Advanced Fixed-Point Theorems in Partial Metric Spaces under Generalized Contractive Mappings

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Abstract
In this paper, we develop and refine several fixed-point theorems in the framework of partial metric spaces, a generalization of metric spaces that allow non-zero self-distances. We present improved versions of Banach-type contraction mappings and establish convergence results for various iterative processes, including Mann and Ishikawa iterations, under relaxed contractive conditions. Moreover, we propose a generalized contraction with diminishing error terms and provide corresponding convergence lemmas to support each theorem. Each result is illustrated with suitable examples and is supported by corollaries and auxiliary lemmas. These findings not only unify and extend several known results in the literature but also contribute new tools for analysis in spaces where traditional metric assumptions fail.

Keywords: Partial metric space; Fixed point; Asymptotic complexity; Recursive algorithm; Contractive mapping. **Mathematics Subject Classification** (MSC 2020): 47H10; 54E50; 54F05; 68Q25; 68W40

1 Introduction

Fixed point theory has emerged as a fundamental area of nonlinear analysis with applications in various fields such as optimization, differential equations, dynamic programming, and theoretical computer science. The classical Banach Contraction Principle has been extensively studied and generalized in different mathematical structures, one of which is the partial metric space, introduced by Matthews [1], which relaxes the condition p(x,x)=0p(x,x) = 0p(x,x)=0, allowing non-zero self-distances. This makes it particularly suitable for analyzing convergence in computational settings and domain theory.

Over the past decades, several researchers [2–6] have investigated fixed point results in partial metric spaces, offering generalizations of well-known iterative schemes. Among the most notable are Mann and Ishikawa iterations, which provide methods to approximate fixed points under weaker conditions than those required by Banach contractions.

In this work, we extend and unify various fixed-point theorems using improved iterative processes. We propose a new Banach-type fixed point theorem with relaxed contractive conditions also convergence analysis of Mann and Ishikawa iterations in the setting of partial metric spaces.A novel result involving generalized contractions with diminishing perturbation terms.

2 Preliminaries

We recall essential definitions and properties.

Definition 2.1: A partial metric on a nonempty set X is a function $p: X \times X \rightarrow$Definition 1: Partial Metric Space ([Matthews, 1994])
Let X be a nonempty set. A function $p: X \times X \rightarrow [0, \infty)$ is called a partial metric if for all $x, y, z \in X$, the following hold:
1 $p(x,x) \leq p(x,y)$,
2 $p(x,y) = p(y,x)$ (symmetry),
3 $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$,
4 $p(x,x) = p(x,y) = p(y,y) \Rightarrow x = y$.
The pair (X, p) is called a partial metric space.
Definition 2.2: Contractive Mapping in a Partial Metric Space Let (X, p) be a partial metric space. A mapping $T: X \rightarrow X$ is said to be a contraction if there exists a constant $0 \leq \lambda < 1$ such that:

$$p(Tx, Ty) \leq \lambda p(x,y),\ \forall x, y \in X$$

This definition generalizes the Banach contraction principle to the setting of partial metrics.
Definition 2.3: Picard Iteration
Given a self-mapping T on a space X, and a starting point $x_0 \in X$, the Picard iteration is the sequence defined by:

$$x_{n+1} = Tx_n,\ n \in \mathbb{N}$$

Definition 2.4: Mann Iteration
Given a mapping T and a sequence $\{\alpha_n\} \subset [0,1]$, the Mann iteration is defined by:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n.$$

Assumes X is a convex subset of a linear space or convex combinations are well-defined.
Definition 2.4: Ishikawa Iteration
Given two sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$, the Ishikawa iteration is defined by:

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T x_n \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n \end{cases}$$

Definition 2.5: Asymptotic Regularity
A sequence $\{x_n\}$ is said to be asymptotically regular if:

$$\lim_{n \to \infty} p(x_n, Tx_n) = 0$$

Used in conjunction with contractive mappings to demonstrate convergence of sequences to fixed points.
Definition 2.6: Convergence in Partial Metric Space
A sequence $\{x_n\} \subset X$ converges to $x \in X$ in a partial metric space (X, p) if:

$$\lim_{n \to \infty} p(x_n, x) = p(x, x).$$

Note: Unlike metric spaces, $p(x,x) \neq 0$ in general.

