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Exploring the Techniques of Lagrangian Reduction and Congruence Transformation in Matrix Analysis

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ABSTRACT
The study explores special cases of symmetric matrices, quadratic forms, triangular matrices, Hermitian, and skew-Hermitian matrices by employing the techniques of Lagrangian reduction and congruence transformation to determine the congruence of two matrices. The study thus provides insights into the intricacies of these matrices and their quadratic forms. The findings of this study contribute to the broader field of matrix analysis and offer potential application to various areas of science and engineering.

Keywords: Lagrangian reduction, Congruence transformation, Hermitian and skew-Hermitian Matrices, Quadratic forms, Symmetric Matrices, Triangular Matrices

INTRODUCTION
The study of special cases of matrix analysis using the techniques of Lagrangian reduction and congruence transformation is a fascinating field of research in mathematics. The techniques offer a powerful tool for simplifying the dynamics of complex systems with symmetry. The techniques have found widespread applications in various areas of physics including classical mechanics, quantum mechanics, and field theory.
Lagrangian reduction first introduced by Joseph-Louis Lagrange in the 18th century focuses on exploiting the symmetries present in a system to simplify its equation of motion. Symmetries can arise from various sources such as rotational, translations, or gauge symmetry (Arnold, 1989).
Let the quadratic form $Q(x) = x^T \boldsymbol{A} x$ where $\boldsymbol{A}$ is symmetric matrix and x is a vector.
The concept of Lagrangian reduction consists essentially in repeated completion of the square on each variable of the quadratic form to reduce it to a form
$D_1\, l_1^2 + \frac{D_2}{D_1} l_2^2 + \cdots + \frac{D_n}{D_{n-1}} l_n^2$,
where l_i are the linear combinations of the variables and the D_i are principal minors of the quadratic form.
In contrast, congruence transformation refers to the process of transforming a given matrix $\boldsymbol{A}$ into another matrix $\boldsymbol{B}$ that is congruent with $\boldsymbol{A}$ (Horn. and Johnson 2013; Voyevodin 1983).
The significance of the concept of symmetry has numerous applications in various fields. In Physics, for example, reduction theory in symmetry systems has its roots in the classical works of Jacobi, Lagrange, Hamilton, and others, is crucial. Routh reduction is often associated with cyclic variables in Lagrangian systems. Another important component of Lagrangian reduction is the development of the Euler-Poincare

equations (Poincare, 1901). Lagrangian reduction techniques have proven to be extremely beneficial in optimal control situations (Fernández, Tori, and Zuccalli 2016; Marsden, Ratiu, and Scheurle 2000; Mestdag and Crampin 2008).

Overall, the application of Lagrangian reduction techniques in various fields of physics has yielded profound insights into the behavior of systems with symmetry. By simplifying the analysis, identifying conserved quantities, and reducing the degrees of freedom. Lagrangian reduction provides a powerful framework for understanding the dynamics of complex physical systems. The study aims to contribute to this body of knowledge by exploring special cases of using Lagrangian reduction and demonstrating their applicability in different contexts of quadratic forms.

Congruence transformation is a mathematical technique that involves transforming matrices through similarity transformations while preserving certain properties. The key aspects of congruence transformation include its mathematical foundation, properties, and applications in the context of matrix analysis.

The two matrices $\boldsymbol{A}$ and $\boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P}$ are said to be congruent if they are equivalently related that is $\boldsymbol{A} = \boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P}$. A congruence transformation provides a way to relate matrices with similar structures or properties, enabling the exploration of matrix properties and simplification of computations. Analyzing quadratic forms is one of the main uses for congruence transformation. **Theorem 1:** A square matrix $\boldsymbol{P}$ is said be semi positive definite if and only if there exists a square matrix $\boldsymbol{U}$ such that $\boldsymbol{P} = \boldsymbol{U}^T\boldsymbol{D}\boldsymbol{U}$ where $\boldsymbol{U}$ is a unitary matrix and $\boldsymbol{D}$ is the diagonal matrix of $\boldsymbol{P}$. When $\boldsymbol{D} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$ where λ_i are the eigenvalues of $\boldsymbol{P}$. The quadratic form is positive (or semi-positive) definite if $\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} > (or \geq)0$. This implied that the eigenvalues of the quadratic form are positive (or non-negative) values. The quadratic form is negative (or semi negative) definite if $\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} < (or \leq)0$ (Chabrillac and Crouzeix 1984; Koh 2022).

