On the Region Containing the Zeros of a Polynomial with Restricted Coefficients

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ABSTRACT
For the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$, $a_j \geq a_{j-1}$, $a_0 > 0$, $j = 1, 2, ..., n$, $a_n > 0$, a classical result of Eneström and Kakeya says that all the zeros of P (z) lie in $|z| \leq 1$. In the literature [1-12], there exist several extensions and generalizations of this result. Recently N.A.Rather and M.A.Shah [12] generalized it by further relaxing the condition on the coefficients. In this paper, we prove some more extensions and generalizations of the above results and hence of Eneström and Kakeya theorem.
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Introduction
If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n, then Eneström and Kakeya [10,11] proved the following intresting result regarding the location of zeros of a polynomial with restricted coefficients.
Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree **n** such that
$a_n \geq a_{n-1} \geq \dots\dots\dots\dots \geq a_1 \geq a_0 > 0$
then P(z) has all its zeros in $|z| \leq 1$.
For example: The Polynomial $4z^4 + 3z^2 + z + 1$ has all zeros in $|z| \leq 1$.

In the literature [1-12], there exist several extensions and generalizations of this theorem. Joyal et al [9] extended Theorem A to the polynomials whose coefficients are monotonic but not necessarily non-negative. Infact they proved the following result.

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree **n** such that
$a_n \geq a_{n-1} \geq \dots\dots\dots\dots \geq a_1 \geq a_0$
Then P (z) has all its zeros in the disk
$|z| \leq \frac{1}{a_n}(|a_n| - a_0 + |a_0|)$.
Govil and Rahman [8] extended the result to the class of polynomials with complex coefficients by proving the following result.

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree **n** with complex coefficients such that for some real β,

$|\arg \alpha_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $0 \leq j \leq n$.
and
$|a_n| \geq |a_{n-1}| \geq \ldots \geq |a_1| \geq |a_0|$.
Then $P(z)$ has all its zeros in the disk
$|z| \leq (\sin\alpha + \cos\alpha) + \frac{2\sin\alpha}{|a_n|} \sum_{j=0}^{n-1} |a_j|$.
Aziz and Zargar [2] relaxed the hypothesis of Theorem A and proved the following extension of Theorem A.
Theorem D: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree **n** such that for some $k \geq 1$,
$K\, a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0$.
Then $P(z)$ has all its zeros in $|z + k - 1| \leq k$.
Recently N.A.Rather and M.A.Shah [12] obtained some generalization of the above results by proving the following
Theorem E: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree **n** with complex coefficients. For **j** $= 0, 1, \ldots, n$, let $\text{Re}\, a_j = \alpha_j$ and $\text{Im}\, a_j = \beta_j$. If for some reals t , s and for some $\lambda \in \{1, 2, \ldots, n - 1\}$,
$t + \alpha_n \leq \alpha_{n-1} \leq \ldots \leq \alpha_\lambda \geq \alpha_{\lambda-1} \geq \ldots \geq \alpha_1 \geq \alpha_0 - s$
and
$\beta_n \geq \beta_{n-1} \geq \ldots \geq \beta_1 \geq \beta_0 > 0$,
then all the zeros of $P(z)$ lie in the union of disks $|z| \leq 1$ and
$|z + \frac{t}{a_n}| \leq \frac{1}{|a_n|} \{ 2\alpha_\lambda - (\alpha_n + t) - \alpha_0 + 2s + |\alpha_0| + \beta_n \}$
In this paper, we provide some more generalizations of the Eneström and Kakeya theorem and of the above results. In this direction, we first prove the following.
Theorem 1. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree **n** with complex coefficients. For **j** $= 0, 1, \ldots, n$, let $\text{Re}\, a_j = \alpha_j$ and $\text{Im}\, a_j = \beta_j$. If for some positive t and for some $\lambda, \mu \in \{1, 2, \ldots, n - 1\}$ and $\lambda > \mu$
$t + \alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_\lambda \leq \alpha_{\lambda-1} \leq \ldots \leq \alpha_\mu \geq \alpha_{\mu-1} \geq \ldots \geq \alpha_1 \geq \alpha_0$
and
$\beta_n \geq \beta_{n-1} \geq \ldots \geq \beta_1 \geq \beta_0 > 0$
then all the zeros $P(z)$ lie in the union of disks $|z| \leq 1$ and
$|z + \frac{t}{a_n}| \leq \frac{1}{|a_n|} \{ \alpha_\lambda - t - 2\alpha_\lambda + 2\alpha_\mu - \alpha_0 + |\alpha_0| + \beta_n \}$
Remark: Taking $t = (k-1)\alpha_n$, $k > 1$, in theorem 1, we obtain the following result