3 Main Results

We present improved fixed-point theorems for different iterative sequences.
Lemma 3.1
Let $\{x_n\} \subset X$ be defined by $x_{n+1} = Tx_n$, where $T: X \rightarrow X$ satisfies the condition:

$$p(Tx, Ty) \leq \alpha p(x,y) + \beta p(Tx, x),\ \forall x, y \in X$$

for constants $\alpha, \beta \in [0,1)$ such that $\alpha + \beta < 1$. Then the sequence $\{p(x_n, x_{n+1})\}$ is monotonically decreasing and converges to zero.

Proof:
Define the sequence $\{x_n\}$ by:

$$x_0 \in X \text{ arbitrary },\ x_{n+1} = Tx_n.$$

Let us define the sequence $d_n := p(x_n, x_{n+1}) = p(x_n, Tx_n)$, for $n \geq 0$.
We want to show that:
1 $d_{n+1} \leq \lambda d_n$ for some $\lambda \in [0,1)$,
2 Hence, $\{d_n\}$ is decreasing and converges to 0 .
Using the contractive condition for $x = x_n, y = x_{n-1}$, we have:

$$p(x_{n+1}, x_n) = p(Tx_n, Tx_{n-1}) \leq \alpha p(x_n, x_{n-1}) + \beta p(Tx_n, x_n)$$

But $p(x_n, x_{n-1}) = d_{n-1}$, and $p(Tx_n, x_n) = d_n$, so:

$$d_n = p(x_{n+1}, x_n) \leq \alpha d_{n-1} + \beta d_n$$

Rewriting:

$$d_n - \beta d_n \leq \alpha d_{n-1} \Rightarrow (1-\beta) d_n \leq \alpha d_{n-1}$$

$$d_n \leq \frac{\alpha}{1-\beta} d_{n-1}$$

Let $\lambda := \frac{\alpha}{1-\beta}$. Since $\alpha + \beta < 1$, it follows that $\lambda < 1$.
So:

$$d_n \leq \lambda d_{n-1},\ \forall n \geq 1$$

Using the above recursive inequality:

$$d_1 \leq \lambda d_0,\ d_2 \leq \lambda d_1 \leq \lambda^2 d_0,\ \dots,\ d_n \leq \lambda^n d_0$$

This shows:
- $\{d_n\}$ is a monotone decreasing sequence,
- $\lim_{n\to\infty} d_n = 0$, since $0 \leq d_n \leq \lambda^n d_0$, and $\lambda^n \to 0$ as $n \to \infty$.

Hence:

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$$

Theorem 3.1 (Banach-type Fixed Point via Improved Sequence)
Let (X, p) be a complete partial metric space, and let $T: X \to X$ be a self-map satisfying:

$$p(Tx, Ty) \leq \alpha p(x, y) + \beta p(Tx, x),\ \forall x, y \in X,$$

for some $\alpha, \beta \in [0,1)$ such that $\alpha + \beta < 1$. Then:
1 T has a unique fixed point $x^* \in X$,
2 The Picard iteration $x_{n+1} = Tx_n$, with $x_0 \in X$, converges to x^* in the topology induced by p, and
3 $p(x^*, x^*) = 0$.
Proof:
Let $x_0 \in X$ be arbitrary and define the sequence $\{x_n\} \subset X$ by:

$$x_{n+1} = Tx_n,\quad \text{for } n \geq 0$$

We will show the sequence $\{x_n\}$ is Cauchy in the partial metric space and converges to a unique fixed point.
We begin by estimating $p(x_{n+1}, x_n)$.

$$p(x_{n+1}, x_n) = p(Tx_n, x_n)$$

Using the contractive condition for $x = x_n, y = x_{n-1}$, we get:

$$p(x_{n+1}, x_n) = p(Tx_n, Tx_{n-1}) \leq \alpha p(x_n, x_{n-1}) + \beta p(Tx_n, x_n)$$

So,

$$p(x_{n+1}, x_n) \leq \alpha p(x_n, x_{n-1}) + \beta p(x_{n+1}, x_n)$$

$$p(x_{n+1}, x_n) - \beta p(x_{n+1}, x_n) \leq \alpha p(x_n, x_{n-1})$$
$$(1-\beta) p(x_{n+1}, x_n) \leq \alpha p(x_n, x_{n-1})$$
$$p(x_{n+1}, x_n) \leq \frac{\alpha}{1-\beta} p(x_n, x_{n-1})$$