Let: $q(x,y) = ax^2 + 2bxy + cy^2$

$$= a\left(x^2 + 2\frac{b}{a}xy\right) + cy^2$$

$$= a\left(x^2 + 2\frac{b}{a}xy + \left(\frac{b}{a}y\right)^2 - \left(\frac{b}{a}y\right)^2\right) + cy^2$$

$$= a\left(x + \frac{b}{a}y\right)^2 - \frac{b^2}{a}y^2 + cy^2$$

$$= a\left(x + \frac{b}{a}y\right)^2 + \frac{ac - b^2}{a}y^2$$

$$= D_1\left(x + \frac{b}{a}y\right)^2 + \frac{D_2}{D_1}y^2$$

Where D_1 and D_2 are the principal minors of the quadratic form. This implied that if $D_1 > 0$ and $D_2 > 0$ then the quadratic form is positive definite but if $D_1 < 0$ and $D_2 > 0$ then the quadratic form is negative definite (Koh 2022). The Sylvester criterion is used to determine the positive (or negative) definite of quadratic form. According to Sylvester criterion a quadratic form is positive (or negative) definite if all the symmetric matrix leading principal minors are greater than zero (or alternate in term of signs).The nature of the quadratic form has also been described in terms of the Sylvester criterion (Conrad 2020; Giorgio 2017; Sree et al. 2022).

Hermitian and Skew-Hermitian Matrices

If a square complex matrix $\boldsymbol{A}$ equals its conjugate transpose, $\boldsymbol{A} = \boldsymbol{A}^*$ then $\boldsymbol{A}$ is Hermitian matrix, if $\boldsymbol{A} = -\boldsymbol{A}^*$ then $\boldsymbol{A}$ is a Skew-Hemitian matrix. The conjugate transpose matrix represented by $\boldsymbol{A}^*$ is also known as the adjoint (or Hermitian adjoint). For example, $\boldsymbol{A} = \begin{pmatrix} 4 & 5-2i \\ 5+2i & 2 \end{pmatrix}$ and $\overline{\boldsymbol{A}} = \begin{pmatrix} 4 & 5+2i \\ 5-2i & 2 \end{pmatrix}$ is a conjugate matrix. Also, $\overline{\boldsymbol{A}}^T = \boldsymbol{A}^*$ is a conjugate transpose matrix of $\boldsymbol{A}$. If $\boldsymbol{A} = a_{ij} \in M_n(C)$, then the Hermitian conjugate transpose $\boldsymbol{A}^*$ of $\boldsymbol{A}$ is defined by $\boldsymbol{A}^* = C_{ij}$ where $C_{ij} = a_{ij}$ (Herstein and Winter 1988; Horn. and Johnson 2013; Ikramov 2019).

Triangular Matrices

A particular kind of triangular matrix which can be either a lower (or left) triangular matrix with all zero members above the primary diagonal of the matrix or an upper (or right) triangular matrix with zero members below the main diagonal of the matrix. Let $\boldsymbol{L}$ and $\boldsymbol{U}$ be left and right triangular matrices respectively, $L = (a_{ij})$, $a_{ij} = 0$, $i > j$ and $U = (a_{ij})$, $a_{ij} = 0, i < j$. The n −square matrix can be reduced to an upper triangular matrix form using similarity transformation such that if S is an invertible matrix then $\boldsymbol{S}^{-1}\boldsymbol{AS}$ is an upper triangular matrix. Triangular matrices are special kind of matrices applied in mathematics (Birkenmeier et al. 2000; Fox 1970; Gerrish 1980).

Definition 1: The signature of a diagonal matrix $\boldsymbol{D}$, is the number of positive entries minus the number of negative entries in $\boldsymbol{D}$. The number of positive elements in $\boldsymbol{D}$ is called index. The total number of non-zero elements in $\boldsymbol{D}$ is called rank.

Theorem 2: If and only if the diagonal matrices of $\boldsymbol{A}$ and $\boldsymbol{B}$ share the same rank, index, and signature, then the two matrices are congruent (Charles R. Johnson 2001; Simon, Chaturvedi, and Srinivasan 1999).

PROCEDURES AND METHODS

The objectives of the paper is outlined and are as follows:

1. To prove that positive (or negative) definite symmetric matrices in quadratic forms are congruent.

For objective 1, suppose the matrix $\boldsymbol{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive (or negative) definite symmetric matrix and congruent to matrix $\boldsymbol{B}$ such that $\boldsymbol{B} = \boldsymbol{P}^T\boldsymbol{AP}$ where $\boldsymbol{B}$ is also positive (or negative) definite symmetric matrix and $\boldsymbol{P}$ is a non-singular matrix.

