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Aspects and Applications of P-Integer-Based Meyer-König-Zeller Durrmeyer Operators

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Abstract:
Within this paper, our study presents a novel class of Meyer-König-Zeller Durrmeyer (MKZD) type operators integrated with the concept of ppp-integers, which adds a new dimension to the approximation process in the context of positive linear operators. The introduction of ppp-integers in the construction of MKZD operators allows greater flexibility and adaptability in handling different functional behaviors, particularly in scenarios requiring finer approximation properties. This generalization opens up new avenues for theoretical development and practical applications in numerical analysis and approximation theory.
We rigorously define the newly constructed operators and investigate their basic properties, including linearity, positivity, and preservation of certain test functions. The core objective of this work is to analyze the rate of convergence of these ppp-MKZD operators, especially in terms of the modulus of continuity and Peetre's K-functional. Emphasis is placed on the confluence behavior of the operators — that is, how effectively they approximate continuous functions as the parameters approach their limiting values. We derive upper bounds for the approximation error and support our theoretical results with illustrative examples.
Additionally, the study highlights how the involvement of ppp-integers enhances the convergence behavior of the operators under various function spaces, thereby demonstrating their applicability in solving practical problems in computational mathematics. This work not only deepens the understanding of MKZD-type operators but also offers a foundation for future investigations into more generalized operator forms and their applications. The proposed operators could be particularly useful in approximation scenarios involving functions with varying smoothness and bounded variation.

Keywords: Lipschitz class, p-integers, p- Meyer-König-Zeller Durrmeyer (MKZD) style operators.

1. INTRODUCTION

[1] established the Meyer-König-Zeller Durrmeyer (MKZD) operators as a collection of

$$D_n(f;x) = \sum_{k=0}^{\infty} m_{n,k}(x) \int_0^1 b_{n,k}(t) f(t) dt,\ 0 \le x < 1, \qquad (1)$$

Through which region

$$m_{n,k}(x) = \binom{n+k-1}{k} x^k (1-x)^n.$$

It is actually,

$$b_{n,k}(t) = n\binom{n+k}{k} t^k (1-t)^{n-1}.$$

There have been several recent studies on the p-MKZ operators that have been conducted by [2], [3], [4], and [5]. Taking this right into profile, our team has launched a crossbreed operator of the MKZ inverted style in purchase to handle along with this complication. [6] and [7] define the operators based on the p-integer which can be found before we introduce the operators. This assumption can be made because n, k, and $n \geq k \geq 0$ possess p-binomial coefficients and, therefore, it is assumed that:

Assuming that k is a non-negative integer, then the p- integer $[k]$ is also a non-negative integer, and the p- factorial $[k]!$ is defined as follows:

$$[k] := \begin{cases} (1-p^k)/(1-p), & p \neq 1 \\ k, & p = 1 \end{cases},$$

As well as

$$[k]! := \begin{cases} [k][k-1]\cdots[1], & k \geq 1 \\ 1, & k = 0 \end{cases}.$$

Because of the truth that the numbers n, k rewarding $n \geq k \geq 0$, possess p- binomial coefficients, it is assumed that:

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]!\,[n-k]!}$$

A few notations that we use in our work are as follows:

$$(a+b)_p^n = \prod_{j=0}^{n-1} (a + p^j b) = (a+b)(a+pb)\cdots(a+p^{n-1}b)$$

As well as

$$(t;p)_0 = 1, (t;p)_n = \prod_{j=0}^{n-1} (1-p^j t), (t;p)_\infty = \prod_{j=0}^{\infty} (1-p^j t).$$

Additionally, it is also evident that this is the case when we take a look at the fact that

$$(a;p)_n = \frac{(a;p)}{(ap^n;p)_\infty}.$$

This can be summarized as a summary of what p- beta is according to its definition:

$$B_p(m,n) = \int_0^1 t^{m-1}(1-pt)_p^{n-1} d_p t$$

In the case of $m, n \in \mathbb{N}$ and we have

$$B_p(m,n) = \frac{[m-1]!\,[n-1]!}{[m+n-1]!}. \qquad (2)$$

Consequently, it is easy to verify that the results are accurate as a result

$$\prod_{j=0}^{n-1} (1-p^j x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k = 1. \qquad (3)$$

We would like to give you a brief introduction to the p-MKZD operator as follows:

$$D_{n,p}(f;x) = \sum_{k=0}^{\infty} m_{n,k,p}(x) \int_0^1 b_{n,k,p}(t) f(pt) pt, 0 \leq x < 1 \qquad (4)$$
$$= \sum_{k=0}^{\infty} m_{n,k,p}(x) A_{n,k,p}(f), \qquad (5)$$

Assume that $0 < p < 1$ and

$$m_{n,k,p}(x) = P_{n-1}(x) \begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k, \tag{6}$$

$$b_{n,k,p}(t) = \frac{[n+k]!}{[k]![n-1]!} t^k (1-pt)_p^{n-1}. \tag{7}$$

The following is

$$P_{n-1}(x) = \prod_{j=0}^{n-1} (1 - p^j x)$$

Remark No. 1. In [8] for $\alpha = 1$, it can be seen that the MKZD operator modified by the p-MKZD operator becomes the operator in the MKZD operator of $p \to 1^-$.