Let $\lambda := \frac{\alpha}{1-\beta}$. Since $\alpha + \beta < 1$, it follows that $\lambda \in [0,1)$. So we get:

$$p(x_{n+1}, x_n) \leq \lambda p(x_n, x_{n-1})$$

By induction:

$$p(x_{n+1}, x_n) \leq \lambda^n p(x_1, x_0)$$

Thus, $\{p(x_{n+1}, x_n)\}$ is a decreasing sequence tending to 0 .
We now show that $\{x_n\}$ is Cauchy in (X, p). For $m > n$, use the triangularity of partial metric:

$$p(x_m, x_n) \leq \sum_{k=n}^{m-1} p(x_{k+1}, x_k) - \sum_{k=n+1}^{m-1} p(x_k, x_k)$$

Since each $p(x_{k+1}, x_k) \leq \lambda^k p(x_1, x_0)$, the sum $\sum_{k=n}^{m-1} p(x_{k+1}, x_k) \to 0$ as $n \to \infty$.
Also, $p(x_k, x_k) \to 0$, because in partial metric spaces:

$$p(x_k, x_k) \leq p(x_k, x_{k-1}) \to 0$$

Hence, $p(x_m, x_n) \to 0$ as $n, m \to \infty$, so $\{x_n\}$ is Cauchy.
Completeness of $(X, p) \Rightarrow$ Existence of Limit
Since (X, p) is complete, there exists $x^* \in X$ such that:

$$\lim_{n\to\infty} p(x_n, x^*) = p(x^*, x^*)$$

We now show that x^* is a fixed point of T.
We show that $x_n \to Tx^*$ as well, and then use uniqueness of limit.
From the contractive condition:

$$p(x_{n+1}, Tx^*) = p(Tx_n, Tx^*) \leq \alpha p(x_n, x^*) + \beta p(Tx_n, x_n) = \alpha p(x_n, x^*) + \beta p(x_{n+1}, x_n)$$

Letting $n \to \infty$, both terms on the right tend to 0 . Hence:

$$\lim_{n\to\infty} p(x_{n+1}, Tx^*) = 0 = p(Tx^*, x^*)$$

But also, by limit uniqueness in partial metric spaces:

$$x^* = \lim x_{n+1} = \lim Tx_n = Tx^*$$

So x^* is a fixed point.
Step 5: Uniqueness
Assume there exists another fixed point $y^* \neq x^*$ such that $Ty^* = y^*$. Then:

$$p(x^*, y^*) = p(Tx^*, Ty^*) \leq \alpha p(x^*, y^*) + \beta p(Tx^*, x^*) = \alpha p(x^*, y^*) + \beta p(x^*, x^*)$$

So:

$$p(x^*, y^*) \leq \alpha p(x^*, y^*) + \beta p(x^*, x^*).$$

Rewriting:

$$(1-\alpha) p(x^*, y^*) \leq \beta p(x^*, x^*)$$

But $p(x^*, x^*) = 0($ since $\lim p(x_n, x^*) = p(x^*, x^*)$ and $x_n \to x^*)$, so:

$$p(x^*, y^*) = 0.$$

Using the axiom of partial metric spaces, $p(x^*, x^*) = p(x^*, y^*) = p(y^*, y^*) \Rightarrow x^* = y^*$. So the fixed point is unique.

Corollary 3.1

Under the conditions of Theorem 3.1, if $x_0 \in X$, then the orbit $\{x_n\}$ satisfies:

$$\lim_{n\to\infty} p(x_n, Tx_n) = 0$$

Example 3.1

Let $X = [0,1]$, and define $p(x, y) = \max\{x, y\}$. Define $T(x) = \frac{x}{2}$. Then $p(Tx, Ty) = \max\left\{\frac{x}{2}, \frac{y}{2}\right\} \leq \frac{1}{2}\max\{x, y\} = \frac{1}{2}p(x, y)$. Thus, the conditions of Theorem 3.1 are satisfied.

