International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

Numerical Simulation of KdV Burgers' in Dusty Plasmas

Harekrishna Deka[1], Jnanjyoti Sarma[2]

[1]Assistant Professor, Department of Mathematics, K.K. Handiqui State Open University
[2]Associate Professor (Retd.), Department of Mathematics, R.G. Baruah College

Abstract
In this paper, the reductive perturbation method is used to derive the KdV Burgers' equation in dusty plasmas in the presence of the Boltzmann distribution of electrons and ions and charged dust grains. A numerical solution to the KdV Burgers' equation has been obtained by using the explicit finite difference method, and the solitary and shock structures have been studied at various values of the dispersion coefficient and the dissipation coefficient. A comparison of solitary and shock structures is also shown by plotting the analytical and numerical solutions. The accuracy and efficiency of the present method have been evaluated by comparing the absolute error of the numerical results obtained with the analytical solution. An analysis of Von Neumann stability is conducted and it shows that the scheme is unconditionally stable.

Keywords: Dusty plamas, Reductive perturbation method, KdV Burgers' Equation,Finite difference explicit method,Von Neumann stability analysis

1. Introduction

The branch of exploring the nonlinear wave phenomena propagating through dusty plasma has been the fastest-growing branch in recent years since dusty plasma is essential for understanding various forms of collaborative processes in the space environment, including the lower and upper mesospheres, radiofrequency plasma discharge, cometary tails, planetary rings, plasma crystals, asteroid zones, planetary magnetosphere, interplanetary spaces, interstellar medium, and the environment on the Earth, etc. [1,2].The appearance of unusual electrostatic wave types, such as solitary or shock waves like dust acoustic waves (DAWs) [3],dust ion acoustic waves (DIAWs) [4],and dust lattice waves (DLWs),[5] is thought to be caused by the presence of charged dust grains, which are typically much larger and heavier than the plasma particle in a two-component electron ion plasma. There are several uses for the propagation of nonlinear waves, particularly solitary or shock waves [6, 7], in space as well as in laboratory dusty plasmas [8, 9]. To analyse the properties of solitary and shock waves in dusty plasmas, the KdVB equation has therefore been extensively employed [10, 11, 12, 13, 14]. Exploring the modest influences of dispersion, dissipation, and nonlinearity in waves propagating through a liquid-filled elastic tube results in the formulation of the KdV Burgers' equation, as originally introduced by Su and Gardner [15]. The Korteweg-de Vries Burgers' (KdVB) equation is the best description of a medium with considerable dissipative effect and dispersion, which favours favours the generation of both shock waves and solitary waves. The Burgers' term in the nonlinear KdVB equation results from dissipative phenomena such as wave-particle interactions, turbulence, dust charge fluctuations in a dusty plasma, multi-ion streaming, Landau damping, anomalous viscosity, etc [16, 17, 18]. Typically, the Burgers' equation [19] and the KdV equation [20] are combined to form the Korteweg-deVries Burgers'(KdVB) equation.

The Korteweg-de Vries-Burgers'(KdV- Burgers') equation is a fundamental nonlinear partial differential equation that models various phenomena,including fluid dynamics and plasma physics.Numerous authors have studied numerical methods for solving KdVB equations over the past few decades. Parumasur et al. [21] introduced the orthogonal collocation on finite elements (OCFE) method,utilizing quadratic and cubic B-splines with quasilinearization. This approach is subsequently applied to solve Burgers' equation, along with the modified Burgers' and KdV–Burgers' equations.

Cao et al. [22] compared the space-time polynomial particular solutions method (ST-MPPS) to the Fourier spectral method for solving the KdV and KdV–Burgers equations across different final times.Ahmad et al [23] proposed an improved version of the Variational Iteration Algorithm-II (VIA-II), specifically designed to solve nonlinear evolution equations such as the Burgers equation, Korteweg–de Vries equation, and Korteweg–de Vries-Burgers equation.Koroche and Chemeda [24] introduced a sixth-order compact finite difference method to solve the one-dimensional KdV-Burgers equation.Datta et al. [25] developed numerical solutions for the KdV and KdV-Burger equations using an innovative approach based on the differential quadrature method. Chentouf and Guesmia [26] examined the stability and well-posedness of the KdV Burgers and Kuramoto-Sivashinsky equations, highlighting their applicability to modeling long-term behavior in plasma systems. Their work provides insights into the mathematical properties essential for simulating sustained dynamics in plasma environments.El-Tantawy et al. [27] examines solutions to the damped nonplanar KdV-Burgers equation using homotopy perturbation methods, analyzing nonlinear structures in strongly coupled dusty plasmas.Kumar and Jana[28] investigated approximate analytical solutions for solitons and shock waves in the damped Korteweg-de Vries (DKdVB) Burgers' model, specifically considering the effects of acoustic dust-ion particles. Their study contributes to understanding the complex dynamics of wave interactions in dusty plasma environments.Ballav et al [29] derived the KdV–Burger equation, the investigation of shock fronts in plasma caused by explosive events associated with Gamma-Ray Bursts (GRBs) takes place, providing insight into a variety of space plasma phenomena. Solutions are achieved by applying the" hyperbolic-tangent method" in addition to the" Cole-Hoff transformation," which enables a detailed examination of the dynamic properties of shock solitons. Shargatov et al [30] suggested the Korteweg-de Vries-Burgers equation to explore traveling wave solutions. Dissipation coefficients with a smoothed step-like profile, which fluctuate in both space and time, are included in this equation. Understanding the effects of small-scale dissipation and dispersion processes is emphasized, especially in high-gradient zones.Tanwar and Wazwaz [31] examine the nonlinear behavior of ion acoustic waves in a plasma composed of superthermal electrons and isothermal positrons.They analyze the KdV–Burgers' equation with dissipation in dusty plasmas, constructing Lie symmetries, infinitesimal generators, and commutation relations based on the invariance properties of Lie group transformations.Korkut et al. [32] proposed a novel approach that combines a mesh-free technique, known as the Taylor wavelet method, with the Euler method to approximate solutions to
the general form of the KdV-Burgers' equation.Roy et al. [33] investigate progressive solitary and shock solutions for dust-ion-acoustic waves (DIAWs) in a collisional, unmagnetized dusty plasma. This plasma consists of negatively charged dust grains, positive ions, neutral particles, and Maxwellian electrons. In

this research work, the KdVB equation is numerically solved to examine some important characteristics of dust acoustic shock and solitary waves that occur in dusty plasmas.

The organization of the paper proceeds as follows: The fundamental equations that describe the plasma model are covered in Section 2. In Section 2, the modified Burgers' equation in dusty plasmas is also obtained. The existence and uniqueness result of a solution of the KdV Burgers' equation is described in Section 3 The explicit finite difference approach is introduced in Section 4. Stability analysis of the numerical scheme is discussed in Section 5. The numerical findings and discussion portion are covered in Section 6, and the conclusion is provided in Section 7.

