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Integral Solutions of the Ternion Quadratic Equation $4(x^2+y^2)-7xy+2(x+y+2)=10z^2$

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Abstract
In order to find its non zero unique integral solutions for the quadratic diophantine equation with three unknowns given by $4(x^2+y^2)-7xy+2(x+y+2)=10z^2$ is analysed. The equation under consideration exhibits multiple patterns of solutions. The solutions are presented with a few fascinating aspects.

Keywords: Quadratic equation with three unknowns, integral solutions, polygonal numbers.

INTRODUCTION
A ternary quadratic Diophantine equation is a type of Diophantine equation involving three variables and quadratic terms. Diophantine equations are named after the ancient Greek mathematician Diophantus, who studied equations where solutions are required to be integers[1-3]. The general form of a ternary quadratic Diophantine equation is:
$ax^2+by^2+cz^2+dxy+exz+fyz+gx+hy+iz+j=0$,where *a, b, c, d, e, f, g, h, i,* and *j* are integer coefficients, and *x, y,* and *z* are the integer variables to be solved for[4-12].
These equations are an important topic in number theory, as they generalize the study of quadratic equations to multiple variables. Unlike linear Diophantine equations, where finding solutions involves relatively straightforward methods, ternary quadratic equations can be significantly more complex and require sophisticated techniques such as the theory of quadratic forms, geometry of numbers, and modular arithmetic. These equations appear in various branches of mathematics, including algebraic number theory, algebraic geometry, and arithmetic geometry. Solving them often involves finding all integer or rational solutions, or proving that no such solutions exist under certain conditions.
This communication deals with yet another fascinating ternary quadratic equation $4(x^2+y^2)-7xy+2(x+y+2)=10z^2$ with three unknown factors that can be used to determine any one of an infinite numbers of non-zero integral solutions.

Notations:
$T_{6,n}=n(2n-1)$ (Hexagonal)
$T_{14,n}=n(6n-5)$ (Tetradecagonal)

$T_{18,n} = n(8n-7)$ (Octadecagonal)

$T_{22,n} = n(10n-9)$ (Icosidigonal number)

$T_{24,n} = n(11n-10)$ (Icositetragonal number)

$Gno_n = 2n-1$ (Gnomonic number)

$Star_n = 6n(n-1)+1$

Method of Analysis:

The ternion quadratic equation to be solved for its non-zero integral solution is

$4(x^2+y^2)-7xy+2(x+y+2)=10z^2$ (1)

various patterns of solutions of (1) are listed below.

Assume $x=u+v, y=u-v$, then one may get

$$(u+2)^2+(\sqrt{15}v)^2=10z^2 \tag{2}$$

Where u and v are non-zero integers

PATTERN 1:

$$\text{Write}10=\frac{(5+i\sqrt{15})(5-i\sqrt{15})}{4} \tag{3}$$

$$z=a^2+15b^2=(a+i\sqrt{15}b)(a-i\sqrt{15}b) \tag{4}$$

Using (3) and (4) in (2), then

$$\left[(u+2)+i\sqrt{15}v\right]\left[(u+2)-i\sqrt{15}v\right]=\frac{1}{3}\left[(5+i\sqrt{15})(5-i\sqrt{15}\right]\left[a+i\sqrt{15}b\right]^2\left[a-i\sqrt{15}b\right]^2$$

If we analyse the positive and negative aspects, one may get

$$\left[(u+2)+i\sqrt{15}v\right]=\frac{1}{3}((5a^2-30ab-75b^2)+i\sqrt{15}(a^2+10ab-15b^2))$$

Equating the actual and fictitious elements, then

$$u+2=\frac{1}{3}(5a^2-30ab-75b^2)\ ,\quad v=\frac{1}{3}(a^2+10ab-15b^2)$$

Since our interest is on finding integer solution

Replace $a=3A, b=3B$, then

$u=15A^2-90AB-225B^2-2$

$v=3A^2+30AB-45B^2$

Substitute $x=u+v, y=u-v,\ z=a^2+15b^2$ then

$x=18A^2-60AB-270B^2-2$

$y=12A^2-120AB-180B^2-2$

$z=9A^2-135B^2$

PROPERTIES

1. $x(A,A)-y(A,A)+3T_{18,n}\equiv 0(\mathrm{mod}21)$

2. $x(A,A)-y(A,A)+z(A,A)-2T_{22,n}\equiv 0(\bmod 18)$

3. $x(A,1)-y(A,1)-Star_A-33Gno_n\equiv 0(\bmod 91)$

4. $y(1,1)-x(1,1)$ is a nasty number

5. $z(1,1)$ is a perfect square

PATTERN 2:

We write 10 in the form as

$$10=\frac{(5+i3\sqrt{15})(5-i3\sqrt{15})}{16} \quad (5)$$

Using (4) and (5) in (2), then

$$\left[(u+2)+i\sqrt{15}v\right]\left[(u+2)-i\sqrt{15}v\right]=\frac{1}{16}\left[(5+i3\sqrt{15})(5-i3\sqrt{15}\right]\left[a+i\sqrt{15}b\right]^2\left[a-i\sqrt{15}b\right]^2$$

If we analyse the positive and negative aspects, one may get

$$\left[(u+2)+i\sqrt{15}v\right]=\frac{1}{16}((5a^2-90ab-75b^2)+i\sqrt{15}(3a^2+10ab-45b^2))$$

Equating the actual and fictitious elements, then

$$u+2=\frac{1}{16}(5a^2-90ab-75b^2) \quad , \quad v=\frac{1}{16}(3a^2+10ab-45b^2)$$

Since our interest is on finding integer solution

Replace $a=16A$, $b=16B$ one may get

$$u=80A^2-1440AB-1200B^2-2$$

$$v=48A^2+160AB-720B^2$$

Substitute $x=u+v, y=u-v,\ z=a^2+15b^2$ then

$$x=128A^2-1280AB-1920B^2-2$$

$$y=32A^2-1600AB-480B^2-2$$

$$z=9A^2-135B^2$$

PROPERTIES

1. $y(1,1)-x(1,1)$ is a duck number
2. $z(1,1)$ is a perfect square
3. $x(A,A)-y(A,A)+128T_{18,A}\equiv 0(\bmod 896)$
4. $x(A,A)-y(A,A)+z(A,A)+80T_{24,A}\equiv 0(\bmod 800)$
5. $x(A,1)-y(A,1)+16T_{14,A}\equiv 0(\bmod 1240)$

PATTERN 3:

Write $10=\frac{(15+i3\sqrt{15})(15-i3\sqrt{15})}{36}$ (6)

Using (4) and (6) in (2), and analyse the positive and negative aspects, one may get

$$\left[(u+2)+i\sqrt{15}v\right]=\frac{1}{36}\left[(15+i3\sqrt{15})\right]\left[a+i\sqrt{15}b\right]^2$$

Equating the actual and fictitious elements, then

$$u+2=\frac{1}{36}(15a^2-90ab-225b^2)\ ,\ v=\frac{1}{36}(3a^2+30ab-45b^2)$$

Since our interest is on finding integer solution

Replace $a=36A\,,b=36B$ then

$$u=540A^2-3240AB-8100B^2-2$$

$$v=108A^2+1080AB-1620B^2$$

Apply $x=u+v, y=u-v,\quad z=a^2+15b^2$

$$x=648A^2-2160AB-9720B^2-2$$

$$y=432A^2-4320AB-6480B^2-2$$

$$z=9A^2+135B^2$$

PROPERTIES

1. $x(A,A)-y(A,A)+417T_{6,A}\equiv 0(\mathrm{mod}417)$
2. $z(A,A)-12T_{26,A}\equiv 0(\mathrm{mod}132)$
3. $y(A,A)-z(A,A)-x(A,A)+345T_{6,A}\equiv 0(\mathrm{mod}345)$
4. $y(1,1)$ is a duck number
5. $2(y(1,1)-x(1,1))$ is a sad number

Discussions:

Equation (2) is written as

$$(u+2)^2+(\sqrt{15}v)^2=10z^2*1$$

Write 1 as

$$1=\frac{(5+i\sqrt{15})(5-i\sqrt{15})}{40}$$

$$1=\frac{(5+i3\sqrt{15})(5-i3\sqrt{15})}{160}$$

$$1=\frac{(15+i3\sqrt{15})(15-i3\sqrt{15})}{360}$$

And 10 as

$$10=\frac{(5+i\sqrt{15})(5-i\sqrt{15})}{4}$$

$$10=\frac{(5+i3\sqrt{15})(5-i3\sqrt{15})}{16}$$

$$10=\frac{(15+i3\sqrt{15})(15-i3\sqrt{15})}{36}$$

As we proceed earlier, one may get different integrals solutions of (1).

CONCLUSION

For the Ternion quadratic equation $4(x^2+y^2)-7xy+2(x+y+2)=10z^2$ we have given numerous non-zero unique integral solutions patterns. To conclude, one can look for further options for solutions and their respective attributes among the various choices.

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