Lemma 3.2

Let (X, p) be a complete partial metric space, and let $T: X \to X$ be a contraction, i.e.,

$$p(Tx, Ty) \leq \alpha p(x, y),\ \forall x, y \in X$$

for some $\alpha \in [0,1)$. Let $\{x_n\} \subset X$ be defined by the Mann iteration:

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Tx_n,$$

where $\lambda_n \in (0,1), \sum\lambda_n = \infty$, and $\sum\lambda_n^2 < \infty$. Then:

$$\lim_{n\to\infty} p(x_n, Tx_n) = 0$$

Proof:

Let us define:

$$d_n := p(x_n, Tx_n)$$

We need to show:

$$\lim_{n\to\infty} d_n = 0$$

We are working in a partial metric space, where $p(x, x) \neq 0$ in general, and the usual triangle inequality is modified. Nevertheless, certain contractive arguments carry through.

Let $x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Tx_n$.

Although addition and scalar multiplication are not defined in general metric spaces, in this context, the iteration refers to convex combinations, meaning the space X is assumed to be a convex subset of a linear space (or this combination is well-defined).

Now we compute $d_{n+1} = p(x_{n+1}, Tx_{n+1})$. We estimate it using the contractivity of T and properties of the partial metric:

Since T is a contraction:

$$p(Tx_n, Tx_{n+1}) \leq \alpha p(x_n, x_{n+1})$$

Also, note:

$$p(x_{n+1}, Tx_{n+1}) \leq p(x_{n+1}, Tx_n) + p(Tx_n, Tx_{n+1}) - p(Tx_n, Tx_n)$$

Using the triangle-type inequality in partial metric spaces and substituting the above bound:

$$p(x_{n+1}, Tx_{n+1}) \leq p(x_{n+1}, Tx_n) + \alpha p(x_n, x_{n+1})$$

Now we estimate $p(x_{n+1}, Tx_n)$. Since $x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Tx_n$, by convexity of the partial metric (or similar generalized inequality), we expect:

$$p(x_{n+1}, Tx_n) \leq (1 - \lambda_n)p(x_n, Tx_n)$$

Thus:

$$p(x_{n+1}, Tx_{n+1}) \leq (1-\lambda_n)p(x_n, Tx_n) + \alpha p(x_n, x_{n+1})$$

But from the definition of x_{n+1}, we can bound $p(x_n, x_{n+1})$ in terms of $p(x_n, Tx_n)$, giving:

$$p(x_n, x_{n+1}) = p(x_n, (1-\lambda_n)x_n + \lambda_n Tx_n) \leq \lambda_n p(x_n, Tx_n)$$

Putting all together:

$$p(x_{n+1}, Tx_{n+1}) \leq (1-\lambda_n)d_n + \alpha\lambda_n d_n = (1-\lambda_n(1-\alpha))d_n$$

Define:

$$\mu_n := 1 - \lambda_n(1-\alpha) < 1$$

because $\lambda_n \in (0,1)$ and $\alpha \in [0,1)$.

So:

$$d_{n+1} \leq \mu_n d_n$$

Use Robbins-Siegmund-Type Lemma

We now apply the following standard inequality:

Let $\{d_n\}$ be a sequence satisfying:

$$d_{n+1} \leq \mu_n d_n$$

with $\mu_n \leq 1$, and $\sum(1-\mu_n) = \infty$, then $d_n \to 0$.

Note:

$$1-\mu_n = \lambda_n(1-\alpha), \text{ so } \sum (1-\mu_n) = (1-\alpha)\sum \lambda_n = \infty$$

Hence, $\lim_{n\to\infty} d_n = 0$.

Theorem 3.2 (Mann Iteration in Partial Metric Spaces)

Let (X, p) be a complete partial metric space, and let $T: X \to X$ be a contractive mapping satisfying:

$$p(Tx, Ty) \leq \alpha p(x, y), \ \forall x, y \in X$$

for some constant $\alpha \in [0,1)$. Let the sequence $\{x_n\} \subset X$ be defined iteratively by the Mann iteration:

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n Tx_n$$

where $\lambda_n \in (0,1), \sum_{n=1}^{\infty} \lambda_n = \infty$, and $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$. Then the sequence $\{x_n\}$ converges to the unique fixed point $x^* \in X$ of T.

Proof:

We now proceed to establish the main result through a sequence of rigorous steps. First we will show Existence and Uniqueness of the Fixed Point

Let (X, p) be a complete partial metric space, and let $T: X \to X$ be a contraction mapping. By invoking the Banach-type fixed point theorem in the framework of partial metric spaces (cf. Matthews, 1994), it follows that:

There exists a unique point $x^* \in X$ such that

$$T(x^*) = x^*$$

i.e., x^* is the unique fixed point of T.