2. Derivation of KdV Burgers' equation and discussion

The basic equations, governing the dust charge grains that are in fluid description, the equations of continuity and momentum which can be written in the following form:

$$\frac{\partial n_d}{\partial t}+\frac{\partial}{\partial x}(n_d u_d)=0, \tag{1}$$

$$\frac{\partial u_d}{\partial t}+u\frac{\partial u_d}{\partial x}+\frac{\sigma_d}{n_d}\frac{\partial p_d}{\partial x}=z_d\frac{\partial \phi}{\partial x}+\eta\frac{\partial^2 u_d}{\partial x^2}, \tag{2}$$

$$\frac{\partial p_d}{\partial t}+u\frac{\partial p_d}{\partial x}+3p_d\frac{\partial u_d}{\partial x}=0, \tag{3}$$

Supplemented by the Poisson's equation as

$$\frac{\partial^2 \phi}{\partial x^2}=z_d\mu n_d+(1-\mu)n_e-n_i, \tag{4}$$

The electron and ion-density may be described by a Boltzmann distribution i.e.

$$n_e=n_{e_0}exp(\phi) \tag{5}$$

$$n_i=n_{i_0}exp(-\gamma\phi) \tag{6}$$

Where n_d,n_e,n_i,u_d,p_d,ϕ,x,t are the dust particle number density, electron number density, ion number density, dust fluid velocity, dust fluid pressure, electrostatics potential, space variable and time, respectively and they have been normalized by n_{d_0}(unperturbed dust particle number density),n_{e_0}(unperturbed electron particle number density) and n_{i_0}(unperturbed ion particle number density);$\mu=\frac{n_{d_0}}{n_{i_0}}$,$\gamma=\frac{T_e}{T_i}$,$\sigma_d=\frac{T_d}{T_e}$ where T_d,T_e,T_i are the temperature for dust, electron and ion.μ_d is the fluid velocity normalized to the dust acoustic speed $C_d=\left(\frac{z_d n_{d_0}e\mu+3\sigma_d K_B T_e q}{m_d q}\right)^{\frac{1}{2}}$ with $q=(1-\mu)n_{e_0}+\gamma n_{i_0}$ and K_B,m_d and z_d being the Boltzmann constant, dust acoustic mass and charged number of dust particles.p_d is the pressure normalized to $n_{d_0}K_B T_d$;ϕ is the electrostatic wave potential normalized by $\left(\frac{K_B T_i}{e}\right)$,with e being the electron charged;the space variable normalized to the dust Debye length $\lambda_d=\left(\frac{3\sigma_d K_B T_e m_d}{4\pi n_{d_0}(z_d^2+qe)}\right)^{\frac{1}{2}}$ and the time variable is normalized to the dust period ω_{pd}^{-1}=$\left(\frac{m_d}{4\pi n_{d_0}z_d^2 e^2}\right)^{\frac{1}{2}}$.The coefficient of viscosity η is a normalized quantity given by $\omega_{pm}\lambda_m^2 m_d n_{d_0}$.

The overall charge neutrality condition has been maintained throughout the plasma system by the following relation:

$$z_d\mu n_{d_0}+(1-\mu)n_{e_0}=n_{i_0} \tag{7}$$

In order to derive the KdV Burgers' equation, the following stretched coordinates are used:

$$\xi=\varepsilon^{\frac{1}{2}}(x-\lambda t),\tau=\varepsilon^{\frac{3}{2}}t,\ \eta=\varepsilon^{\frac{1}{2}}\eta_0 \tag{8}$$

Where λ is the phase velocity of the wave along the x direction and normalized by acoustic velocity and ε is a smallness dimensionless expansion parameter which measuring strength of the dispersion.
The physical variables of plasma parameters namely n_d,u_d,p_d,ϕ are expanded in power series written in general form as:

$$n_d = 1 + \varepsilon n_d^{(1)} + \varepsilon^2 n_d^{(2)} + \varepsilon^3 n_d^{(3)} + \cdots, \quad (9)$$

$$u_d = \varepsilon u_d^{(1)} + \varepsilon^2 u_d^{(2)} + \varepsilon^3 u_d^{(3)} + \cdots, \quad (10)$$

$$p_d = 1 + \varepsilon p_d^{(1)} + \varepsilon^2 p_d^{(2)} + \varepsilon^3 p_d^{(3)} + \cdots, \quad (11)$$

$$\phi = \varepsilon \phi^{(1)} + \varepsilon^2 p_d^{(2)} + \varepsilon^3 p_d^{(3)} + \cdots, \quad (12)$$

Substituting Eqs. (8) – (12) into the basic Eqs. (1) – (6), and thereafter, equating the coefficient of lower order of ε

$$n_d^{(1)} = -\left\{\frac{(1-\mu)n_{e_0} + \gamma n_{i_0}}{z_d\mu}\right\}\phi^{(1)}$$

$$u_d^{(1)} = -\lambda\left\{\frac{(1-\mu)n_{e_0} + \gamma n_{i_0}}{z_d\mu}\right\}\phi^{(1)}$$

$$p_d^{(1)} = -3\left\{\frac{(1-\mu)n_{e_0} + \gamma n_{i_0}}{z_d\mu}\right\}\phi^{(1)}$$

$$\lambda^2 = 3\sigma_d + \frac{z_d}{m_d}\left\{\frac{z_d\mu}{(1-\mu)n_{e_0} + \gamma n_{i_0}}\right\} \quad (13)$$

For the next higher order of ε,we get

$$\frac{\partial n_d^{(1)}}{\partial \tau} - \lambda\frac{\partial n_d^{(2)}}{\partial \xi} + \frac{\partial u_d^{(2)}}{\partial \xi} + \frac{\partial\left(n_d^{(1)}u_d^{(1)}\right)}{\partial \xi} = 0 \quad (14)$$

$$\frac{\partial n_d^{(1)}}{\partial \tau} - \lambda\frac{\partial u_d^{(2)}}{\partial \xi} + \sigma_d\frac{\partial p_d^{(2)}}{\partial \xi} = \frac{z_d}{m_d}\frac{\partial \phi^{(2)}}{\partial \xi} + \frac{z_d n_d^{(1)}}{m_d}\frac{\partial \phi^{(1)}}{\partial \xi} + \eta_0\frac{\partial^2 u_d^{(1)}}{\partial \xi^2} \quad (15)$$

$$\frac{\partial p_d^{(1)}}{\partial \tau} - \lambda\frac{\partial p_d^{(2)}}{\partial \xi} + u_d^{(1)}\frac{\partial p_d^{(1)}}{\partial \xi} + 3p_d^{(1)}\frac{\partial u_d^{(1)}}{\partial \xi} + 3\frac{\partial u_d^{(2)}}{\partial \xi} = 0 \quad (16)$$

$$\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} = n_{e_0}(1-\mu)\phi^{(2)} + n_{e_0}(1-\mu)\frac{1}{2}\left(\phi^{(1)}\right)^2 + z_d\mu n_d^{(2)} + \gamma n_{i_0}\phi^{(2)} - \frac{\gamma}{2}n_{i_0}\left(\phi^{(1)}\right)^2$$
(17)

Eliminating $n_d^{(2)}$,$u_d^{(2)}$,$\phi^{(2)}$,$p_d^{(2)}$ from Eqs. (14) – (17) and using Eq. (13), the KdV Burgers' equation is derived as

$$\frac{\partial \phi^{(1)}}{\partial \tau} + A\phi^{(1)}\frac{\partial \phi^{(1)}}{\partial \xi} + B\frac{\partial^3 \phi^{(1)}}{\partial \xi^3} = C\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} \quad (18)$$

Where $A = \frac{z_d\mu\{\gamma^2 n_{i_0} - (1-\mu)n_{e_0}\}(\lambda - 3\sigma_d) - 3(\lambda^2+\sigma_d)\{(1-\mu)n_{e_0} + \gamma n_{i_0}\}^2}{2\lambda z_d\mu\{(1-\mu)n_{e_0} + \gamma n_{i_0}\}}$

$$B = \frac{\lambda^2 - 3\sigma_d}{2\lambda\{(1-\mu)n_{e_0} + \gamma n_{i_0}\}}$$

and $C = \frac{\eta_0}{2\lambda}$

The KdVB equation (18) contains both dispersive as well as dissipative terms. A particular type of solution of the KdVB equation exhibits the monotonic shock structure. However, it produces a dispersive shock wave in plasma when wave breaking due to nonlinearity is balanced by the combined action of dispersion and dissipation. In the absence of dissipation, we recover the KdV equation which exhibits soliton structure. On the other hand, when dissipation dominates, the shock front exhibits a

monotonic transition of the plasma density, while the shock transition is of oscillatory nature when the dissipation is weak.
The stationary shock wave solution of K-dV Burgers' equation (18) is obtained by using the transformation $\chi = \xi - U\tau$ where U is the speed of the shock waves.
The analytical solution of Eq (18) has been obtained using the well-known tanh-method and is given by

$$\phi^1(\xi,\tau) = \frac{14B}{A} - \frac{12C}{5A} tanh(\xi - U\tau) + \frac{12B}{A} sech^2(\xi - U\tau) \quad (19)$$

This analytical solution elucidates a combination of a soliton wave with a Burgers' shock wave.

