightarrow m\ddot{x} - kx = 2m\ddot{x} \Rightarrow m\ddot{x} + kx = 0 \Rightarrow \ddot{x} + \frac{k}{m}x = 0 \Rightarrow \ddot{x} + \omega_0^2 x = 0$

Also from 3rd canonical equation $\frac{\partial C}{\partial \ddot{q}_j} = p_j$ which simply gives $\frac{\partial C}{\partial \ddot{q}_j} = \frac{\partial C}{\partial \ddot{x}} = m\dot{x} = p_x = p_j$

b) Oscillation of Simple Pendulum:

Here Lagrangian for oscillation of Simple Pendulum is

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - (-mgl\ Cos\theta) = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\ Cos\theta, \qquad [As\ P.E = V = -mgl\ Cos\theta]$$

So here we get generalized momentum $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$ and $C = \frac{dL}{dt} = ml^2\dot{\theta}\ddot{\theta} - mgl\ Sin\theta\dot{\theta}$

Thus from Citician canonical equations[6]

$$\frac{\partial C}{\partial q_j} = \dot{p}_j \Rightarrow \frac{\partial C}{\partial \theta} = \dot{p}_\theta \Rightarrow -mglCos\theta\dot{\theta} = \frac{d}{dt}(ml^2\ddot{\theta}) \Rightarrow \frac{d}{dt}(ml^2\ddot{\theta} + mglSin\theta) = 0$$

$$\Rightarrow ml^2\ddot{\theta} + mglSin\theta = c_2 \ \ (\text{where } c_2 = \text{constant of integration})$$

But here at $\theta = 0$, $\ddot{\theta} = 0$, we get $c_2 = 0$ and we get $ml^2\ddot{\theta} + mgl\theta = 0$ (for $\theta \to 0$)

And finally we get $\ddot{\theta} + \frac{g}{l}\theta = 0$. This is general equation of motion for pendulum oscillation. Again from another canonical equation

$$\frac{\partial C}{\partial \dot{q}_j} = 2\dot{p}_j \Rightarrow \frac{\partial C}{\partial \dot{\theta}} = 2\dot{p}_\theta \Rightarrow ml^2\ddot{\theta} - mgl\ Sin\theta = 2\ ml^2\ddot{\theta} \Rightarrow ml^2\ddot{\theta} + mgl\ Sin\theta = 0$$

$\Rightarrow ml^2\ddot{\theta} + mgl\ \theta = 0$ (where $\theta \to 0$) $\Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$ which is also equation of motion for pendulum oscillation. Again from another canonical equation we get

$$\frac{\partial C}{\partial \ddot{q}_j} = p_j \Rightarrow \frac{\partial c}{\partial \ddot{\theta}} = p_\theta \Rightarrow p_\theta = ml^2\dot{\theta}$$

c) Particle Motion under Attractive Central Force:

Here we get for particle motion under attractive inverse square law[7] force $F = -\frac{k}{r^2} \Rightarrow V = -\frac{k}{r}$ and Lagrangian of the system

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{k}{r} \text{ where } p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \text{ and } p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \text{ and we get}$$

$$C = \frac{dL}{dt} = m\left(\dot{r}\ddot{r} + r^2\dot{\theta}\ddot{\theta} + r\dot{\theta}^2\right) - \frac{k}{r^2}\dot{r} \text{ and from Citician canonical equation}$$

$$\frac{\partial C}{\partial r} = \dot{p}_r \Rightarrow 2mr\dot{\theta}\ddot{\theta} + \dot{r}\dot{\theta}^2 + \frac{2k}{r^3}\dot{r} = \frac{d}{dt}(m\ddot{r})$$

$$\Rightarrow 2mr\dot{\theta}\ddot{\theta} + \dot{r}\dot{\theta}^2 - \frac{d}{dt}\left(\frac{k}{r^2}\right) = \frac{d}{dt}(m\ddot{r}) \Rightarrow \frac{d}{dt}\left(m\ddot{r} + \frac{k}{r^2}\right) = 2mr\dot{\theta}\ddot{\theta} + \dot{r}\dot{\theta}^2 \ \ldots\ldots\ldots (1)$$

Again we have from another canonical equation $\frac{\partial C}{\partial \dot{r}} = 2\dot{p}_r \Rightarrow m\ddot{r} + mr\dot{\theta}^2 - \frac{k}{r^2} = 2\ m\ddot{r}$

$$\Rightarrow m\ddot{r} - mr\dot{\theta}^2 = -\frac{k}{r^2} \Rightarrow m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{k}{r^2} = F(r) \Rightarrow m\left(\ddot{r} - r\dot{\theta}^2\right) = F(r)$$

This is the general equation of motion for particle motion under attractive central force. Also we have from another canonical equation

$$\frac{\partial C}{\partial \theta} = \dot{p}_\theta \Rightarrow 0 = \frac{d^2}{dt^2}(mr^2\dot{\theta}) \Rightarrow \frac{d}{dt}\left(mr^2\dot{\theta}\right) = const = c_1(say) \ldots\ldots\ldots (2)$$

$$\Rightarrow m\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right) = c_1 \ldots\ldots\ldots (3)$$