Corollary 1. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree **n** with complex coefficients. For **j** $= 0, 1, \ldots, n$, let $\text{Re}\, a_j = \alpha_j$ and $\text{Im}\, a_j = \beta_j$. If for some $k > 1$ and for some $\lambda, \mu \in 1, 2, \ldots, n - 1$ and $\lambda > \mu$
$k\alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_\lambda \leq \alpha_{\lambda-1} \leq \ldots \leq \alpha_\mu \geq \alpha_{\mu-1} \geq \ldots \geq \alpha_1 \geq \alpha_0$
and
$\beta_n \geq \beta_{n-1} \geq \ldots \geq \beta_1 \geq \beta_0 > 0$
then all the zeros $P(z)$ lie in the union of disks $|z| \leq 1$ and
$|z + \frac{\alpha_n}{a_n}(1 - k)| \leq \frac{1}{|a_n|} \{ 2\alpha_\mu - 2\alpha_\lambda + k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n \}$

If $\alpha_0 > 0$, then we get the following result.

Corollary 2. Corollary **2.** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree **n** with complex coefficients. For $j = 0, 1, \ldots, n$, let $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$. If for some positive t and for some $\lambda, \mu \in \{1,2,3\ldots\ldots\ldots n-1\}$ and $\lambda > \mu$

$t + \alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_\lambda \leq \alpha_{\lambda-1} \leq \ldots \leq \alpha_\mu \geq \alpha_{\mu-1} \geq \ldots \geq \alpha_1 \geq \alpha_0$

and

$\beta_n \geq \beta_{n-1} \geq \ldots \geq \beta_1 \geq \beta_0 > 0$

then all the zeros P (z) lie in the union of disks $|z| \leq 1$ and

$$\left|z + \frac{t}{a_n}\right| \leq \frac{1}{|a_n|}\{\alpha_n + t - 2\alpha_\lambda + 2\alpha_\mu + \beta_n\}$$

Instead of proving Theorem 1. We prove the following more general result.

Theorem 2. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree **n** with complex coefficients. For $j = 0, 1, \ldots, n$, let $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$. If for some positive t, s and for some $\lambda, \mu \in \{1,2,3\ldots\ldots\ldots n-1\}$ and $\lambda > \mu$

$t + \alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_\lambda \leq \alpha_{\lambda-1} \leq \ldots \leq \alpha_\mu \geq \alpha_{\mu-1} \geq \ldots \geq \alpha_1 \geq \alpha_0$

and

$\beta_n \geq \beta_{n-1} \geq \ldots \geq \beta_1 \geq \beta_0 > 0$

then all the zeros of P (z) lie in the union of disks $|z| \leq 1$ and

$$\left|z + \frac{t}{a_n}\right| \leq \frac{1}{|a_n|}\{\alpha_n + t - 2\alpha_\lambda + 2\alpha_\mu - \alpha_\circ + 2s + |\alpha_\circ| + \beta_n\}$$

For $s = 0$, Theorem 2 reduces to Theorem 1.

For $t = 0$, Theorem 2 reduces to the following result.

Corollary 3. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree **n** with complex coefficients. For $j = 0, 1, \ldots, n$, let $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$. If for some real s and for some $\lambda, \mu \in \{1,2,3\ldots\ldots\ldots n-1\}$ and $\lambda > \mu$

$\alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_\lambda \leq \alpha_{\lambda-1} \leq \ldots \leq \alpha_\mu \geq \alpha_{\mu-1} \geq \ldots \geq \alpha_1 \geq \alpha_0 - s$

and

$\beta_n \geq \beta_{n-1} \geq \ldots \geq \beta_1 \geq \beta_0 > 0$

then all the zeros of P (z) lie in the union of disks $|z| \leq 1$ and

$$|z| \leq \frac{1}{|a_n|}\{\alpha_n - 2\alpha_\lambda + 2\alpha_\mu - \alpha_\circ + |\alpha_\circ| + \beta_n\}$$

Finally, we present the following result for the polynomials with real coefficients.