1. In order to calculate for matrices $\boldsymbol{B}$ and $\boldsymbol{P}$ we need to transform matrix $\boldsymbol{A}$ to a quadratic form. That is $q(u, v) = au^2 + 2buv + cv^2$.
2. Using Lagrangian reduction method on the quadratic form to obtain this result $q(u, v) = a\left(u + \frac{b}{a}v\right)^2 + \frac{ac-b^2}{a}v^2, a \neq 0$. Where $l_1 = u + \frac{b}{a}v$ and $l_2 = v$ which are the linear combinations of the variables u and v. Then, $\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ and the matrix $P = \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}, a \neq 0$.
3. Therefore, matrix $\boldsymbol{B} = \boldsymbol{P}^T\boldsymbol{AP}$ such that $B = \begin{pmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 2b \\ 2b & \frac{3b^2+ac}{a} \end{pmatrix}, a \neq 0$.
4. Now, congruence transformation is needed to reduce both matrices to their diagonal forms and to verify that they are congruent.

2. To demonstrate that every right triangular matrix is congruent with left triangular matrix.

In objective 2, suppose $\boldsymbol{L}$ represents a left triangular marix and $\boldsymbol{U}$ represents a right triangular matrix for example $\boldsymbol{U} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. The following procedures are considered to demonstrate that right triangular matrix is congruent to left triangular matrix.

1. Convert the right triangular matrix $\boldsymbol{U} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ to a quadratic form such that $q(x, y) = ax^2 + bxy + cy^2$.
2. Then, by the Lagrangian reduction technique $q(x, y) = \left(x + \frac{b}{2a}y\right)^2 + \frac{4ac-b^2}{4a}y^2, a \neq 0.$
3. Based on the linear combinations of l_1 and l_2 of x and y such that $l_1 = x + \frac{b}{2a}y$ and $l_2 = y$ where the matrix $\boldsymbol{P} = \begin{pmatrix} 1 & \frac{b}{2a} \\ 0 & 1 \end{pmatrix}, a \neq 0.$
4. To obtain the matrix $\boldsymbol{L}$ using the formula $\boldsymbol{L} = \boldsymbol{P}^{\boldsymbol{T}}(\boldsymbol{PU})^{\boldsymbol{T}}$ then $\boldsymbol{L} = \begin{pmatrix} a & 0 \\ 2a & c \end{pmatrix}$.
5. We can conclude that both matrices are congruent if they satisfy Sylvester's law of inertia.

3. To verify that Hermitian (or Skew-Hermitian) matrices are congruent and also congruent to real (or purely imaginary) diagonal matrix.

The following procedures will be used to verify that Hermitian (or Skew-Hermitian) matrices are congruent also congruent to real (or purely imaginary) diagonal matrix.

1. For objective 3, let covert both $\boldsymbol{A} = \begin{pmatrix} a & b-i \\ b+i & c \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} -ai & b+i \\ -b+i & ci \end{pmatrix}$ be Hermitain and skew-Hermitain matrices respectively to a quadratic form.
2. Applying Lagrangian reduction method to reduce both matrices to determine the non-singular matrices P_A and P_B.
3. Calculating for Hermitian matrix $\boldsymbol{A_1}$ such that $\boldsymbol{A_1} = \boldsymbol{P_A}^*\boldsymbol{AP_A}$ and Skew-Hermitian $\boldsymbol{B_1}$ such that $\boldsymbol{B_1} = \boldsymbol{P_B}^*\boldsymbol{BP_B}$.
4. Applying congruence transformation to reduce all matrices to their diagonal forms to verify that the matrices are congruent and congruent to real (or purely imaginary) diagonal matrix.

RESULS AND DISCUSSION

In order to examine the objectives, we establish some preliminary results.

Preliminary Results

Lemma 1

If a matrix is positive (or negative) definite and symmetric, then it is congruent to a symmetric positive (or negative) definite matrix.

Proof

If $\boldsymbol{P}$ is a non-singular matrix such that $\boldsymbol{B} = \boldsymbol{P^TAP}$, then $\boldsymbol{B}$ is congruent to a symmetric positive (or negative) definite matrix $\boldsymbol{A}$. This implies that;

$$\boldsymbol{B^T} = \boldsymbol{P^TA^TP}$$

The matrix $\boldsymbol{A}$ is a symmetric matrix then $\boldsymbol{A^T} = \boldsymbol{A}$ and $\boldsymbol{B^T} = \boldsymbol{P^TAP}$. Since $\boldsymbol{B}$ is the symmetric matrix then we apply congruence transformation to reduce $\boldsymbol{B}$ to a diagonal matrix form to confirm that matrix $\boldsymbol{B}$ is positive (or negative) definite matrix. This completes the proof.