Again we have

$$\frac{\partial C}{\partial \dot{\theta}} = 2\dot{p}_\theta \Rightarrow m\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right) = 2\frac{d}{dt}\left(mr^2\dot{\theta}\right) \Rightarrow m\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right) = 2\left[\left(mr^2\ddot{\theta}\right) + 2mr\dot{r}\dot{\theta}\right)]$$

$$\Rightarrow m\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right) = 0 \ldots\ldots\ldots (4)$$

Comparing equation (3) *with equation* (4) we get $c_1 = 0$. So finally from equation (3) or (4)

$\frac{d}{dt}(mr^2\dot{\vartheta}) = 0 \implies mr^2\dot{\theta} = Constant \implies mr^2\dot{\theta} = mh = H \implies r^2\dot{\theta} = h = constant.$

This is also relevant condition for particle motion under central force which gives areal velocity constant for such motion.

d) Motion of Single Atwood Machine:

For a single Atwood Machine with masses m_1 and m_2 suspended from a massless pulley through a massless spring, we get Lagrangian of the system $L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + mgx_1 + m_2gx_2$

$But\ x_1 + x_2 + \pi r = L_0 = const \implies x_2 = L_0 - \pi r - x_1 \implies \dot{x}_2 = -\dot{x}_1 \implies \ddot{x}_2 = -\ddot{x}_1$

Thus we get

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + (m_1 - m_2)\, gx_1 + m_2gL_{10} \quad \text{where} \quad L_{10} = L_0 - \pi r = constant$$

Here the generalized momentum $p_{x_1} = \frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\,\dot{x}_1$. Also here Cictician of the system is given by $C = \frac{dL}{dt} = (m_1 + m_2)\dot{x}_1\ddot{x}_1 + (m_1 - m_2)g\dot{x}_1$ and from Cictician canonical equation

$\frac{\partial C}{\partial x_1} = \dot{p}_{x_1} \quad \implies \quad 0 = \frac{d}{dt}\,[(m_1 + m_2)\ddot{x}_1] = 0 \quad \implies \quad \ddot{x}_1 = const = -\ddot{x}_2$

Again from another canonical equation

$$\frac{\partial C}{\partial \dot{x}_1} = 2\dot{p}_{x_1} \implies (m_1 + m_2)\ddot{x}_1 + (m_1 - m_2)g = 2(m_1 + m_2)\ddot{x}_1$$

$$\implies \ddot{x}_1 = \frac{(m_1 - m_2)g}{(m_1 + m_2)} = -\ddot{x}_2 = \text{Acceleration of mass in a single Atwood machine}$$

In similar fashion, Cictician dynamics can be applied to solve and analysis motion of several conservative dynamical systems.

3. Characteristics of Cictician

Here the characteristics of Cictician are

i) For Lagrangian of the system $L = L(q_i, \dot{q}_j)$ we have $C = C(q_i, \dot{q}_j, \ddot{q}_j)$ where $\frac{\partial C}{\partial t} = 0$ for $\frac{\partial L}{\partial t} = 0$

ii) For $L \neq L(q_k)$, $p_k = constant,\ \dot{p}_k = 0\ and\ \ddot{p}_k = 0$. This gives $\frac{\partial C}{\partial q_k} = \frac{\partial C}{\partial \dot{q}_k} = 0$. Thus if Lagrangian of the system be cyclic in q_k , the corresponding Cictician $'C'$ is also cyclic in q_k and $\dot{q}_k$.

iii) The 3rd Cictician canonical equation $\frac{\partial C}{\partial \ddot{q}_j} = p_j \implies \frac{\partial C}{\partial \ddot{q}_j} = \frac{\partial L}{\partial \dot{q}_j} = p_j$. So 3rd canonical equation in Cictician dynamics gives generalized momentum.

iv) Since $\frac{dL}{dt} = \sum \dot{p}_j\dot{q}_j + \sum p_j\ddot{q}_j$ we should have

$\frac{\partial}{\partial q_j}\left(\frac{dL}{dt}\right) = \ddot{q}_j\frac{\partial p_j}{\partial q_j} = \ddot{q}_j\frac{\partial}{\partial q_j}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \ddot{q}_j\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial L}{\partial q_j}\right) = \frac{d\dot{q}_j}{dt}\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial L}{\partial q_j}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial q_j}\right) = \frac{d}{dt}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right] = \frac{d}{dt}(\dot{p}_j) = \ddot{p}_j$

So finally we get $\frac{\partial C}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\frac{dL}{dt}\right) = \ddot{p}_j$. This 1st canonical equation in Cictician dynamics which can also be obtained directly from Lagrangian dynamics

v) In Lagrangian dynamics we should have $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \neq \frac{\partial}{\partial \dot{q}_j}\left(\frac{dL}{dt}\right)$. Because as we have