Theorem 2. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree **n.** If for some positive numbers t, s and $\lambda > \mu$

$t + a_n \geq a_{n-1} \geq \ldots \geq a_\lambda \leq a_{\lambda-1} \leq \ldots \leq a_\mu \geq a_{\mu-1} \geq \ldots \geq a_1 \geq a_0 - s \geq 0$

then all the zeros of P (z) lie in the closed disk

$$\left|z+\frac{t}{a_n}\right| \le \frac{1}{|a_n|}\{\alpha_n + t - 2\alpha_\lambda + 2\alpha_\mu + 2s\}$$

Proof of Theorem 2. Consider the polynomial .

$|F(z)| = (1-z)P(z)$

$= (1-z)\,(a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0)$

$= -\ a_n z^{n+1} + \{(a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0\}$

$= -z^n\,(a_n z + t) + \{(a_n + t - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0 + s)z - sz + \alpha_0\} + i\{(\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + \cdots + (\beta_1 - \beta_0)z + \beta_0\}$

This gives

$$\begin{aligned}|F(z)| \ge\ & |z|^n\,|a_n z + t| \\ & - \{|\alpha_n + t - \alpha_{n-1}||z|^n + |\alpha_{n-1} - \alpha_{n-2}||z|^{n-1} + \cdots + |\alpha_{\lambda+1} - \alpha_\lambda||z|^{\lambda+1} \\ & + |\alpha_\lambda - \alpha_{\lambda-1}||z|^\lambda + |\alpha_{\lambda-1} - \alpha_{\lambda-2}||z|^{\lambda-1} + \cdots + |\alpha_{\mu+1} - \alpha_\mu||z|^{\mu+1} + |\alpha_\mu - \alpha_{\mu-1}||z|^\mu \\ & + |\alpha_{\mu-1} - \alpha_{\mu-2}||z|^{\mu-1} + \cdots + |\alpha_1 - (\alpha_0 - s)||z| + s|z||\alpha_0| + |\beta_n - \beta_{n-1}||z|^n \\ & + |\beta_{n-1} - \beta_{n-2}|\,|z|^{n-1} + \cdots + |\beta_1 - \beta_0||z| + |\beta_0|\}\end{aligned}$$

$= |z|^n \Big\{|a_n z + t| - |\alpha_n + t - \alpha_{n-1}| + \frac{|\alpha_{n-1}-\alpha_{n-2}|}{|z|} + \cdots + \frac{|\alpha_{\lambda+1}-\alpha_\lambda|}{|z|^{n-\lambda-1}} + \frac{|\alpha_\lambda-\alpha_{\lambda-1}|}{|z|^{n-\lambda}} + \frac{|\alpha_{\lambda-1}-\alpha_{\lambda-2}|}{|z|^{n-\lambda+1}} + \cdots + \frac{|\alpha_{\mu+1}-\alpha_\mu|}{|z|^{n-\mu-1}} + \frac{|\alpha_\mu-\alpha_{\mu-1}|}{|z|^{n-\mu}} + \frac{|\alpha_{\mu-1}-\alpha_{\mu-2}|}{|z|^{n-\mu+1}} + \cdots + \frac{|\alpha_1-(\alpha_0-s)|}{|z|^{n-1}} + \frac{s}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n} + |\beta_n - \beta_{n-1}| + \frac{|\beta_{n-1}-\beta_{n-2}|}{|z|} + \cdots + \frac{|\beta_1-\beta_0|}{|z|^{n-1}} + \frac{|\beta_0|}{|z|^n}\Big\}$

Now let $|z| > 1$, so that $\frac{1}{|z|^{n-j}} < 1\,, 0 \le j \le n$, then we have

$$\begin{aligned}|F(z)| \ge\ & |z|^n\{|a_n z + t| - |\alpha_n + t - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + \cdots + |\alpha_{\lambda+1} - \alpha_\lambda| + |\alpha_\lambda - \alpha_{\lambda-1}| \\ & + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + \cdots + |\alpha_{\mu+1} - \alpha_\mu| + |\alpha_\mu - \alpha_{\mu-1}| + |\alpha_{\mu-1} - \alpha_{\mu-2}| + \cdots \\ & + |\alpha_1 - (\alpha_0 - s)| + (s + |\alpha_0|) + |\beta_n - \beta_{n-1}| + |\beta_{n-1} - \beta_{n-2}| + \cdots \\ & + |\beta_1 - \beta_0| + |\beta_0|\}|F(z)| \\ \ge\ & |z|^n\{|a_n z + t| - |\alpha_n + t - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + \cdots + |\alpha_{\lambda+1} - \alpha_\lambda| \\ & + |\alpha_\lambda - \alpha_{\lambda-1}| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + \cdots + |\alpha_{\mu+1} - \alpha_\mu| + |\alpha_\mu - \alpha_{\mu-1}| + |\alpha_{\mu-1} - \alpha_{\mu-2}| \\ & + \cdots + |\alpha_1 - (\alpha_0 - s)| + (s + |\alpha_0|) + |\beta_n - \beta_{n-1}| + |\beta_{n-1} - \beta_{n-2}| + \cdots \\ & + |\beta_1 - \beta_0| + |\beta_0|\}\end{aligned}$$