2. Basic Result

Lemma 1. For $F_s(t) = t^s, s = 0,1,2, \ldots$, we have

$$\int_0^1 b_{n,k,p}(t) g_s(pt) d_p t = p^s \frac{[n+k]![k+s]!}{[k][[k+s+n]!}. \tag{8}$$

Proof. It is very easy to verify the above lemma using the p-Beta function (2). The following two lemmas are introduced here, as confirmed in [9]:

Lemma 2. With regards to $r = 0,1,2, \ldots$ and $n > r$, we posses

$$P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{x^k}{[n+k-1]^r} = \frac{\prod_{j=1}^{r} (1-p^{n-j}x)}{[n-1]^r}, \tag{9}$$

where $[n-1]]^r = [n-1][n-2] \cdots [n-r]$.

Lemma 3. The identity

$$\frac{1}{[n+k+r]} \leq \frac{1}{p^{r+1}[n+k-1]}, \ r \geq 0 \tag{10}$$

holds.

3. Direct result

Theorem 1. With regards to everything all $x \in [0,1], n \in \mathbb{N}$ and also $p \in (0,1)$, we posses

$$D_{n,p}(e_0; x) = 1, \tag{11}$$

$$D_{n,p}(e_1; x) \leq x + \frac{(1-p^{n-1}x)}{p[n-1]}, \tag{12}$$

$$D_{n,p}(e_1; x) \geq \left(1 - \frac{(1+p^{n-2})}{[n+1]}\right) x + p^{n-2}(1-p)x^2, \tag{13}$$

$$D_{n,q}(e_2; x) \leq x^2 + \frac{(1+q)^2}{q^3} \frac{(1-q^{n-1}x)}{[n-1]} x + \frac{(1+q)}{q^4} \frac{(1-q^{n-1}x)(1-q^{n-2}x)}{[n-1][n-2]}. \tag{14}$$

Proof. We have to estimate $D_{n,p}(e_s; x)$ for $s = 0,1,2$. The outcome could be simply confirmed for $s = 0$. Applying the above lemmas as well as situation (3), our experts secure connections (12) and also (13) as observes

$$D_{n,p}(e_1, x) = pP_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{[k+1]}{[n+k+1]} x^k$$

$$\leq pP_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{p[k]+1}{p^2[n+k-1]} x^k$$

$$= xP_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k + \frac{P_{n-1}(x)}{p} \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{x^k}{[n+k-1]}$$

$$= x + \frac{(1 - p^{n-1}x)}{p[n-1]}.$$

The result can be easily verified for

Also,

$$D_{n,p}(e_1, x) \quad = pP_{n-1}(x) \sum_{k=1}^{\infty} \begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix} \frac{[k+1]}{[k]} \frac{[n+k-1]}{[n+k+1]} x^k$$

$$\geq P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \left(\frac{[n+k+1]-1}{[n+k+2]} \right) x^{k+1}$$

$$\geq P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \left(\frac{[n+k+1]}{[n+k+2]} - \frac{1}{[n+1]} \right) x^{k+1}$$

$$\geq P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \left(1 - \frac{p^{n+k+1}}{[n+k+2]} \right) x^{k+1} - \frac{1}{[n+1]} x$$

$$\geq P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \left(1 - \frac{p^{n-2}(1-(1-p)[k])}{[n+k-1]} \right) x^{k+1} - \frac{1}{[n+1]} x$$

$$= x - \frac{p^{n-2}x}{[n+1]} + p^{n-2}(1-p)x^2 P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k - \frac{1}{[n+1]} x$$

$$= \left(1 - \frac{(1+p^{n-2})}{[n+1]} \right) x + p^{n-2}(1-p)x^2.$$

As a result of similar calculations, (14) can be found as follows:

$$D_{n,p}(e_2, x) = p^2 P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{[k+1][k+2]}{[n+k+1][n+k+2]} x^k$$

$$\leq \frac{1}{p^4} P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{p^3[k]^2 + (2p+1)p[k] + (p+1)}{[n+k-1][n+k-2]} x^k$$

$$= \frac{P_{n-1}(x)}{p} \sum_{k=0}^{\infty} \frac{[n+k-2]!}{[k]!\,[n-1]!} (p[k]+1) x^{k+1}$$

$$+ \frac{P_{n-1}(x)(2p+1)x}{p^3} \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{x^k}{[n+k-1]}$$

$$+ \frac{P_{n-1}(x)(1+p)}{p^4} \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{x^k}{[n+k-1]^2}$$

$$= x^2 P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k$$

$$+ x \frac{P_{n-1}(x)}{p} \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{x^k}{[n+k-1]} + x \frac{(2p+1)}{p^3} \frac{(1-p^{n-1}x)}{[n-1]}$$

$$+ \frac{(1+p)}{p^4} \frac{(1-p^{n-1}x)(1-p^{n-2}x)}{[n-1][n-2]}$$

$$= x^2 + \frac{(1+p)^2}{p^3}\frac{(1-p^{n-1}x)}{[n-1]}x + \frac{(1+p)}{p^4}\frac{(1-p^{n-1}x)(1-p^{n-2}x)}{[n-1][n-2]}.$$