3. Existence and uniqueness of a solution of the KdV Burgers' equation

Bouteraa [34] discussed the existence and uniqueness results of a general class of Zakharov–Kuznetsov–Burgers' equation and he utilize the concept of the fixed-point theorem. In In this work, we discuss the existence and uniqueness results of the KdV Burgers' equation. Kuznetsov–Burgers' equation.
For simplicity, we consider $\phi^1(\xi,\tau) = u(x,t) \cong u(i\Delta x, j\Delta t) \cong u_{i,j}$
The equation (18) can be expressed as

$$\frac{\partial u}{\partial t} + Au\frac{\partial u}{\partial x} + B\frac{\partial^3 u}{\partial x^3} = C\frac{\partial^2 u}{\partial x^2} \quad (20)$$

Equation (20) is considered as a combination of the Burgers' equation and the KdV equation.
The equation is the standard form of wave equation in which the term $Au\frac{\partial u}{\partial x}$ represents nonlinearity, $B\frac{\partial^3 u}{\partial x^3}$ represents dispersion and $C\frac{\partial^2 u}{\partial x^2}$ represents dissipation.
The analytical solution (19) can be rewritten as

$$u(x,t) = \frac{14B}{A} - \frac{12C}{5A} tanh(x - Ut) + \frac{12B}{A} sech^2(x - Ut) \quad (21)$$

Let us take the initial condition as

$$u(x,0) = \frac{14B}{A} - \frac{12C}{5A} tanhx + \frac{12B}{A} sech^2 x \quad (22)$$

The KdV Burgers' equation is given by:

$$\frac{\partial u}{\partial t} + Au\frac{\partial u}{\partial x} + B\frac{\partial^3 u}{\partial x^3} - C\frac{\partial^2 u}{\partial x^2} = 0, t \in R^+, x \in R, u(x,0) = \phi \quad (23)$$

The integral formulation of the equation is used to prove existence and uniqueness of the KdV Burgers' equation and is expressed as:

$$u(t) = W(t)\phi - \frac{1}{2}\int_0^t W(t - t')\partial_x\left(u^2(t')\right)dt', t \geq 0 \quad (24)$$

where $W(t)$ is a solution operator and ϕ is the initial condition function.
The existence and uniqueness are established using the approach introduced by Molinet and Ribaud [35]. They show that the equation has unique solutions in the Sobolev space H^s for $s > -1$.
A mathematical technique called a fixed-point argument is used. This technique helps in proving that a function (in this case, the solution to the KdV Burgers' equation) exists and is unique.
Applying a fixed-point argument to the integral formulation:

$$u(t) = \psi(t)\left[W(t)\phi - \chi_{R^+(t)}\frac{1}{2}\int_0^t W(t - t')\partial_x\left(\psi_T^2(t')u^2(t')\right)dt'\right] \quad (25)$$

Where $\psi(t)$ is a time cut off function ensuring smoothness and $\chi_{R^+(t)}$ is the characteristic function.
Theorem 1: Let $\phi \in H^s$, $s > -1$.For any $T > 0$,there exists a unique solution u of (6.24) in $Z_T = C([0,T], H^s)_T^{\frac{1}{2},s}$.The map ϕ is smooth from $H^s(R)$ to Z_T and u belongs to $C(]0,+\infty[, H^s(R))$.

Proposition 1: Let $s \in R$.There exists $C > 0$ such that

$$\|\psi(t)W(t)\phi\|_{X^{\frac{1}{2},s}} \leq C\|\phi\|_{H^s}\ \forall\phi \in H^s(R)$$

Proposition 2: For $\omega \in \delta(R^2)$,we consider k_ξ defined on R by

$$k_\xi = \psi(t)\int \frac{e^{it\tau} - e^{-\xi^2|t|}}{i\tau + \xi^2}\widehat{\omega}(\tau)d\tau$$

Then, it holds for all $\xi \in R$ that

$$\left\|\langle i\tau + \xi^2\rangle^{\frac{1}{2}}\left(k_\xi(t)\right)\right\|_{L^2(R)}^2 \leq C\left[\left(\int \frac{|\widehat{\omega}(\tau)|}{\langle i\tau + \xi^2\rangle}\widehat{\omega}(\tau)d\tau\right)^2 + \left(\int \frac{|\widehat{\omega}(\tau)|}{\langle i\tau + \xi^2\rangle}\widehat{\omega}(\tau)d\right)\right]$$

Proposition 3: Let $s \in R$,

(1) There exists $C > 0$ such that for all $\nu \in \ \delta(R^2)$,

$$\left\|X_{R^+(t)}\psi(t)\int_0^t W(t - t^/)\nu(t^/)dt^/\right\|_{X^{\frac{1}{2},s}} \leq C\left[\|\nu\|_{X^{-\frac{1}{2},s}} + \left(\int\langle\xi\rangle^{2s}\left(\int \frac{|\hat{\nu}(\tau)|}{\langle i\tau + \xi^2\rangle}d\tau\right)^2 d\xi\right)^{\frac{1}{2}}\right]$$

(2) For any $0 < \delta < 1$,there exists $C_\delta > 0$ such that for all $\nu \in X^{-\frac{1}{2}+\delta,s}$

$$\left\|X_{R^+(t)}\psi(t)\int_0^t W(t - t^/)\nu(t^/)dt^/\right\|_{X^{\frac{1}{2},s}} \leq C_\delta\|\nu\|_{X^{-\frac{1}{2}+\delta,s}}$$

Proposition 4: Let Let $s \in R$ and $\delta > 0$.For all $f \in X^{-\frac{1}{2}+\delta,s}$,

$$t \to \int_0^t W(t - t^/)f(t^/)dt^/ \in C(R_+, H^{s+2\delta})$$

Moreover, if (f_n) is a sequence with $f_n \to 0$ as $n \to 0$ in $X^{-\frac{1}{2}+\delta,s}$,then
$\left\|\int_0^t W(t - t^/)f(t^/)dt^/\right\|_{L^\infty(R_+,H^{s+2\delta})} \to 0$ as $n \to 0$.

We first prove the existence of a solution of the integral formulation of (24) of the KdVB equation on some interval [0, T] for $T < 1$.Clearly, if u is a solution of the integral equation $u = F(u)$ with

$$F(u) = \psi(t)\left[W(t)u_0 - \chi_{R^+(t)}\frac{1}{2}\int_0^t W(t - t^/)\partial_x\left(\psi_T^2(t^/)u^2(t^/)\right)dt^/\right] \qquad (26)$$

Then u is a solution of (24) on [0, T], We need to run a fixed-point argument in the space

$$Z = \left\{u \in X^{\frac{1}{2},s}: \|u\|_z = \|u\|_{X^{\frac{1}{2},s_C^+}} + \nu\|u\|_{X^{\frac{1}{2},s}} < \infty\right\} \qquad (27)$$

Where $s_C^+ \in]-1, min(0,1)[$ is fixed and where the constant ν is defined for all nontrivial φ by

$$\nu = \frac{\|\varphi\|_{H^{s_C^+}}}{\|\varphi\|_{H^s}}$$

There exists δ, μ depending on s_C^+ such that

$$\|F(u)\|_{X^{\frac{1}{2},s_C^+}} \leq C\|\varphi\|_{H^{s_C^+}} + CT^\mu\|u\|_{X^{\frac{1}{2},s_C^+}}^2$$

$$\|F(u)\|_{X^{\frac{1}{2},s}} \leq C\|\varphi\|_{H^s} + CT^\mu\|u\|_{X^{\frac{1}{2},s_C^+}}\|u\|_{X^{\frac{1}{2},s}}$$

Combining the above two, it becomes that

$$\|F(u)\|_Z \leq C\left(\|\varphi\|_{H^{s_C^+}} + \nu\|\varphi\|_{H^s}\right) + CT^\mu\|u\|_Z^2 \qquad (28)$$