Lemma 2

If the diagonal matrices of symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ have equal upper-left 2×2 submatrices, then $\boldsymbol{A}$ and $\boldsymbol{B}$ are congruent.

Proof

Suppose by basic row and column operations, to reduce matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ may be reduced to their diagonal form $\boldsymbol{D_A}$ and $\boldsymbol{D_B}$ such that $\boldsymbol{D_A} = \begin{pmatrix} \Lambda_{a1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Lambda_{an} \end{pmatrix}$ and $\boldsymbol{D_B} = \begin{pmatrix} \Lambda_{b1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Lambda_{bn} \end{pmatrix}$. Where $\begin{pmatrix} \Lambda_{a1} & 0 \\ 0 & \Lambda_{a2} \end{pmatrix} = \begin{pmatrix} \Lambda_{b1} & 0 \\ 0 & \Lambda_{b2} \end{pmatrix}$ represents 2×2 upper-left sub-matrices of their diagonal matrices which is equal to show that matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are congruent. The proof is now completed.

Lemma 3

Every left triangular matrix $\boldsymbol{L}$ is congruent to a right triangular matrix $\boldsymbol{U}$ for some non-singular matrix $\boldsymbol{P}$ such that $\boldsymbol{L} = \boldsymbol{P}^T(\boldsymbol{PU})^T$ if $\boldsymbol{U} = \boldsymbol{P}^{-1}\boldsymbol{L}^T\boldsymbol{P}^{-1}$.

Proof

If $\boldsymbol{L}$ and $\boldsymbol{U}$ are left and right triangular matrices for some $\boldsymbol{P}$,

$$\boldsymbol{L} = \boldsymbol{P}^T\boldsymbol{U}^T\boldsymbol{P}^T$$

$$\boldsymbol{L}^T = (\boldsymbol{P}^T\boldsymbol{U}^T\boldsymbol{P}^T)^T$$

$$= \boldsymbol{P}^{TT}\boldsymbol{U}^{TT}\boldsymbol{P}^{TT}$$

$$= \boldsymbol{PUP}$$

$$\boldsymbol{P}^{-1}\boldsymbol{L}^T\boldsymbol{P}^{-1} = \boldsymbol{P}^{-1}(\boldsymbol{PUP})\boldsymbol{P}^{-1}$$

$$\boldsymbol{U} = \boldsymbol{P}^{-1}\boldsymbol{L}^T\boldsymbol{P}^{-1}$$

Therefore, the matrix L is congruent to the matrix U. This proof is completed.

Lemma 4

If a matrix is Hermitian (or Skew-Hermitian) then it is congruent to a Hermitian (or Skew-Hermitian)

Proof

For a non-singular matrix $\boldsymbol{P}$, if matrix $\boldsymbol{B}$ is congruent to the matrix $\boldsymbol{A}$ such that $\boldsymbol{A} = \boldsymbol{A}^*$ and $\boldsymbol{A} = -\boldsymbol{A}^*$ then $\boldsymbol{B} = \boldsymbol{P}^*\boldsymbol{AP}$. This implies that;

$$\boldsymbol{B}^* = (\boldsymbol{P}^*\boldsymbol{AP})^*$$

$$= \boldsymbol{P}^*\boldsymbol{A}^*\boldsymbol{P}$$

If $\boldsymbol{A}$ is Hermitian matrix then $\boldsymbol{A} = \boldsymbol{A}^*$ therefore, $\boldsymbol{B}^* = \boldsymbol{P}^*\boldsymbol{AP}$ where $\boldsymbol{B}$ is Herrmitian matrix. If $\boldsymbol{A}^* = -\boldsymbol{A}$ is Skew-Hermitian matrix then $\boldsymbol{B}^* = \boldsymbol{P}^* - \boldsymbol{AP}$, this implied that $\boldsymbol{B}^* = -(\boldsymbol{P}^*\boldsymbol{AP})$. Therefore, $\boldsymbol{B}^* = -\boldsymbol{B}$ which is Skew-Hermitian completes the proof.

Main Result

Our goal in this section is to systematically present the solution methods for each objective.

We seek to prove that positive (or negative) definite symmetric matrix of a quadratic form is congruent to a matrix.

Consider two matrices $\boldsymbol{A_1} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ be a positive definite symmetric matrix and $\boldsymbol{A_2} = \begin{pmatrix} -2 & -1 & 2 \\ -1 & -2 & 1 \\ 2 & 1 & -4 \end{pmatrix}$ be a negative definite symmetric matrix. Converting to quadratic forms.