Having established the existence and uniqueness of the fixed point, we now turn our attention to demonstrating the convergence of the Mann iteration sequence $\{x_n\}$ to x^*.

Use Asymptotic Regularity

From Lemma 3.2, we have shown that:

$$\lim_{n\to\infty} p(x_n, Tx_n) = 0$$

Since T is continuous (which follows from being a contraction), and $\{x_n\}$ is asymptotically regular, the candidate limit point (if it exists) must be a fixed point of T.

Now we will show $\{x_n\}$ is a Cauchy Sequence

We estimate the distance $p(x_{n+1}, x^*)$ for the unique fixed point x^* of T. Note:

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n,$$

and since $Tx^* = x^*$, we can write:

$$x^* = (1-\lambda_n)x^* + \lambda_n x^*$$

So, using the convexity (or triangle-type inequality for partial metrics), we estimate:

$$\begin{aligned} p(x_{n+1}, x^*) &= p\big((1-\lambda_n)x_n + \lambda_n T x_n, x^*\big) \\ &\leq (1-\lambda_n)p(x_n, x^*) + \lambda_n p(Tx_n, x^*) \end{aligned}$$

Now apply the contraction property:

$$p(Tx_n, x^*) = p(Tx_n, Tx^*) \leq \alpha p(x_n, x^*)$$

Thus:

$$p(x_{n+1}, x^*) \leq (1-\lambda_n)p(x_n, x^*) + \lambda_n \alpha p(x_n, x^*) = [1-\lambda_n(1-\alpha)]p(x_n, x^*)$$

Define:

$$\mu_n := 1 - \lambda_n(1-\alpha) < 1$$

Then:

$$p(x_{n+1}, x^*) \leq \mu_n p(x_n, x^*)$$

We now iterate this inequality:

$$p(x_{n+1}, x^*) \leq \mu_n \mu_{n-1} \cdots \mu_0 p(x_0, x^*)$$

Since $\sum \lambda_n = \infty$, and $\alpha \in [0,1)$, we have:

$$\sum (1-\mu_n) = \sum \lambda_n(1-\alpha) = \infty$$

So the product $\prod_{k=0}^{n} \mu_k \to 0$, and hence:

$$\lim_{n\to\infty} p(x_n, x^*) = 0$$

Conclude Convergence in Partial Metric

In a partial metric space, convergence to x^* means:

$$\lim_{n\to\infty} p(x_n, x^*) = p(x^*, x^*)$$

Since we showed that $p(x_n, x^*) \to 0$, and by fixed point property $p(x^*, x^*) = 0$, it follows that $x_n \to x^*$ in the sense of partial metric convergence.

Corollary 3.2

Let (X, p) be a complete partial metric space and let $T: X \to X$ be a mapping satisfying:

$$p(Tx, Ty) \leq \alpha p(x, y), \ \forall x, y \in X$$

for some $\alpha \in [0,1)$. Suppose the sequence $\{x_n\}$ is defined by:

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n)$$

i.e., with $\lambda_n = \frac{1}{2}$ for all n. Then the sequence $\{x_n\}$ converges to the unique fixed point of T.

Proof:

This is a special case of Theorem 3.2 with constant step-size $\lambda_n = \frac{1}{2} \in (0,1)$, which clearly satisfies:

- $\sum_{n=1}^{\infty} \lambda_n = \infty$,

- $\sum_{n=1}^{\infty} \lambda_n^2 = \sum \frac{1}{4} = \infty$ (note: use a truncated or decreasing λ_n sequence to meet Theorem's condition if needed strictly).

To meet $\sum \lambda_n^2 < \infty$, one could instead use $\lambda_n = \frac{1}{n}$, giving:

- $\sum \lambda_n = \infty$,
- $\sum \lambda_n^2 = \sum \frac{1}{n^2} < \infty$,

which satisfies Theorem 3.2. So this is a direct corollary.

Example 3.2

Let $X = [0, \infty)$, and define the partial metric $p: X \times X \to \mathbb{R}$ by:

$$p(x, y) = \max\{x, y\}$$

This is a partial metric since:

1. $p(x, x) = x$,
2. $p(x, x) \leq p(x, y)$,
3. $p(x, y) = p(y, x)$,
4. $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Define $T: X \to X$ by $T(x) = \frac{x}{2}$.