Next since $\partial_x(u^2) - \partial_x(v^2) = \partial_x[(u - v)(u + v)]$,we get the same way that

$$\|F(u)-F(v)\|_{X^{\frac{1}{2},s_C^+}} \leq CT^{\mu}\|u-v\|_{X^{\frac{1}{2},s_C^+}}\|u+v\|_{X^{\frac{1}{2},s_C^+}}$$
$$\|F(u)-F(v)\|_{X^{\frac{1}{2},s}} \leq CT^{\mu}\left(\|u-v\|_{X^{\frac{1}{2},s_C^+}}\|u+v\|_{X^{\frac{1}{2},s}}\|u+v\|_{X^{\frac{1}{2},s_C^+}}\|u-v\|_{X^{\frac{1}{2},s}}\right)$$

Combining the above two, we conclude that

$$\|F(u)-F(v)\|_Z \leq CT^{\mu}\|u-v\|_Z\|u+v\|_Z \tag{29}$$

Considering $T=\left(4C^2\left(\|\varphi\|_{H^{s_C^+}}+\nu\|\varphi\|_{H^s}\right)\right)^{-\frac{1}{\mu}}$ which leads by definition of ν to $T=\left(8C^2\|\varphi\|_{H^{s_C^+}}\right)^{-\frac{1}{\mu}}$,we infer from (28) and (29) that F is strictly contractive on the ball of radius $4C\|\varphi\|_{H^{s_C^+}}$ in Z.This proves the existence of a solution $u\in X^{\frac{1}{2},s}$ to KdV Burgers' equation on the time interval $[0,T]$ with $T=T\left(\|\varphi\|_{H^{s_C^+}}\right)>0$.

Let $u_1,u_2\in X_T^{\frac{1}{2},s}$be two solutions of the integral equation (24) on the time $[0,T]$.Because of propositions 3 and 4, $u_1,u_2\in C\big([0,T];H^s(R)\big)$.For $0<\delta<\frac{T}{2}$,we define $\tilde{u}_i$,$i=1,2$ by

$$\tilde{u}_i(t)=\begin{cases} u_i(t) & on\ [0,\delta] \\ u_i(2\delta-t) & on\ \ [\delta,2\delta] \\ \phi & elsewhere \end{cases}$$

Since $t\to\tilde{u}_i(t)$ is continuous at $t=0,t=\delta$,and $t=2\delta$ with value in $H^s(R)$,it is clear that $\tilde{u}_i(t)$ is locally in $X^{\frac{1}{2},s}$.Moreover,$\tilde{u}_1-\tilde{u}_1\equiv 0$ on $R/[0,2\delta]$,therefore by propositions 2 and 4,

$$\|u_1-u_2\|_{X_\delta^{\frac{1}{2},s}} \leq \left\|X_{R^+(t)}\psi(t)\int_0^t W(t-t^{/})\partial_x\left(\psi_\delta(\tilde{u}_1(t^{/})-\tilde{u}_2(t^{/})\right)\left(\tilde{u}_1(t^{/})-\tilde{u}_2(t^{/})\right)\nu(t^{/})dt^{/}\right\|_{X_\delta^{\frac{1}{2},s}}$$
$$\leq C\left\|\partial_x\left(\psi_\delta(\tilde{u}_1(t^{/})-\tilde{u}_2(t^{/})\right)\left(\tilde{u}_1(t^{/})-\tilde{u}_2(t^{/})\right)\right\|_{X^{-\frac{1}{2}+\delta,s}}$$
$$\leq CT^{\mu}\|\tilde{u}_1-\tilde{u}_2\|_{X^{\frac{1}{2},s}}\|\tilde{u}_1-\tilde{u}_2\|_{X_T^{\frac{1}{2},s}}$$

For some $\mu>0$.But it is easy to check by construction

$$\|\tilde{u}_1-\tilde{u}_2\|_{X^{\frac{1}{2},s}} \leq 2\|u_1-u_2\|_{X_\delta^{\frac{1}{2},s}}$$

Hence,

$$\|u_1-u_2\|_{X_\delta^{\frac{1}{2},s}} \leq 2CT^{\mu}\left(\|u_1\|_{X_T^{\frac{1}{2},s}}+\|u_2\|_{X_T^{\frac{1}{2},s}}\right)\|u_1-u_2\|_{X_\delta^{\frac{1}{2},s}}$$

Taking $\delta\leq\left[4C\left(\|u_1\|_{X_T^{\frac{1}{2},s}}+\|u_2\|_{X_T^{\frac{1}{2},s}}\right)\right]^{-\mu}$,it forces $u_1\equiv u_2$ on $[0,\delta]$.Iterating this argument, we extend the uniqueness result on the whole interval $[0,T]$.

4. Explicit finite difference method

Finite differences for partial derivatives can be represented as

$$\frac{\partial u}{\partial x}=\frac{u_{i+1,j}-u_{i-1,j}}{2h}+O(h^2) \tag{30}$$
$$\frac{\partial^2 u}{\partial x^2}=\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}+O(h^2) \tag{31}$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}}{2h^3} + O(h^4) \tag{32}$$

$$\frac{\partial u}{\partial t} = \frac{u_{i,j} - u_{i,j-1}}{k} + O(k) \tag{33}$$

The non-linear term $u\frac{\partial u}{\partial x}$ is expressed in the form $\frac{1}{2}\frac{\partial u^2}{\partial x}$ and using central difference

$$u\frac{\partial u}{\partial x} \approx \frac{1}{4h}\left(\left(u_{i+1,j}\right)^2 - \left(u_{i-1,j}\right)^2\right) \tag{34}$$

Neglecting the terms $O(k), O(h), O(h^2)$ and $O(h^4)$ and substituting equations (6.30), (31), (32), (33) in (20), we get

$$\frac{u_{i,j} - u_{i,j-1}}{k} + A\frac{1}{4h}\left(\left(u_{i+1,j}\right)^2 - \left(u_{i-1,j}\right)^2\right) + B\frac{u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}}{2h^3} = C\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

which simplifies

$$u_{i,j} = u_{i,j-1} + \frac{Ak}{4h}\left(\left(u_{i-1,j}\right)^2 - \left(u_{i+1,j}\right)^2\right) - \frac{Bk}{2h^3}\left(u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}\right) + \frac{Ck}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) \tag{35}$$

The KdVB equation is a nonlinear evolution equation that involves of nonlinearity, dissipation and dispersion. If C tends to zero, we should get the KdVB equation tends to behave like the KdV equation. Whereas, if we let B tends to zero, we should get the KdVB equation tends to behave like the Burgers' equation.

5. Stability analysis of the explicit finite difference method

In this section, we will study the stability of explicit finite difference method using Von Neumann stability analysis. We investigate the stability of the numerical scheme (35) for the KdVB equation in the linearized form.

We can rewrite the scheme (35) in the linearized form as

$$u_{i,j} = u_{i,j-1} + \frac{Ak}{4h}\left(u_{i-1,j} - u_{i+1,j}\right) - \frac{Bk}{2h^3}\left(u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}\right) + \frac{Ck}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) \tag{36}$$

Which may be write in the form

$$u_{i,j} = u_{i,j-1} + p\left(u_{i-1,j} - u_{i+1,j}\right) - q\left(u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}\right) + r\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) \tag{37}$$

Where $p = \frac{Ak}{4h}, q = \frac{Bk}{2h^3}, r = \frac{Ck}{h^2}$.