We have

$$q(x,y,z) = (x \quad y \quad z)\begin{pmatrix}1 & 2 & 1\\ 2 & 5 & 1\\ 1 & 1 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$= x^2 + 4xy + 2xz + 5y^2 + 2yz + 3z^2$$

$$= x^2 + 2x(2y+z) + 5y^2 + 2yz + 3z^2$$

$$= (x+2y+z)^2 + (y-z)^2 + z^2$$

To the form $q(w) = D_1\, l_1^2 + \frac{D_2}{D_1} l_2^2 + \frac{D_3}{D_2} l_3^2$

Where $l_1 = x + 2y + z$, $l_2 = y - z$ and $l_3 = z$. Therefore, $\begin{pmatrix}l_1\\ l_2\\ l_3\end{pmatrix} = \begin{pmatrix}1 & 2 & 1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$, $l = \boldsymbol{P}w$.

Where $\boldsymbol{P} = \begin{pmatrix}1 & 2 & 1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}$, $l = (l_1, l_2, l_3)$ and $w = (x, y, z)$. Then a matrix congruent to $\boldsymbol{A}$ is $\boldsymbol{B_1} = \boldsymbol{P^T A_1 P}$

$$= \begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 1 & -1 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 1\\ 2 & 5 & 1\\ 1 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 4 & 0\\ 4 & 17 & -2\\ 0 & -2 & 5\end{pmatrix}$$

Similarly, for $\dot{\boldsymbol{A}}_2$

$$q(x,y,z) = (x \quad y \quad z)\begin{pmatrix}-2 & -1 & 2\\ -1 & -2 & 1\\ 2 & 1 & -4\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$= -2x^2 - 2xy + 4xz - 2y^2 + 2yz - 4z^2$$

$$= -2\left(x + \frac{1}{2}y - z\right)^2 - \frac{3}{2}y^2 + 4yz - 2z^2$$

$$= -2\left(x + \frac{1}{2}y - z\right)^2 - \frac{3}{2}\left(y - \frac{4}{3}z\right)^2 + \frac{2}{3}z^2$$

To the form $q(x,y,z) = D_1\, l_1^2 + \frac{D_2}{D_1} l_2^2 + \frac{D_3}{D_2} l_3^2$. Matrix of coefficient of l_i is given as $\boldsymbol{P} = \begin{pmatrix}1 & \frac{1}{2} & -1\\ 0 & 1 & \frac{-4}{3}\\ 0 & 0 & \frac{2}{3}\end{pmatrix}$.

Therefore, the matrix congruent to $\boldsymbol{A_2}$ is

$\boldsymbol{B_2} = \boldsymbol{P^T A_2 P}$

$$= \begin{pmatrix}1 & 0 & 0\\ \frac{1}{2} & 1 & 0\\ -1 & \frac{-4}{3} & 1\end{pmatrix}\begin{pmatrix}-2 & -1 & 2\\ -1 & -2 & 1\\ 2 & 1 & -4\end{pmatrix}\begin{pmatrix}1 & \frac{1}{2} & -1\\ 0 & 1 & \frac{-4}{3}\\ 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}-2 & -2 & \frac{16}{3}\\ -2 & \frac{-7}{2} & \frac{22}{3}\\ \frac{16}{3} & \frac{22}{3} & \frac{-170}{9}\end{pmatrix}$$

By the method of congruence transformation, all the matrices are reduced to their diagonal matrix forms by elementary row and column operations.

Thus,

$$\begin{pmatrix}1&2&1\\2&5&1\\1&1&3\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\boldsymbol{A_1}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\-2&1&0\\-3&1&1\end{pmatrix}\boldsymbol{A_1}\begin{pmatrix}1&-2&-3\\0&1&1\\0&0&1\end{pmatrix}$$

$$\boldsymbol{D_{A_1}}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}1&4&0\\4&17&-2\\0&-2&5\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\boldsymbol{B_1}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\-4&1&0\\-8&2&1\end{pmatrix}\boldsymbol{B_1}\begin{pmatrix}1&-4&-8\\0&1&2\\0&0&1\end{pmatrix}$$

$$\boldsymbol{D_{B_1}}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}-2&-1&2\\-1&-2&1\\2&1&-4\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\boldsymbol{A_2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}-2&0&0\\0&-6&0\\0&0&-2\end{pmatrix}=\begin{pmatrix}1&0&0\\1&-2&0\\1&0&1\end{pmatrix}\boldsymbol{A_2}\begin{pmatrix}1&1&1\\0&-2&0\\0&0&1\end{pmatrix}$$