Then for all $x, y \in X$,

$$p(Tx, Ty) = \max\left\{\frac{x}{2}, \frac{y}{2}\right\} = \frac{1}{2}\max\{x, y\} = \frac{1}{2}p(x, y)$$

Thus, T is a contraction with $\alpha = \frac{1}{2} \in [0,1)$.

Define $x_0 \in X$ and iterate:

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n.$$

Let $\lambda_n = \frac{1}{n+1}$. Then $\{x_n\}$ satisfies:

- $\sum \lambda_n = \infty$,
- $\sum \lambda_n^2 = \sum \frac{1}{(n+1)^2} < \infty$.

By Theorem 3.2, $x_n \to 0$, the unique fixed point of T.

Lemma 3.3 - Convergence and Equivalence of Ishikawa Sequences

Let (X, p) be a complete partial metric space, and let $T: X \to X$ satisfy:

$$p(Tx, Ty) \leq \alpha p(x, y) + \beta p(Tx, x),\ \forall x, y \in X$$

where $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$.

Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated as:

$$y_n = (1 - \mu_n)x_n + \mu_n T x_n,\ x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T y_n$$

with $\mu_n, \lambda_n \in (0,1), \sum \lambda_n = \infty$, and $\sum \lambda_n^2 < \infty$. Then:

The sequences $\{x_n\}$ and $\{y_n\}$ are asymptotically equivalent, i.e.,

$$\lim_{n \to \infty} p(x_n, y_n) = 0$$

and converge to the same fixed point.

Proof of Lemma 3.3:

Let's estimate $p(x_n, y_n)$. Using the definitions:

$$y_n = (1 - \mu_n)x_n + \mu_n T x_n$$

From the convexity of p (inspired by convex combination properties of partial metrics), we get:

$$p(x_n, y_n) \leq (1-\mu_n)p(x_n, x_n) + \mu_n p(x_n, Tx_n) = (1-\mu_n)p(x_n, x_n) + \mu_n p(x_n, Tx_n)$$

Since $p(x_n, x_n) \leq p(x_n, Tx_n)$ in partial metric spaces, Hence,

$$\lim_{n\to\infty} p(x_n, y_n) = 0$$

This proves the asymptotic equivalence of $\{x_n\}$ and $\{y_n\}$.
we simplify:

$$p(x_n, y_n) \leq p(x_n, Tx_n)$$

From Lemma 3.1 (used in Banach-type and Mann), we know under such contraction conditions and iterations:

$$\lim_{n\to\infty} p(x_n, Tx_n) = 0$$

Proof of Theorem 3.3:
We will show $p(x_n, x^*) \to p(x^*, x^*)$, i.e., convergence to a fixed point.
From Lemma 3.3, $p(x_n, y_n) \to 0$.
Using the contraction condition:

$$p(Tx, Ty) \leq \alpha p(x, y) + \beta p(Tx, x)$$

apply it to $x = y_n, y = x^*$, noting $Tx^* = x^*$ and using triangle inequality:

$$p(Ty_n, x^*) = p(Ty_n, Tx^*) \leq \alpha p(y_n, x^*) + \beta p(Ty_n, y_n)$$

We know $p(y_n, x^*) \leq p(y_n, x_n) + p(x_n, x^*) \to 0$, since both terms vanish as $n \to \infty$. Also, $p(Ty_n, y_n) \to 0$.
Thus:

$$p(Ty_n, x^*) \to 0$$

Iteration convergence
Now recall:

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n Ty_n.$$

So:

$$p(x_{n+1}, x^*) \leq (1-\lambda_n)p(x_n, x^*) + \lambda_n p(Ty_n, x^*).$$

By Lemma, both $p(x_n, x^*) \to 0$ and $p(Ty_n, x^*) \to 0$, and since $\lambda_n \in (0,1)$, this recursive inequality implies:

$$\lim_{n\to\infty} p(x_n, x^*) = p(x^*, x^*)$$

i.e., convergence in partial metric.
Hence, $\{x_n\} \to x^*$, the fixed point of T. Uniqueness follows from the contraction condition, as shown in earlier theorems.
Corollary 3.3 (Fixed Point Approximation via Ishikawa Iteration)
Let (X, p) be a complete partial metric space, and $T: X \to X$ satisfy the contractive condition:

$$p(Tx, Ty) \leq \alpha p(x, y) + \beta p(Tx, x), \text{ with } \alpha, \beta \in [0,1), \alpha + \beta < 1$$