To study the stability of the equation (37), we apply the Von-Neumann analysis. Let $u_{i,j} = e^{at}e^{Ik_m x}$ to get

$$\begin{aligned} e^{at}e^{Ik_m x} &= e^{a(t-\Delta t)}e^{Ik_m x} + p\left(e^{at}e^{Ik_m(x-\Delta x)} - e^{at}e^{Ik_m(x+\Delta x)}\right) \\ &\quad - q\left(e^{at}e^{Ik_m(x+2\Delta x)} - 2e^{at}e^{Ik_m(x+\Delta x)} + 2e^{at}e^{Ik_m(x-\Delta x)} - e^{at}e^{Ik_m(x+2\Delta x)}\right) \\ &\quad + r\left(e^{at}e^{Ik_m(x+\Delta x)} - 2e^{at}e^{Ik_m x} + e^{at}e^{Ik_m(x-\Delta x)}\right) \end{aligned}$$

Divide by $e^{at}e^{Ik_m x}$,we obtain

$$\begin{aligned} 1 &= e^{-a\Delta t} + p(e^{-Ik_m\Delta x} - e^{Ik_m\Delta x}) - q(e^{2Ik_m\Delta x} - 2e^{Ik_m\Delta x} + 2e^{-Ik_m\Delta x} - e^{-2Ik_m\Delta x}) \\ &\quad + r(e^{Ik_m\Delta x} - 2 + e^{-Ik_m\Delta x}) \end{aligned}$$

$$\begin{aligned} 1 - e^{-a\Delta t} &+ p(e^{Ik_m\Delta x} - e^{-Ik_m\Delta x}) + q(e^{2Ik_m\Delta x} - 2e^{Ik_m\Delta x} + 2e^{-Ik_m\Delta x} - e^{-2Ik_m\Delta x}) \\ &- r(e^{Ik_m\Delta x} + e^{-Ik_m\Delta x}) + 2r = 0 \end{aligned}$$

$$1 - e^{-a\Delta t} + 2Ipsin(k_m\Delta x) + 2Iq\big(sin(2k_m\Delta x) - 2sin(k_m\Delta x)\big) - 2rcos(k_m\Delta x) + 2r = 0$$

Now putting $\alpha = k_m\Delta x$ and the amplification factor is denoted by

$$\xi = e^{a\Delta t} = \frac{1}{1 + 2Ipsin\alpha + 2r(1 - cos\alpha) + 4Iqsin\alpha(cos\alpha - 1)}$$

$$= \frac{1}{\left(1 + 4rsin^2\left(\frac{\alpha}{2}\right)\right) + 2Isin\alpha\left(p - 4qsin^2\left(\frac{\alpha}{2}\right)\right)}$$

When $\alpha > 0 \Rightarrow |\xi| < 1$

For any p, q, r, α the amplification factor $|\xi| < 1$,thus the present method is unconditionally stable.

6. Numerical results and discussion

A numerical approach is used for a comprehensive investigation of the dynamics and behaviours, including solitary waves, shock waves, and their interactions in dusty plasma governed by the KdV Burgers' equation. We also compare the obtained numerical results with known analytical solutions of the KdV Burgers' equation in specific cases to validate the accuracy of our numerical scheme. We also investigate the influence of various dust parameters on the wave dynamics of the KdV Burgers' equation. The behaviour of solitary waves and shock waves in the travelling wave solution of the KdV Burgers' equation is greatly influenced by the dissipation dispersion coefficient B and dissipation coefficient C. The dissipation coefficient introduces dissipative effects in the wave in the solution, whereas the dispersion coefficient enhances the wave's dispersive qualities. In the KdV Burgers' equation, the dispersion coefficient affects the behaviour of the solitary waves, and the dissipation coefficient affects the behaviour of the shock waves.

We assess the precision and effectiveness of the current approach through the evaluation of absolute error, defined as follows:

$$\left|u_i^{Analytical} - u_i^{Numerical}\right|$$

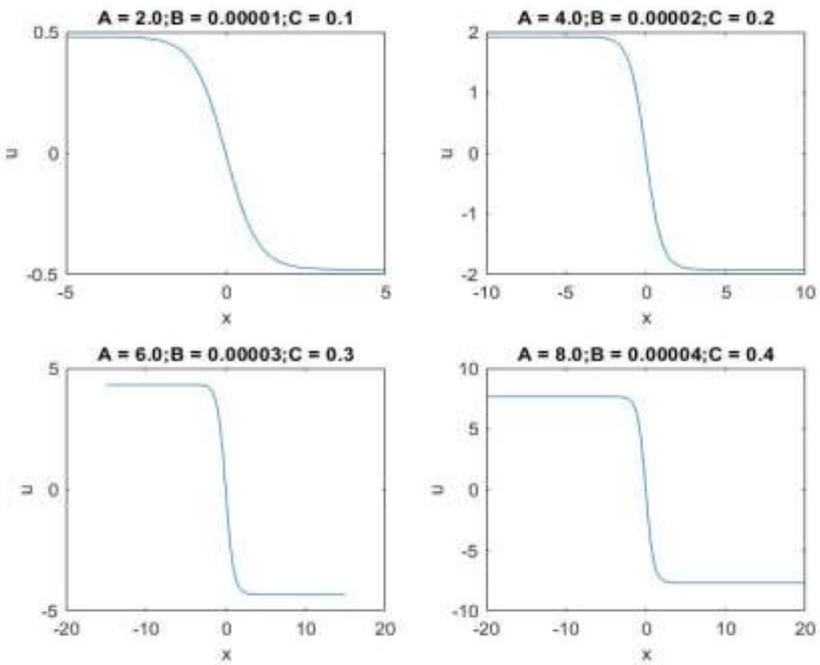

Figure 1: Numerical solution of KdVB equation at (a) A = 2.0; B = 0.0001; C =0.1 (b) A = 4.0; B = 0.00002; C = 0.2 (c) A = 6.0; B = 0.00003; C = 0.3(d)A =8.0; B = 0.00004; C = 0.4.

The behaviour of plasma wave structures has been analysed in Figure 1 to determine the effect of the dominating dissipation term. In this case, when the dissipation coefficient dominates the coefficient of dispersion, the dominant dissipation effect leads to the damping and dissipation of plasma waves. The consequences of particle interactions and collisional processes inside the plasma are represented by the dissipation term in the equation. These processes cause the plasma waves to gradually lose their energy and dissipate. As a result of dominant dissipation, the wave profiles in plasma tend to flatten out and lose their sharpness over time. The amplitude of the waves decreases as energy is dissipated through collisions and other damping mechanisms. Furthermore, in plasma systems, the dominant dissipation can also lead to the generation of shock waves or discontinuities in the wave profiles. These shock waves can arise due to strong dissipative effects overwhelming the dispersive effects, causing localised disturbances and abrupt changes in the plasma wave amplitude. Since the dissipation coefficient relies on several plasma parameters, the behaviour of plasma waves in the presence of dominant dissipation also depends on different plasma parameters, such as electron and ion densities, temperatures, magnetic field strength, and collision frequencies. These parameters influence the strength and nature of the dissipation effects in the plasma, ultimately shaping the wave structures and their evolution. It has been noted that the wave profiles experience damping, flattening, and exhibit shock wave formations due to dominant dissipation effects in the plasma medium in the numerical solution of the KdV Burgers' equation. As the coefficient of dissipation rises, the wave fronts transition from being smooth to sharp. The consequences of dissipation are more noticeable when the term" dissipation" predominates. As a result, the wave profile is dampened and loses energy. Therefore, as time passes, the wave amplitude diminishes, and the wave pattern begins to flatten. As seen in Figure 1, dispersive effects such as wave dispersion and wavefront steepening have minimal impact on the wave profiles. It is clear from Figure 1 that shock waves emerge in wave profiles when the dissipation coefficient C dominates the coefficient of dispersion B and the dissipation effects are more prominent. In this case, the dominant dissipation effect leads to damping and smoothing of the wave profiles. The term" dissipation" acts to dissipate energy from the system, causing the waves to gradually lose their amplitude and become more diffused over time. As a result, the wave profiles tend to flatten out and lose their sharpness. In Figure 1, the dominance of dissipation over dispersion leads to the formation of shocks or discontinuities in the wave profiles. These shocks arise due to the dissipative effects overwhelming the dispersive effects, causing abrupt changes or steep gradients in the wave amplitude. Also, the simulations revealed that shock waves occurred due to the nonlinear convection term in the equation. Figure 1 shows that the nonlinear term in the equation is responsible for the steepening of the waves, leading to the formation of shocks.