Remark 2. Offered by Lemma 3, it is noted that for $p \to 1^-$, we get

$$D_n(e_0;x) = 1,$$
$$D_n(e_1;x) \le x + \frac{(1-x)}{(n-1)},$$
$$D_n(e_1;x) \ge \left(1 - \frac{2}{(n+1)}\right)x,$$
$$D_n(e_2;x) \le x^2 + \frac{4x(1-x)}{(n-1)} + \frac{2(1-x)^2}{(n-1)(n-2)},$$

that are points for a different reason of the MKZ operators for $\alpha = 1$ in [8]. The fundamental points

Theorem 2 The fundamental points of $D_{n,p}$ are

$$D_{n,p}(\psi_0;x) = 1,$$
$$D_{n,p}(\psi_1;x) \le \frac{(1-p^{n-1}x)}{p[n-1]},$$
$$D_{n,p}(\psi_2;x) \le \frac{(1+p)^2}{p^3}\frac{(1-p^{n-1}x)}{[n-1]}x + \frac{(1+p)}{p^4}\frac{(1-p^{n-1}x)(1-p^{n-2}x)}{[n-1][n-2]} + 2\frac{(1+p^{n-2})}{[n+1]}x^2,$$

where $\psi_i(x) = (t-x)^i$ for $i = 0,1,2$.

Proof. Due to the linearity of $D_{n,p}$ and also Theorem 1, our experts straight obtain the 1st 2 main instances. Applying basic estimations, the 3rd instant could be quickly confirmed as uses

$$\begin{aligned} D_{n,p}(\psi_2;x) = {} & D_{n,p}(e_2;x) + x^2 D_{n,p}(e_0;x) - 2x D_{n,p}(e_1;x) \\ \le {} & \frac{(1+p)^2}{p^3}\frac{(1-p^{n-1}x)}{[n-1]}x + \frac{(1+p)}{q^4}\frac{(1-p^{n-1}x)(1-p^{n-2}x)}{[n-1][n-2]} \\ & + \left(1 - \frac{(1+p^{n-2})}{[n+1]}\right)x - p^{n-2}(1-p)x^2 \\ \le {} & \frac{(1+p)^2}{p^3}\frac{(1-p^{n-1}x)}{[n-1]}x + \frac{(1+p)}{p^4}\frac{(1-p^{n-1}x)(1-p^{n-2}x)}{[n-1][n-2]} \\ & + 2\frac{(1+p^{n-2})}{[n+1]}x^2. \end{aligned}$$

Remark 3. For $p \to 1^-$, we get

$$D_n(\psi_2;x) \le \frac{4x}{n-1} + \frac{2(1-x)^2}{(n-1)(n-2)}$$

which is similar to the result in [8].

Theorem 3. The sequence $D_{n,p_n}(f)$ comes together to f consistently on $C[0,1]$ for each and every $f \in C[0,1]$ iff $p_n \to 1$ since $n \to \infty$.

Proof. Due to the Korovkin theorem [10], $D_{n,p_n}(f;x)$ comes together to f consistently on $[0,1]$ since $n \to$

∞ to get $f \in C[0,1]$ if $D_{n,p_n}(t^i;x) \to x^i$ to get $i = 1,2$ consistently on $[0,1]$ since $n \to \infty$.
Coming from the explanation of $D_{n,p}$ as well as Theorem 1, D_{n,p_n} is a direct operator and also follows continuous uses.
Furthermore, being $p_n \to 1$, at that time $[n]_{p_n} \to \infty$, as a result through Theorem1, we obtain

$$D_{n,p_n}(t^i;x) \to x^i$$

When it comes to $i = 0,1,2$.
Consequently, $D_{n,p_n}(f)$comes together to f consistently on $C[0,1]$.
However, intent in which $D_{n,p_n}(f)$ comes together to f consistently on $C[0,1]$ and p_n is of use certainly does not tend to 1 as $n \to \infty$. Thus, there certainly occurs a subsequence (p_{n_k}) of (p_n) s.t. $p_{n_k} \to p_0(p_0 \neq 1)$ as $k \to \infty$.

$$\frac{1}{[n]_{p_{n_k}}} = \frac{1 - p_{n_k}}{1 - {p_{n_k}}^n} \to (1 - q_0)$$

Taking $n = n_k$ and $p = p_{n_k}$ in $D_{n,p}(e_2, x)$, we have

$$D_{n,p_{n_k}}(e_2;x) \leq x + \frac{(1 - p_{n_k}^{n-1}x)(1 - p_0)}{p_{n_k}} \neq x$$

which is a dispute. $p_n \to 1$. This executes the evidence.

Conclusion

The features of approximation are investigated in this study utilizing the general relationship between the Meyer-König, Zeller, and Durrmeyer operators. The geometric features of such complicated operators are still being researched.

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