$$\boldsymbol{D_{A_2}}=\begin{pmatrix}-2&0&0\\0&-6&0\\0&0&-2\end{pmatrix}$$

$$\begin{pmatrix}-2&-2&\frac{16}{3}\\-2&\frac{-7}{2}&\frac{22}{3}\\\frac{16}{3}&\frac{22}{3}&\frac{-170}{9}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\boldsymbol{B_2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}-2&0&0\\0&-6&0\\0&0&-18\end{pmatrix}=\begin{pmatrix}1&0&0\\-2&2&0\\4&4&3\end{pmatrix}\boldsymbol{B_2}\begin{pmatrix}1&-2&4\\0&2&4\\0&0&3\end{pmatrix}$$

$$\boldsymbol{D_{B_2}}=\begin{pmatrix}-2&0&0\\0&-6&0\\0&0&-18\end{pmatrix}$$

As a result, all the matrices satisfy Lemma 2 and Sylvester's law of inertia as they share equal rank, index and signature. Based on this, they are congruent.

This is how to demonstrate the congruence of any right triangular matrix with a left triangular matrix. Let consider the right triangular matrix, $\boldsymbol{U}=\begin{pmatrix}1&2&3\\0&4&5\\0&0&6\end{pmatrix}$ and convert it to a quadratic form. Then, apply

Lagrangian reduction method to obtain matrix $\boldsymbol{P}$ such that the left triangular matrix, $\boldsymbol{L} = \boldsymbol{P}^{\boldsymbol{T}}(\boldsymbol{PU})^{\boldsymbol{T}}$which is congruent to the matrix $\boldsymbol{U}$, if $\boldsymbol{U} = \boldsymbol{P}^{-1}\boldsymbol{L}^{\boldsymbol{T}}\boldsymbol{P}^{-1}$. This implies that;

$$q(x,y,z) = (x \quad y \quad z)\begin{pmatrix}1 & 2 & 3\\0 & 4 & 5\\0 & 0 & 6\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$= x^2 + 2xy + 3xz + 4y^2 + 5yz + 6z^2$$

$$= x^2 + 2x\left(y + \frac{3}{2}z\right) + \left(y + \frac{3}{2}z\right)^2 - \left(y + \frac{3}{2}z\right)^2 + 4y^2 + 5yz + 6z^2$$

$$= \left(x + y + \frac{3}{2}z\right)^2 + 3y^2 + 2yz + \frac{15}{4}z^2$$

$$= \left(x + y + \frac{3}{2}z\right)^2 + 3\left[y^2 + \frac{2}{3}yz + \left(\frac{1}{3}z\right)^2 - \left(\frac{1}{3}z\right)^2\right] + \frac{15}{4}z^2$$

$$= \left(x + y + \frac{3}{2}z\right)^2 + 3\left(y + \frac{1}{3}z\right)^2 + \frac{41}{12}z^2$$

To the form $q(x,y,z) = D_1\, l_1^2 + \frac{D_2}{D_1}l_2^2 \ + \frac{D_3}{D_2}l_3^2$.

Where $l_1 = x + y + \frac{3}{2}z, l_2 = \ y + \frac{1}{3}z$ and $l_3 = z$ are the linear combinations of the variables $x, y. and\ z$.

Matrix of coefficient of l_i is given as $\boldsymbol{P} = \begin{pmatrix}1 & 1 & \frac{3}{2}\\0 & 1 & \frac{1}{3}\\0 & 0 & 1\end{pmatrix}$.

We have $\boldsymbol{L} = \boldsymbol{P}^{\boldsymbol{T}}(\boldsymbol{PU})^{\boldsymbol{T}}$.

$$\boldsymbol{PU} = \begin{pmatrix}1 & 1 & \frac{3}{2}\\0 & 1 & \frac{1}{2}\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\0 & 4 & 5\\0 & 0 & 6\end{pmatrix} = \begin{pmatrix}1 & 6 & 17\\0 & 4 & 7\\0 & 0 & 6\end{pmatrix}$$

$$\boldsymbol{L} = \begin{pmatrix}1 & 0 & 0\\1 & 1 & 0\\\frac{3}{2} & \frac{1}{3} & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\6 & 4 & 0\\17 & 7 & 6\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\7 & 4 & 0\\\frac{41}{2} & \frac{25}{3} & 6\end{pmatrix}$$

Such that;