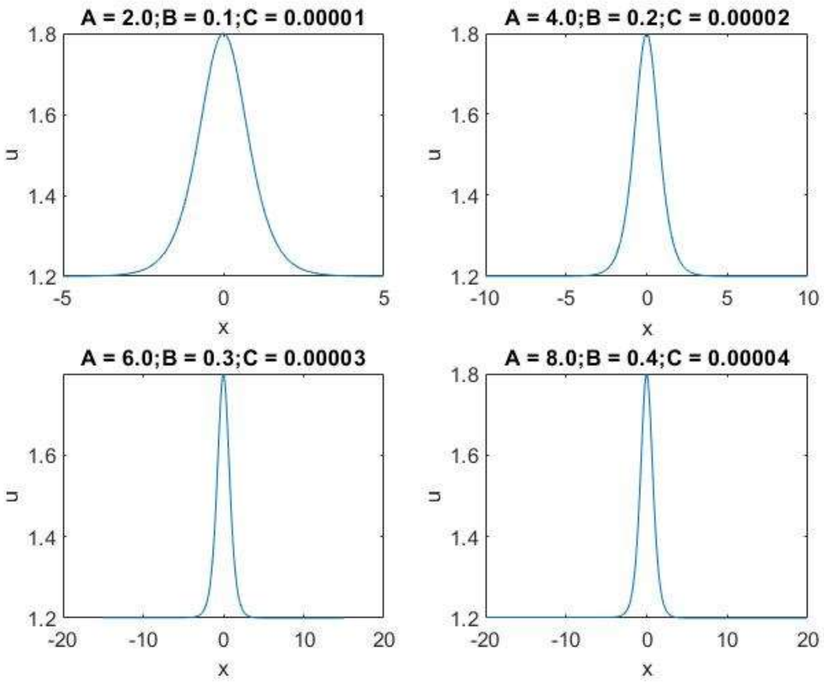

Figure 2: Numerical solution of KdVB equation at (a) A = 2.0; B = 0.1; C =0.0001 (b) A = 4.0; B = 0.2; C = 0.00002 (c) A = 6.0; B = 0.3; C = 0.00003 (d) A =8.0; B = 0.4; C = 0.00004.

Figure 2 depicts the progression of the amplitude of the nonlinear wave as a function of time and space. We observe the formation and propagation of soliton-like structures, indicating the presence of localised disturbances in the dusty plasma system. The amplitude profiles exhibit a distinct shape, characterised by a steep leading edge followed by a decaying tail. It can be seen from Figure 2 that the dominant dispersion effects lead to the formation of solitary wave profiles known as solitons. Because of dispersion effects, plasma waves retain their coherence and exhibit oscillatory behaviour for long periods of time. Also, the dominance of the dispersion term in the KdV Burgers' equation causes the plasma waves to spread out and exhibit wave dispersion. As a result, the wave profiles become broader and more spread out over time.

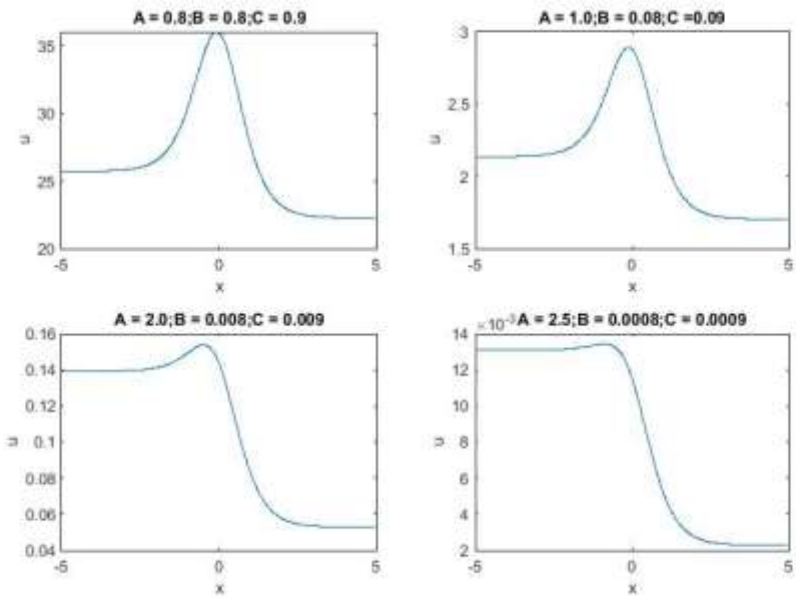

Figure 3: Numerical solution of KdVB equation at (a) A = 0.8; B = 0.8; C = 0.9(b)A = 1.0; B = 0.08; C = 0.09 (c) A = 2.0; B = 0.008; C = 0.009 (d) A = 2.5; B =0.0008; C = 0.0009.

Figure 3 shows that there is a delicate balance between shock waves and solitary waves when dissipation and dispersion coefficients are similar. When the dissipative and dispersive effects are of equivalent magnitude, dissipative-dispersive shock wave formations have been observed in Figure 3. Both damping and dispersion are visible in the wave profiles in this figure, and as the waves move through space, damping and spreading occur simultaneously. Wave breaking, soliton formation, shock formation, and other wave phenomena are all influenced by the interaction between dissipation and dispersion. The precise balance between these two coefficients, as well as the characteristics of the dusty plasma system, determines the specific nature of the wave profiles. As the wave travels, the damping effect progressively lowers the wave amplitude, causing the wave to decay. The waveforms spread out or disperse simultaneously as a result of the dispersion effect. As a result of the spreading effect, the waveforms show oscillating patterns or multiple peaks.

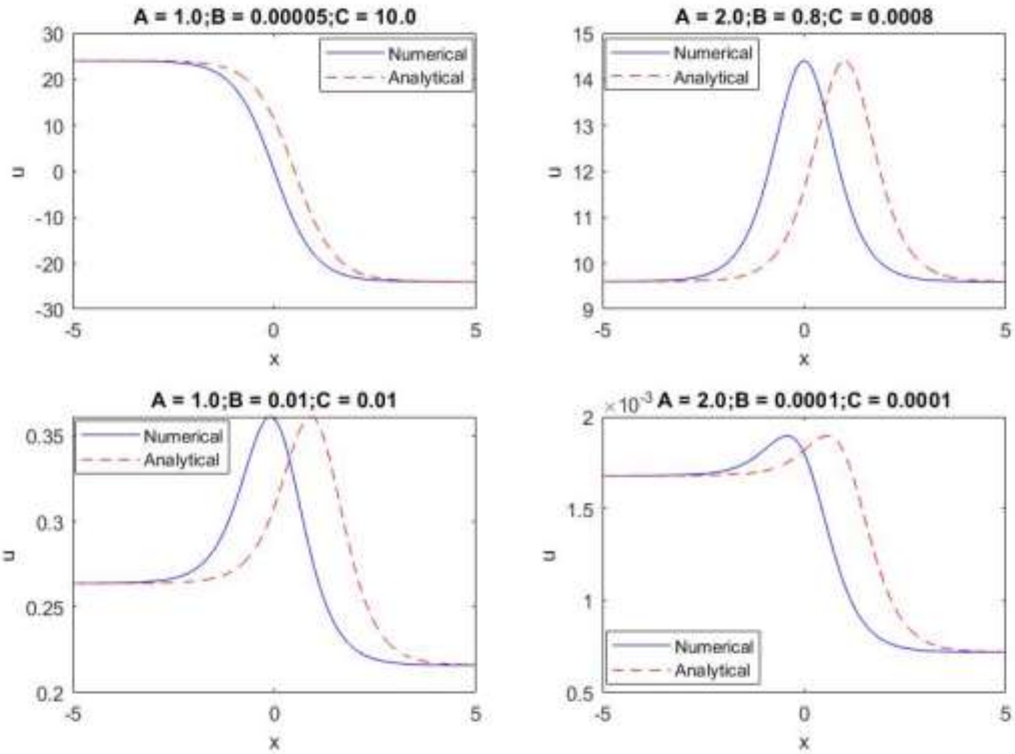

Figure 4: Comparison between numerical and analytical solution of KdVB equation at (a) A = 1.0; B = 0.00005; C = 10.0 (b) A = 2.0; B = 0.8; C = 0.0008 (c) A = 1.0; B = 0.01; C = 0.01 (d) A = 2.0; B = 0.0001; C = 0.0001.

Figure 4 illustrates the comparison between numerical and analytical solutions using a graphical representation at different values of A, B and C. The presented results exhibit good agreement between the numerical and analytical solutions.