$= |z|^n\,[\,|a_n z + t| - \{\alpha_n + t - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + \cdots + \alpha_{\lambda+1} - \alpha_\lambda - \alpha_\lambda + \alpha_{\lambda-1} - \alpha_{\lambda-1} + \alpha_{\lambda-2} + \cdots - \alpha_{\mu+1} + \alpha_\mu + \alpha_\mu - \alpha_{\mu-1} + \alpha_{\mu-1} - \alpha_{\mu-2} + \cdots + \alpha_1 - (\alpha_0 - s) + (s + |\alpha_0|) + \beta_n - \beta_{n-1} + \beta_{n-1} - \beta_{n-2} + \cdots + \beta_1 - \beta_0 + \beta_0\}]$

$= |z|^n\,[\,|a_n z + t| - \{\alpha_n + t - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + \cdots + \alpha_{\lambda+1} - \alpha_\lambda - \alpha_\lambda + \alpha_{\lambda-1} - \alpha_{\lambda-1} + \alpha_{\lambda-2} + \cdots - \alpha_{\mu+1} + \alpha_\mu + \alpha_\mu - \alpha_{\mu-1} + \alpha_{\mu-1} - \alpha_{\mu-2} + \cdots + \alpha_1 - (\alpha_0 - s) + (s + |\alpha_0|) + \beta_n - \beta_{n-1} + \beta_{n-1} - \beta_{n-2} + \cdots + \beta_1 - \beta_0 + \beta_0\}]$

$$= |a_n z + t| > \{\alpha_n + t - 2\alpha_\lambda + 2\alpha_\mu - (\alpha_0 - s) + (s + |\alpha_0|) + \beta_n\}$$

$= |a_n z + t| > \{\alpha_n + t - 2\alpha_\lambda + 2\alpha_\mu - (\alpha_0 - s) + (s + |\alpha_0|) + \beta_n\}$

$i.e$ if $\left|z + \frac{t}{a_n}\right| > \frac{1}{|a_n|}\{\alpha_n + t - 2\alpha_\lambda + 2\alpha_\mu - \alpha_0 + s + s + |\alpha_0| + \beta_n\}$

i.e if $\left|z + \frac{t}{a_n}\right| > \frac{1}{|a_n|}\{\alpha_n + t - 2\alpha_\lambda + 2\alpha_\mu - \alpha_0 + s + s + |\alpha_0| + \beta_n\}$

Thus all the zeros of $F(z)$F(z) whose modulus is greater than or equal to 1 lie in

$$\left|z+\frac{t}{a_n}\right| > \frac{1}{|a_n|}\{\alpha_n + t - 2\alpha_\lambda + 2\alpha_\mu - \alpha_0 + 2s + |\alpha_0| + \beta_n\}$$

$$\left|z+\frac{t}{a_n}\right| > \frac{1}{|a_n|}\{\alpha_n + t - 2\alpha_\lambda + 2\alpha_\mu - \alpha_0 + 2s + |\alpha_0| + \beta_n\}$$

But all the zeros of $P(z)$P(z) are also the zeros of $F(z)$.F(z). Hence it follows that all the zeros of $F(z)$F(z) and hence of $P(z)$P(z) lie in the uniform disks of $|z|$|z| ≤ 1 and

$$\left|z+\frac{t}{a_n}\right| > \frac{1}{|a_n|}\{\alpha_n + t - 2\alpha_\lambda + 2\alpha_\mu - \alpha_0 + 2s + |\alpha_0| + \beta_n\}$$

$$\left|z+\frac{t}{a_n}\right| > \frac{1}{|a_n|}\{\alpha_n + t - 2\alpha_\lambda + 2\alpha_\mu - \alpha_0 + 2s + |\alpha_0| + \beta_n\}$$

This completes the proof of theorem 2.