$\boldsymbol{U} = \boldsymbol{P}^{-1}\boldsymbol{L}^{\boldsymbol{T}}\boldsymbol{P}^{-1}$

$$= \begin{pmatrix}1 & -1 & -\frac{7}{6}\\0 & 1 & -\frac{1}{3}\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 7 & \frac{41}{2}\\0 & 4 & \frac{25}{3}\\0 & 0 & 6\end{pmatrix}\begin{pmatrix}1 & -1 & -\frac{7}{6}\\0 & 1 & -\frac{1}{3}\\0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & -1 & -\frac{7}{6}\\0 & 1 & -\frac{1}{3}\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 6 & 17\\0 & 4 & \frac{21}{3}\\0 & 0 & 6\end{pmatrix}$$

$$= \begin{pmatrix}1 & 2 & 3\\0 & 4 & 5\\0 & 0 & 6\end{pmatrix}$$

As a result, $\boldsymbol{L}$ is congruent to $\boldsymbol{U}$ since it satisfies Lemma 3 and Sylvester's law of inertia.
To verify that Hermitian (or Skew-Hermitian) matrices are congruent to each other, as well as to real (or purely imaginary) diagonal matrices.

To answer this question, let consider matrix $\boldsymbol{A} = \begin{pmatrix} -1 & 1-2i & 0 \\ 1+2i & 0 & -i \\ 0 & i & 1 \end{pmatrix}$ be a Hermitian and the matrix $\boldsymbol{B} = \begin{pmatrix} i & 1+i & 3 \\ 1+2i & 0 & -i \\ -3 & i & 0 \end{pmatrix}$ be a Skew-Hermitian so that both matrices will be converted to their quadratic forms. Then, Lagrangian reduction method will to obtain matrix $\boldsymbol{P}$ from each matrix such that $\boldsymbol{A_1} = \boldsymbol{P^*AP}$ and $\boldsymbol{B_1} = \boldsymbol{P^*BP}$. This implied that;

$$q(x,y,z) = (x \quad y \quad z)\begin{pmatrix} -1 & 1-2i & 0 \\ 1+2i & 0 & -i \\ 0 & i & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= -x^2 + 2xy + z^2$$

$$= -[x^2 - 2xy + (-y)^2 - (-y)^2] + z^2$$

$$= -(x-y)^2 + y^2 + z^2$$

Where $\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and matrix $\boldsymbol{P} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We have

$\boldsymbol{A_1} = \boldsymbol{P^*AP}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1-2i & 0 \\ 1+2i & 0 & -i \\ 0 & i & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1-2i & 0 \\ 1+2i & -1 & -2i-i \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2-2i & 0 \\ 2+2i & -3 & -i \\ 0 & i & 1 \end{pmatrix}$$

$$q(x,y.z) = (x \quad y \quad z)\begin{pmatrix} i & 1+i & 3 \\ 1+2i & 0 & -i \\ -3 & i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= ix^2 + 2ixy + 4iy^2 + 2iyz$$

$$= i(x^2 + 2xy + (y)^2 - (y)^2) + 4iy^2 + 2iyz$$

$$= i(x+y)^2 + 3iy^2 + 2iyz$$

$$= i(x+y)^2 + 3i\left[y^2 + \frac{2}{3}yz + \left(\frac{1}{3}z\right)^2 - \left(\frac{1}{3}z\right)^2\right]$$

$$= i(x+y)^2 + 3i\left(y + \frac{1}{3}z\right)^2 - \frac{1}{3}iz^2$$

Therefore, $\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and matrix of coefficient of l_i is given as $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$.

This implies that;

$\boldsymbol{B_1} = \boldsymbol{P^*BP}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} i & 1+i & 3 \\ 1+2i & 0 & -i \\ -3 & i & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} i & 1+2i & \frac{10+i}{3} \\ -2-i & 7i & \frac{10+8i}{3} \\ -3 & -3+i & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} i & 1+2i & \frac{10+i}{3} \\ -1+2i & 7i & \frac{10+8i}{3} \\ \frac{-10+i}{3} & \frac{-10+8i}{3} & \frac{10i}{9} \end{pmatrix}$$

Now, apply method of congruence transformation to reduce each matrix to a diagonal matrix form.

$$\begin{pmatrix} -1 & 1-2i & 0 \\ 1+2i & 0 & -i \\ 0 & i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{A} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1+2i & 1 & 0 \\ 2-i & -i & 5 \end{pmatrix} \boldsymbol{A} \begin{pmatrix} 1 & 1-2i & 2+i \\ 0 & 1 & i \\ 0 & 0 & 5 \end{pmatrix}$$

$$\boldsymbol{D_A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2-2i & 0 \\ 2+2i & -3 & -i \\ 0 & i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{A_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2+2i & 1 & 0 \\ 2-2i & -i & 5 \end{pmatrix} \boldsymbol{A_1} \begin{pmatrix} 1 & 2-2i & 2+2i \\ 0 & 1 & -i \\ 0 & 0 & 5 \end{pmatrix}$$