It appears from the comparison that waveforms and wave profiles are very similar between numerical and analytical results. It is evident from this figure that the current numerical technique effectively captures the effects of dissipation, dispersion, and nonlinearity. The agreement between the numerical and analytical solutions validated the accuracy and robustness of our numerical scheme in capturing the dynamics of the dusty plasma system.

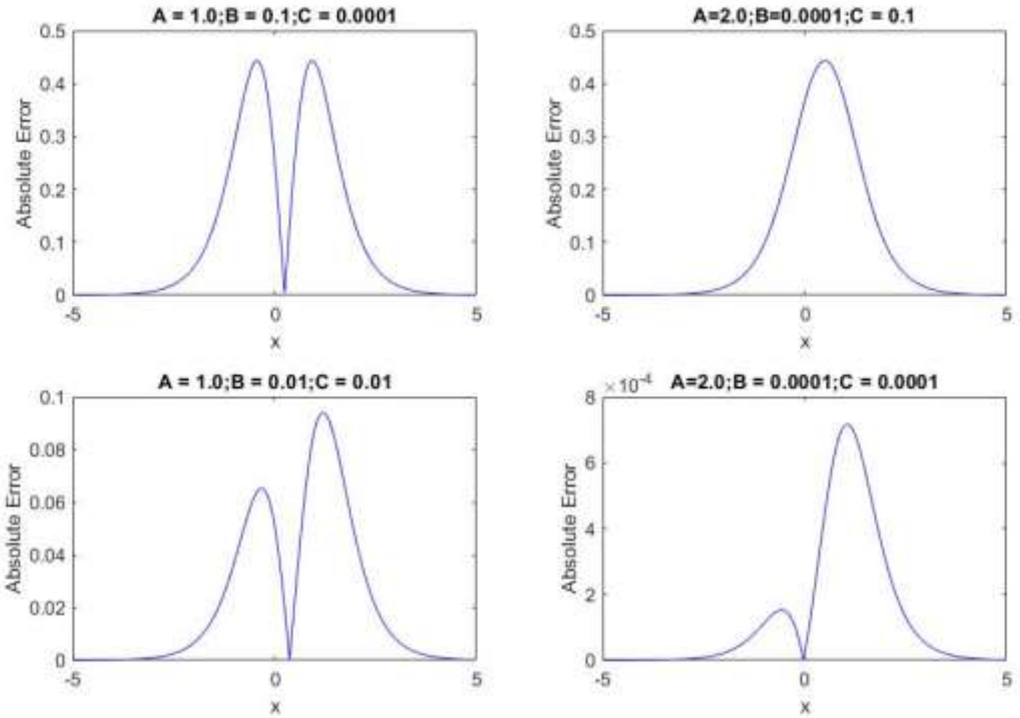

Figure 5: The absolute error between the analytical solution and numerical solutions at (a) A = 1.0; B = 0.1; C = 0.0001 (b) A = 2.0; B = 0.0001; C = 0.1 (c) A = 1.0; B = 0.01; C = 0.01 (d) A = 2.0; B = 0.0001; C = 0.0001.

Table 1: Absolute error between the numerical and analytical values at A = 1.0, B = 0.1, C = 0.0001

x	Numerical value	Analytical value	Absolute error
-5	2.4005	2.4002	0.00021389
-3.75	2.4029	2.4003	0.002603
-2.5	2.4321	2.4008	0.031315
-1.25	20.36	23.929	3.569
-0.625	3.231	2.4252	0.80587
0	3.6	2.485	1.115
0.625	3.2308	2.6713	0.55947
1.25	2.7363	3.1161	0.37976
2.5	2.4317	3.3436	0.91195
3.75	2.4024	2.5363	0.13394
4.375	2.4005	2.4406	0.040062
4.9805	2.4	2.4121	0.012083

Table 2: Absolute error between the numerical and analytical values at A = 2.0, B = 0.0001, C = 0.1

x	Numerical value	Analytical value	Absolute error
-5	0.48116	0.4812	4.2677e-05

-3.75	0.48067	0.48119	0.00051964
-2.5	0.47479	0.48108	0.006291
-1.25	0.40854	0.47976	0.71218
-0.625	0.26782	0.4762	0.20838
0	0.0018	0.46398	0.46218
0.625	-0.26459	0.42365	0.68824
1.25	-0.40581	0.30643	0.71224
2.5	-0.47236	-0.22014	0.25221
3.75	-0.47827	-0.45059	0.027676
4.375	-0.47865	-0.47055	0.0081023
4.9805	-0.47875	-0.47633	0.0024288

Table 3: Absolute error between the numerical and analytical values at A = 1.0, B = 0.01, C = 0.01

x	Numerical value	Analytical value	Absolute error
-5	0.26402	0.264	1.9252e-05
-3.75	0.26424	0.264	0.00023428
-2.5	0.26687	0.26405	0.0028165
-1.25	0.29401	0.26465	0.029361
-0.625	0.3364	0.26624	0.070159
0	0.36	0.27161	0.088385
0.625	0.30978	0.28822	0.021555
1.25	0.25329	0.32683	0.073543
2.5	0.21951	0.32328	0.10377
3.75	0.21629	0.23106	0.014773
4.375	0.21608	0.22049	0.00441
4.9805	0.21602	0.21735	0.0013293

Table 4: Absolute error between the numerical and analytical values at A = 2.0, B = 0.0001, C = 0.0001

x	Numerical value	Analytical value	Absolute error
-5	0.0016801	0.00168	6.4171e-08
-3.75	0.0016808	0.00168	7.8068e-07
-2.5	0.0016895	0.0016802	9.3525e-06
-1.25	0.0017754	0.0016822	9.3269e-05
-0.625	0.0018817	0.0016875	0.00019421
0	0.0018	0.0017051	9.4876e-05
0.625	0.0013492	0.0017579	0.00040862
1.25	0.00096107	0.0018628	0.00090175
2.5	0.00074238	0.0014501	0.00070767
3.75	0.0007286	0.00081643	9.457e-05
4.375	0.00072053	0.00074864	2.8111e-05

4.9805	0.00072016	0.00072862	8.4636e-06

Tables 1, 2, 3 and 4 present a comprehensive comparison between numerically approximated values and analytical values for varying coefficients of nonlinearity, dispersion, and dissipation and a good level of compatibility has been observed between the results. The tables indicate that the numerical results are accurate and precise for smaller values of the coefficients of dissipation and dispersion. As is shown in Table 4, the numerical and analytical results have good agreement with a minor error that tends to zero at smaller values of the coefficients of dissipation and dispersion.

7. Conclusion

In this paper, we have effectively introduced a numerical approach employing an explicit finite difference scheme to derive solutions for the KdVB equation. This method allows us to investigate the emergence of nonlinear structures within dusty plasmas when considering the influence of the Boltzmann distribution of electrons and ions, along with charged dust grains. It is found that the numerical solutions tend to behave like Burgers' equation when the dissipation coefficient dominates over the coefficient of dispersion, whereas KdV-type behaviour has been obtained when dispersion dominates over the coefficient of dissipation. Then a comparison between the numerical results and the analytical solutions is discussed. Stability analysis has been investigated, and it has been proven that the presented scheme is unconditionally stable. As long as the dispersive term and the dissipative term, as well as the nonlinear term, are balanced, the shock wave structure forms; otherwise, the soliton forms due to the balance between the dispersive term and the nonlinear term. With strong dissipation and a weak dispersion coefficient, the shock wave structure becomes steeper. It has also been shown that the plasma parameters are very significant in determining the nature of solitary and shock waves.