$\frac{dL}{dt} = \sum \dot{p}_j\dot{q}_j + \sum p_j\ddot{q}_j$ and $\frac{\partial}{\partial \dot{q}_j}\left(\frac{dL}{dt}\right) = \dot{p}_j + \ddot{q}_j\frac{\partial}{\partial \dot{q}_j}(p_j) = \dot{p}_j + \frac{d\dot{q}_j}{dt}\frac{\partial}{\partial \dot{q}_j}(p_j) = \dot{p}_j + \dot{p}_j = 2\dot{p}_j$

i.e. $\frac{\partial}{\partial \dot{q}_j}\left(\frac{dL}{dt}\right) = \frac{\partial C}{\partial \dot{q}_j} = 2\dot{p}_j$. On the other hand in Lagrangian dynamics $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \dot{p}_j$.

Thus $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \neq \frac{\partial}{\partial \dot{q}_j}\left(\frac{dL}{dt}\right)$ $i.e. \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \neq \frac{\partial C}{\partial \dot{q}_j}$

And in Citician dynamics $\frac{\partial C}{\partial \dot{q}_j} = 2\dot{p}_j = 2\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 2\frac{\partial L}{\partial q_j}$ This 2nd canonical equation in Citician dynamics which can also be obtained directly from Lagrangian dynamics

vi) Since $\frac{dL}{dt} = \sum \dot{p}_j \dot{q}_j + \sum p_j \ddot{q}_j$ we should have $\frac{\partial C}{\partial \ddot{q}_j} = p_j$. This 3rd canonical equation in Citician dynamics which can also be obtained directly from Lagrangian dynamics

vii) The instantaneous power for the dynamical system is $P = \frac{dW}{dt} = \sum Q_j \dot{q}_j$, we can now write down from Lagrange's equation of 1st kind

$$P = \frac{dW}{dt} = \sum Q_j \dot{q}_j = \sum\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\dot{q}_j = \left[\sum \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\dot{q}_j - \sum \frac{\partial T}{\partial q_j}\dot{q}_j\right]$$
$$= \sum \dot{p}_j \dot{q}_j - \sum \frac{\partial T}{\partial q_j}\dot{q}_j$$

Consider that $T = T(q_j, \dot{q}_j) \Rightarrow dT = \sum \frac{\partial T}{\partial q_j} dq_j + \sum \frac{\partial T}{\partial \dot{q}_j} d\dot{q}_j = \sum \frac{\partial T}{\partial q_j} dq_j + \sum p_j\, d\dot{q}_j$

So we get $\frac{dT}{dt} = \sum \frac{\partial T}{\partial q_j}\dot{q}_j + \sum p_j\, \ddot{q}_j \Rightarrow \sum \frac{\partial T}{\partial q_j}\dot{q}_j = \frac{dT}{dt} - \sum p_j\, \ddot{q}_j$

So we get

$$P = \frac{dW}{dt} = \sum \dot{p}_j \dot{q}_j + \sum p_j\, \ddot{q}_j - \frac{dT}{dt} = \frac{d}{dt}\left(\sum p_j\, \dot{q}_j\right) - \frac{dT}{dt} = \frac{d}{dt}(L + H) - \frac{dT}{dt} = \frac{dT}{dt} = -\frac{dV}{dt}$$

But for conservative system with Lagrangian not function of time explicitly, $\frac{dH}{dt} = 0$ Or. $\frac{d(T+V)}{dt} = 0$

So we get $\frac{dT}{dt} = -\frac{dV}{dt}$ and finally we get $P = \frac{dW}{dt} = \frac{d}{dt}(L + H) - \frac{dT}{dt} = -\frac{dV}{dt} = \frac{dT}{dt}$. So Citician is actually given by $C = \frac{dL}{dt} = \frac{dT}{dt} + \left(-\frac{dV}{dt}\right) = 2\frac{dT}{dt} = 2P =$ Twice the instantaneous power of the dynamic system.

viii) Citician for a dynamic conservative system is actually

$$C = \frac{dL}{dt} = 2\frac{dT}{dt} = -2\frac{dV}{dt} = 2P = 2 \times \text{Power}$$

Concluding Remarks

In summary, Citician represents an extension of Lagrangian mechanics for holonomic conservative systems in classical dynamics. Defined as the total time derivative of a system's Lagrangian, Citician closely mirrors the behaviour of the Hamiltonian. Fundamentally, it encapsulates twice the instantaneous power of a holonomic conservative system where the Lagrangian is not an explicit function of time.

A defining characteristic of Citician is its reliance not only on a single independent variable, as in Lagrangian mechanics, but also on the first and second derivatives of generalized coordinates. This distinction enables the derivation of three canonical equations using conjugate coordinates (q, p) in phase space. In doing so, Citician offers a broader range of possible transformations compared to Hamiltonian mechanics, making it particularly valuable for simplifying and solving complex dynamical systems.

The concept of Citician opens new avenues for research, with the potential for further exploration and refinement to unlock additional applications in classical mechanics and beyond.

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