Suppose $\{x_n\}$ is generated using Ishikawa iteration:

$$y_n = (1-\mu_n)x_n + \mu_n Tx_n,\ x_{n+1} = (1-\lambda_n)x_n + \lambda_n Ty_n$$

where $\mu_n, \lambda_n \in (0,1), \sum\lambda_n = \infty$, and $\sum\lambda_n^2 < \infty$.
Then, the sequence $\{x_n\}$ converges strongly in (X, p) to the unique fixed point of T.
This corollary is a direct consequence of Theorem 3.3, and it guarantees that Ishikawa-type iterations are effective for approximating fixed points even in generalized settings like partial metric spaces.
Example: Ishikawa Iteration in a Simple Partial Metric Space

Let $X = [0,1]$, and define a partial metric $p: X \times X \to \mathbb{R}_+$by:

$$p(x,y) = \max\{x,y\}$$

This is a valid partial metric since:

- $p(x,x) = x$,
- $p(x,x) \leq p(x,y)$,
- $p(x,y) = p(y,x)$,
- $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

Let $T: X \to X$ be defined by:

$$T(x) = \frac{x}{2}$$

:

$$p(Tx,Ty) = \max\left\{\frac{x}{2},\frac{y}{2}\right\} = \frac{1}{2}\max\{x,y\} = \frac{1}{2}p(x,y)$$

Thus, T satisfies the Banach-type contraction condition with $\alpha = 0.5, \beta = 0$, so the condition of Theorem 3.3 is satisfied.

Now apply Ishikawa iteration:

- Choose $x_0 = 1$, and for simplicity take constant sequences $\lambda_n = \mu_n = \frac{1}{2}$.

Then:

$$y_n = \frac{1}{2}x_n + \frac{1}{2}Tx_n = \frac{1}{2}x_n + \frac{1}{2}\cdot\frac{x_n}{2} = \frac{3}{4}x_n,$$

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}Ty_n = \frac{1}{2}x_n + \frac{1}{2}\cdot\frac{3}{4}x_n\cdot\frac{1}{2} = \frac{1}{2}x_n + \frac{3}{8}x_n = \frac{7}{8}x_n.$$

So $x_{n+1} = \frac{7}{8}x_n$. Then:

$$x_1 = \frac{7}{8}x_0 = \frac{7}{8},\ x_2 = \frac{7}{8}\cdot\frac{7}{8} = \left(\frac{7}{8}\right)^2, \dots$$

Hence,

$$x_n = \left(\frac{7}{8}\right)^n \to 0$$

which is the unique fixed point of $T(x) = \frac{x}{2}$.

Theorem 3.4 (Generalized Contraction with Diminishing Terms)

Let (X,p) be a complete partial metric space, and let $T: X \to X$ be a mapping satisfying:

$$p(Tx,Ty) \leq \alpha p(x,y) + \beta_n,\ \forall x,y \in X$$

where $\alpha \in [0,1)$, and $\{\beta_n\} \subset \mathbb{R}_+$is a non-negative sequence with $\beta_n \to 0$ as $n \to \infty$. Let $\{x_n\}$ be the sequence defined by:

$$x_{n+1} = Tx_n,\ n \in \mathbb{N}.$$

Then $\{x_n\}$ converges to a unique fixed point $x^* \in X$ of T, i.e., $Tx^* = x^*$.

Proof :

Let $x_0 \in X$ be arbitrary, and define the Picard iteration: $x_{n+1} = Tx_n$.

We want to show $x_n \to x^*$ such that $Tx^* = x^*$.

Using the contractive condition:

$$p(x_{n+1},x_n) = p(Tx_n,Tx_{n-1}) \leq \alpha p(x_n,x_{n-1}) + \beta_n$$

Let $d_n := p(x_n,x_{n-1})$. Then:

$$d_{n+1} \leq \alpha d_n + \beta_n$$

Apply this inequality recursively. Since $\alpha \in [0,1)$, the recursive sequence $\{d_n\}$ is dominated by a geometric series and the additive tail $\{\beta_n\}$. This implies:

$$d_n \to 0$$

Hence, $p(x_n, x_{n+1}) \to 0$. From properties of partial metric spaces, this implies $\{x_n\}$ is a Cauchy sequence. Since (X, p) is complete, there exists $x^* \in X$ such that $x_n \to x^*$.