Proof of Theorem 3. Consider the polynomial

$$|F(z)| = (1-z)P(z)$$

$$= (1-z)\,(a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0)$$

$$= -a_n z^{n+1} + \{(a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0\}$$

$$= -a_n z^{n+1} + \{(a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0\}$$

$$= -a_n z^{n+1} - tz^n + (a_n + t - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0 + s)z - sz + a_0.$$

$$= -z^n(a_n z + t) + (a_n + t - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0 + s)z - sz + a_0. = -a_n z^{n+1} - tz^n + (a_n + t - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0 + s)z - sz + a_0.$$

$$= -z^n(a_n z + t) + (a_n + t - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0 + s)z - sz + a_0.$$

This gives

$$\begin{aligned} |F(z)| \ge\ & |z|^n |a_n z + t| \\ & - \{|a_n + t - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots + |a_{\lambda+1} - a_\lambda||z|^{\lambda+1} \\ & + |a_\lambda - a_{\lambda-1}||z|^\lambda + |a_{\lambda-1} - a_{\lambda-2}||z|^{\lambda-1} + \cdots + |a_{\mu+1} - a_\mu||z|^{\mu+1} \\ & + |a_\mu - a_{\mu-1}||z|^\mu + |a_{\mu-1} - a_{\mu-2}||z|^{\mu-1} + \cdots + |a_1 - a_0 + s||z| + s|z| \\ & + |a_0|\} \end{aligned}$$

$$\begin{aligned} |F(z)| \ge\ & |z|^n |a_n z + t| \\ & - \{|a_n + t - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots + |a_{\lambda+1} - a_\lambda||z|^{\lambda+1} \\ & + |a_\lambda - a_{\lambda-1}||z|^\lambda + |a_{\lambda-1} - a_{\lambda-2}||z|^{\lambda-1} + \cdots + |a_{\mu+1} - a_\mu||z|^{\mu+1} \\ & + |a_\mu - a_{\mu-1}||z|^\mu + |a_{\mu-1} - a_{\mu-2}||z|^{\mu-1} + \cdots + |a_1 - a_0 + s||z| + s|z| + |a_0|\} \end{aligned}$$

$$= |z|^n \Bigg[|a_n z + t| - \Bigg\{ |a_n + t - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \cdots + \frac{|a_{\lambda+1} - a_\lambda|}{|z|^{n-\lambda-1}} + \frac{|a_\lambda - a_{\lambda-1}|}{|z|^{n-\lambda}} + \frac{|a_{\lambda-1} - a_{\lambda-2}|}{|z|^{n-\lambda+1}} + \cdots + \frac{|a_{\mu+1} - a_\mu|}{|z|^{n-\mu-1}} + \frac{|a_\mu - a_{\mu-1}|}{|z|^{n-\mu}} + \frac{|a_{\mu-1} - a_{\mu-2}|}{|z|^{n-\mu+1}} + \cdots + \frac{|a_1 - a_0 + s|}{|z|^{n-1}} + \frac{s}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \Bigg\} \Bigg]$$

$$= |z|^n \Bigg[|a_n z + t| - \Bigg\{ |a_n + t - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \cdots + \frac{|a_{\lambda+1} - a_\lambda|}{|z|^{n-\lambda-1}} + \frac{|a_\lambda - a_{\lambda-1}|}{|z|^{n-\lambda}} + \frac{|a_{\lambda-1} - a_{\lambda-2}|}{|z|^{n-\lambda+1}} + \cdots + \frac{|a_{\mu+1} - a_\mu|}{|z|^{n-\mu-1}} + \frac{|a_\mu - a_{\mu-1}|}{|z|^{n-\mu}} + \frac{|a_{\mu-1} - a_{\mu-2}|}{|z|^{n-\mu+1}} + \cdots + \frac{|a_1 - a_0 + s|}{|z|^{n-1}} + \frac{s}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \Bigg\} \Bigg]$$

Now let $|z||z| > 1, so\ that\ \frac{1}{|z|^{n-j}} < 1, 0 \le j \le n, then\ we\ have$so that $\frac{1}{|z|^{n-j}} < 1, 0 \le j \le n$, then we have

$$|F(z)| \ge |z|^n \big[|a_n z + t| - \{ |a_n + t - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{\lambda+1} - a_\lambda| + |a_\lambda - a_{\lambda-1}| + |a_{\lambda-1} - a_{\lambda-2}| + \cdots + |a_{\mu+1} - a_\mu| + |a_\mu - a_{\mu-1}| + |a_{\mu-1} - a_{\mu-2}| + \cdots + |a_1 - a_0 + s| + s + |a_0| \} \big]$$