References

1. C.Goertz,Dusty plasmas in the solar system Reviews of Geophysics. 1989. 27 (2). 271–292. https://doi.org/10.1029/RG027i002p00271.
2. M.Hor´anyi, Charged dust dynamics in the solar system. Annual review of astronomy and astrophysics. 1996.34(1).383-418. https://doi.org/10.1146/annurev.astro.34.1.383.
3. N.N.Rao,P.K.Shukla and M.Yu Yu, Dust-acoustic waves in dusty plasmas. Planetary and space science. 1990. 38 (4).543-546. https://doi.org/10.1016/0032-0633(90)90147-I.
4. P.K.Shukla and V.P.Silin, Dust ion-acoustic wave. Physica Scripta, 1992. 45 (5).508. DOI 10.1088/0031-8949/45/5/015.
5. F.Melandso , Lattice waves in dust plasma crystals. Astronomy and Astrophysics.1996.3(1).3890-3901. https://doi.org/10.1063/1.871577.
6. P.K.Shukla,M.Yu Yu and R.Bharuthram, Linear and nonlinear dust drift waves. Journal of Geophysical Research: Space Physics. 1991. 96 (A12).21343-21346. https://doi.org/10.1029/91JA02331.
7. P.K.Shukla and A.A.Mamun, Introduction to dusty plasma physics. CRC press. 2015.
8. A.A.Mamun and P.K.Shukla, Cylindrical and spherical dust ion–acoustic solitary waves. Physics of Plasmas. 2002. 9 (4).1468-1470. https://doi.org/10.1063/1.1458030.
9. W.F.El-Taibany,N.A.El-Bedwehy and E.F.El-Shamy, Three-dimensional stability of dust-ion acoustic solitary waves in a magnetized multicomponent dusty plasma with negative ions. Physics of

Plasmas. 2011. 18 (3).033703. https://doi.org/10.1063/1.3570662.

10. P.K.Shukla and A.A.Mamun, Dust-acoustic shocks in a strongly coupled dusty plasma. EEE transactions on plasma science. 2001. 29 (2).221-225. https://DOI: 10.1109/27.923698.
11. A.A.Mamun,B. Eliasson and P.K.Shukla, Dust-acoustic solitary and shock waves in a strongly coupled liquid state dusty plasma with a vortex-like ion distribution. Physics Letters A. 2004. 332 (5-6).412-416. https://doi.org/10.1016/j.physleta.2004.10.012.
12. S.Ghosh and M.R.Gupta, Charging-delay effect on longitudinal dust acous tic shock wave in strongly coupled dusty plasma. Physics of plasmas. 2005. 12 (9). 092306. https://doi.org/10.1063/1.2041649.
13. H.R.Pakzad, Dust acoustic shock waves in strongly coupled dusty plasmas with kappa-distributed ions. Indian Journal of Physics. 2012. 86 (8). 743 747. https://doi.org/10.5281/zenodo.1072575.
14. A.M.El-Hanbaly,E.K.El-Shewy,M. Sallah and H.F.Darweesh, Linear and nonlinear analysis of dust acoustic waves in dissipative space dusty plas mas with trapped ions. Journal of Theoretical and Applied Physics. 2015. 9 (3).167-176. https://DOI 10.1088/0253-6102/65/5/606.
15. C.H.Su and C.S.Gardner, Korteweg-de Vries equation and generaliza tions. III. Derivation of the Korteweg-de Vries equation and Burg ers equation. Journal of Mathematical Physics .1969. 10 (3).536-539. https://doi.org/10.1063/1.1664873.
16. A.A.Mamun and P.K.Shukla, Cylindrical and spherical dust ion–acoustic solitary waves. Physics of Plasmas .2002. 9 (4).1468-1470. https://doi.org/10.1063/1.1458030.
17. J.K.Xue,Cylindrical and spherical dust–ion acoustic shock waves. Physics of Plasmas .2003. 10 (12).4893-4896. https://doi.org/10.1063/1.1622954.
18. B.Sahu and R. Roychoudhury, Quantum ion acoustic shock waves in pla nar and nonplanar geometry. Physics of plasmas .2007. 14 (7).072310. https://doi.org/10.1063/1.2753741.
19. J.M.Burgers, Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion. Springer Netherlands, 1995.
20. D.J.Korteweg and G. De Vries,XLI. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science .1895. 39 (240). 422443.https://doi.org/10.1080/14786449508620739.
21. N. Parumasur,R.A.Adetona and P. Singh, Efficient solution of burgers', modified burgers' and KdV–burgers' equations using B-spline approximation functions. Mathematics .2023. 11 (8). 1847.https://doi.org/10.3390/math11081847.
22. Y. Cao,X.Wu and Z. Jia. Space Time Method for Solving KdV and KdV Burgers' Equation. Mechanics of Solids.2024.59(1).268-279.
23. H.Ahmad,A.R.Seadawy and T.A.Khan , Numerical solution of Korteweg–de Vries-Burgers equation by the modified variational iteration algorithm II arising in shallow water waves. Physica Scripta .2020. 95 (4).045210. https://DOI 10.1088/1402-4896/ab6070.
24. K.A.Koroche and H.M.Chemeda , Sixth-order compact finite dif ference method for solving KDV-Burger equation in the applica tion of wave propagations. Physica Scripta .2022. 12 (2).277-300. https://doi.org/10.22067/ijnao.2021.72366.1058.
25. D. Datta,S. Mishra,S. ˙ S. Rajest,M. Sakthivanitha,D. ˙ K. Hanirex and S. ˙ S. Priscila, modelling solitary wave via numerical solution of Ko rteweg de Vries and KdV-Burger equation using

differential quadra ture. International Journal of Applied Nonlinear Science.2024.4(3).191 209.https://doi.org/10.1504/IJANS.2024.141367.

26. B.Chentouf and A.Guesmia . Well-posedness and stability results for the Korteweg–de Vries–Burgers and Kuramoto–Sivashinsky equations with infinite memory: A history approach. Nonlinear Analysis: Real World Applications.2022.65. 103508.https://doi.org/10.1016/j.nonrwa.2022.103508.
27. S. ˙ A. El-Tantawy,A. ˙ H. Salas and M. ˙ R. Alharthi, On the analytical and numerical solutions of the damped nonplanar Shamel Korteweg–de Vries Burgers equation for modeling nonlinear structures in strongly coupled dusty plasmas: Multistage homotopy perturbation method. Physics of Fluids.2021.33(4). https://doi.org/10.1063/5.0040886.
28. M. Kumar and R. ˙ K. Jana, Analysis of Dust-Ion Acoustic Soliton and ShockWaves of Damped KdV Burgers' Equation in Superthermal Plasmas: Adomian Decomposition Approach. Brazilian Journal of Physics.2024.54(4).98.
29. S. Ballav,A. Das,S.Pramanick and S. Chandra, Plasma shock wave in gamma-ray bursts: Nonlinear phenomena and radiative process. IEEE Transactions on Plasma Science.2021.50(6).1488-1494.
30. V. ˙ A. Shargatov, and A. ˙ P. Chugainova . Stability analysis of traveling wave solutions of a generalized Korteweg–de Vries–Burgers' equation with vari able dissipation parameter. Journal of Computational and Applied Mathe matics.2021.397. 113654.https://doi.org/10.1016/j.cam.2021.113654.
31. D. ˙ A. Tanwar, and A. ˙ M. Wazwaz. Lie symmetries and exact solutions of KdV–Burgers' equation with dissipation in dusty plasma. Qualitative theory of dynamical systems.2022.21(4).164.
32. S. ˙ O. Korkut, and N. ˙ Imamo˘glu Karaba¸ s. A reliable explicit method to ap proximate the general type of the kdv–burgers' equation. Iranian Journal of Science and Technology. Transactions A: Science.2022.1488-1494.
33. S.Roy,R. ˙ R. Kairi and S.Raut . Cylindrical and spherical dust–ion-acoustic shock solitary waves by Korteweg–de Vries–Burgers' equation. Brazilian Journal of Physics,.51(6).2021. 1651-1660.
34. N. Bouteraa, EXISTENCE AND UNIQUENESS OF ZA KHAROV–KUZNETSOV–BURGERS EQUATION WITH CA PUTO–FABRIZIO FRACTIONAL DERIVATIVE. Memoirs on Differential Equations and Mathematical Physics.2024.92.59-67.
35. L.Molinet and F.Ribaud . On the low regularity of the Korteweg de Vries-Burgers equation. International Mathematics Research Notices.2002.37.1979-2005.https://doi.org/10.1155/S1073792802112104.