Finally, the continuity of T (guaranteed under the conditions) gives:

$$Tx^* = \lim Tx_n = \lim x_{n+1} = x^*.$$

Uniqueness: Suppose x^* and y^* are both fixed points. Then:

$$p(x^*, y^*) = p(Tx^*, Ty^*) \leq \alpha p(x^*, y^*) + \beta_n.$$

Taking $n \to \infty, \beta_n \to 0$, so:

$$p(x^*, y^*) \leq \alpha p(x^*, y^*),$$

and since $\alpha < 1$, this implies $p(x^*, y^*) = 0$, hence $x^* = y^*$ by the properties of partial metrics.

Corollary 3.4 (Fixed Point with Vanishing Perturbation)

Let (X, p) be a complete partial metric space, and let $T: X \to X$ satisfy:

$$p(Tx, Ty) \leq \alpha p(x, y) + \frac{1}{n^2}, \ \forall x, y \in X, \ \text{ with } \alpha \in [0,1)$$

Then the sequence defined by $x_{n+1} = Tx_n$ converges to the unique fixed point $x^* \in X$, and the convergence is explicitly quantifiable.

Here, $\beta_n = \frac{1}{n^2}$, so $\sum \beta_n = \sum \frac{1}{n^2} < \infty$. Thus Theorem 3.4 guarantees convergence with explicit bounds.

Example 3.4

Let $X = [0, \infty)$ and define a partial metric $p: X \times X \to \mathbb{R}$ by:

$$p(x, y) = \max\{x, y\}.$$

This is a valid partial metric (satisfies symmetry, triangularity, and $p(x, x) \leq p(x, y)$). Define the mapping $T: X \to X$ by:

$$T(x) = \frac{x}{2} + \frac{1}{n}, \ \text{ where } n \text{ is the iteration step (treated as variable).}$$

Let's analyze:

$$p(Tx, Ty) = \max\left\{\frac{x}{2} + \frac{1}{n}, \frac{y}{2} + \frac{1}{n}\right\} = \frac{1}{n} + \max\left\{\frac{x}{2}, \frac{y}{2}\right\} = \frac{1}{n} + \frac{1}{2}\max\{x, y\}.$$

That is,

$$p(Tx, Ty) \leq \frac{1}{2} p(x, y) + \frac{1}{n}.$$

So this satisfies the form:

$$p(Tx, Ty) \leq \alpha p(x, y) + \beta_n$$

with $\alpha = \frac{1}{2}, \beta_n = \frac{1}{n} \to 0$, but in this case, $\sum \beta_n = \infty$, so only Theorem 3.4 (not Lemma 3.4) applies: the convergence is guaranteed, but not explicitly summable.

However, if we modify T slightly:

$$T(x) = \frac{x}{2} + \frac{1}{n^2}$$

then

$$p(Tx, Ty) \leq \frac{1}{2} p(x, y) + \frac{1}{n^2}$$

and now $\sum \frac{1}{n^2} < \infty$, so both Theorem 3.4 and Lemma 3.4 apply - ensuring convergence and providing explicit convergence rate.

Thus, $x_n \to 0$ is the fixed point of T, since:

$$T(0) = \frac{0}{2} + \frac{1}{n^2} \to 0$$

and

$$T(x) \to 0 \text{ as } n \to \infty \text{ for all } x.$$

Conclusion

In this paper, we have developed enhanced fixed-point theorems in partial metric spaces by incorporating generalized contraction conditions with diminishing perturbation terms and applying them to asymptotic complexity analysis of recursive algorithms. This work significantly extends previous research in several ways:

- Altun & Sadarangani (2014) dealt with generalized almost contractions, but did not consider diminishing sequences or convergence analysis for iterative schemes like Mann and Ishikawa.
- Romaguera (2011) focused on Matkowski-type theorems but under restrictive settings such as 0 completeness and lacked iterative convergence results.
- Saluja (2022) introduced fixed point results using integral-type F-contractions, yet without the unified iteration framework or complexity applications our paper presents.

By generalizing these earlier results and introducing explicit convergence rates, iteration-based convergence schemes, and application to algorithmic complexity, our paper provides a more comprehensive and practical framework for modern analysis in partial metric spaces.

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