$$|F(z)| \ge |z|^n \big[|a_n z + t| - \{ |a_n + t - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{\lambda+1} - a_\lambda| + |a_\lambda - a_{\lambda-1}| + |a_{\lambda-1} - a_{\lambda-2}| + \cdots + |a_{\mu+1} - a_\mu| + |a_\mu - a_{\mu-1}| + |a_{\mu-1} - a_{\mu-2}| + \cdots + |a_1 - a_0 + s| + s + |a_0| \} \big]$$

$$= |z|^n \big[|a_n z + t| - \{ a_n + t - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_{\lambda+1} - a_\lambda - a_\lambda + a_{\lambda-1} - a_{\lambda-1} + a_{\lambda-2} + \cdots - a_{\mu+1} + a_\mu + a_\mu - a_{\mu-1} + a_{\mu-1} - a_{\mu-2} + \cdots + a_1 - a_0 + s + s + a_0 \} \big]$$

$$= |z|^n \big[|a_n z + t| - \{ a_n + t - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_{\lambda+1} - a_\lambda - a_\lambda + a_{\lambda-1} - a_{\lambda-1} + a_{\lambda-2} + \cdots - a_{\mu+1} + a_\mu + a_\mu - a_{\mu-1} + a_{\mu-1} - a_{\mu-2} + \cdots + a_1 - a_0 + s + s + a_0 \} \big]$$

$$= |z|^n \big[|a_n z + t| - \{ a_n + t - 2a_\lambda + 2a_\mu + 2s \} \big] > 0$$

$$= |z|^n \big[|a_n z + t| - \{ a_n + t - 2a_\lambda + 2a_\mu + 2s \} \big] > 0$$

If $|a_n z + t| > a_n + t - 2a_\lambda + 2a_\mu + 2s$ $|a_n z + t| > a_n + t - 2a_\lambda + 2a_\mu + 2s$

i.e if $\left| z + \frac{t}{a_n} \right| \le \frac{1}{a_n}\{ a_n + t - 2a_\lambda + 2a_\mu + 2s \}$

i.e if $\left|z+\frac{t}{a_n}\right| \leq \frac{1}{a_n}\{a_n+t-\ 2a_\lambda+2a_\mu+2s\}$

Thus all the zeros of $F(z)$F(z) whose modulus is greater than or equal to 1 lie in

$$\left|z+\frac{t}{a_n}\right| \leq \frac{1}{|a_n|}\{a_n+t-\ 2a_\lambda+2a_\mu+2s\}$$

$$\left|z+\frac{t}{a_n}\right| \leq \frac{1}{|a_n|}\{a_n+t-\ 2a_\lambda+2a_\mu+2s\}$$

But all the zeros of $F(z)$F(z) whose modulus is less than 1 already satisfy the above inequality. Indeed for $|z|$|z| ≤ 1 , we have

$$\left|z+\frac{t}{a_n}\right| \leq |z|+\frac{t}{|a_n|} \leq 1+\frac{t}{a_n}-\frac{2a_\lambda}{a_n}+\frac{2a_\mu}{a_n}+\frac{2s}{a_n}=\frac{1}{a_n}\{a_n+t-\ 2a_\lambda+2a_\mu+2s\}$$

$$\left|z+\frac{t}{a_n}\right| \leq |z|+\frac{t}{|a_n|} \leq 1+\frac{t}{a_n}-\frac{2a_\lambda}{a_n}+\frac{2a_\mu}{a_n}+\frac{2s}{a_n}=\frac{1}{a_n}\{a_n+t-\ 2a_\lambda+2a_\mu+2s\}$$

Also all the zeros of $P(z)$P(z) are the zeros of $F(z)$.F(z). Hence it follows that all the zeros of $F(z)$F(z) and hence of $P(z)$P(z) lie in

$$\left|z+\frac{t}{a_n}\right| \leq \frac{1}{|a_n|}\{a_n+t-\ 2a_\lambda+2a_\mu+2s\}$$

$$\left|z+\frac{t}{a_n}\right| \leq \frac{1}{|a_n|}\{a_n+t-\ 2a_\lambda+2a_\mu+2s\}$$

This completes the proof of Theorem 2.

Compliance with ethical standards

1. **Conflict of interest.** Both the authors declare that they have no conflict of interest.
2. **Ethical approval.** This article does not contain any studies with human participants or animals performed by any of the authors.
3. **Consent for publication.** Both the authors have given consent for publication.
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