$$\boldsymbol{D_{A_1}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Following the same procedures the diagonal matrix of matrix $B = \begin{pmatrix} i & 1+i & 3 \\ 1+2i & 0 & -i \\ -3 & i & 0 \end{pmatrix}$ is given as $\boldsymbol{D_B} = \begin{pmatrix} i & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -31i \end{pmatrix}$ and diagonal matrix $\boldsymbol{B_1} = \begin{pmatrix} i & 1+2i & \frac{10+i}{3} \\ -1+2i & 7i & \frac{10+8i}{3} \\ \frac{-10+i}{3} & \frac{-10+8i}{3} & \frac{10i}{9} \end{pmatrix}$ is given as $\boldsymbol{D_{B_1}} = \begin{pmatrix} i & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -279i \end{pmatrix}$. Therefore, observing the diagonal matrices we can conclude that the matrices are congruent and congruent to real and purely diagonal matrices.

To present the general proof of three-variable quadratic form using the method of Lagrangian reduction to the form $D_1 {l_1}^2 + \frac{D_2}{D_1} {l_2}^2 + \frac{D_3}{D_2} {l_3}^2$ where the D_i are the principal minors of the quadratic form. We have

$q(x, y, z) = ax^2 + 2dxy + 2exz + by^2 + 2fyz + cz^2$

$= a\left[x^2 + \frac{2}{a}x(dy + ez)\right] + by^2 + 2fyz + cz^2$

$= \ a\left[x^2 + \frac{2}{a}x(dy + ez) + \left(\frac{d}{a}y + \frac{e}{a}z\right)^2 - \left(\frac{d}{a}y + \frac{e}{a}z\right)^2\right] + by^2 + 2fyz + cz^2$

$= a\left(x + \frac{d}{a}y + \frac{e}{a}z\right)^2 + \frac{ab-d^2}{a}y^2 + \frac{2af-2de}{a}yz + \frac{ac-e^2}{a}z^2$

$= a\left(x + \frac{d}{a}y + \frac{e}{a}z\right)^2 + \frac{ab-d^2}{a}\left[y^2 + 2\frac{af-de}{ab-d^2}yz + \left(\frac{af-de}{ab-d^2}z\right)^2 - \left(\frac{af-de}{ab-d^2}z\right)^2\right] + \frac{ac-e^2}{a}z^2$

$= a\left(x + \frac{d}{a}y + \frac{e}{a}z\right)^2 + \frac{ab-d^2}{a}\left(y + \frac{af-de}{ab-d^2}z\right)^2 + \frac{2adef - a^2f^2 - d^2e^2}{a(ab-d^2)}z^2 + \frac{ac-e^2}{a}z^2$

$= a\left(x + \frac{d}{a}y + \frac{e}{a}z\right)^2 + \frac{ab-d^2}{a}\left(y + \frac{af-de}{ab-d^2}z\right)^2 + \frac{2def - af^2 + abc - be^2 - cd^2}{ab-d^2}z^2$

which is of the form

$q(\boldsymbol{w}) = D_1\, l_1^2 + \frac{D_2}{D_1} l_2^2 + + \frac{D_3}{D_2} l_3^2$,

where $\boldsymbol{l}_1 = \begin{pmatrix} 1 & \frac{d}{a} & \frac{e}{a} \end{pmatrix}$; $\boldsymbol{l}_2 = \begin{pmatrix} 0 & 1 & \frac{af-de}{ab-d^2} \end{pmatrix}$; $\boldsymbol{l}_3 = (0 \quad 0 \quad 1)$.

Thus,

$$\boldsymbol{P} = \begin{pmatrix} 1 & \frac{d}{a} & \frac{e}{a} \\ 0 & 1 & \frac{af-de}{ab-d^2} \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, the quadratic form is also of the form

$$q(\boldsymbol{w}) = w^T \boldsymbol{A} w$$

where $\boldsymbol{A} = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$.

Thus, the congruent matrix to $\boldsymbol{A}$ is given by

$$\boldsymbol{B} = \boldsymbol{p}^T \boldsymbol{A}\boldsymbol{p}.$$

By carrying out the multiplication, we obtain the matrix $\boldsymbol{B}$ as

$$\boldsymbol{B} = \begin{pmatrix} a & 2d & 2e + \frac{d(ef-de)}{ab-d^2} \\ 2d & \frac{3d^2}{a} + b & \frac{4de}{a} + \frac{2b(af-de)}{ab-d^2} \\ 2e + \frac{d(ef-de)}{ab-d^2} & \frac{4de}{a} + \frac{2b(af-de)}{ab-d^2} & K \end{pmatrix}$$

where $K = \frac{4e^2+ac}{a} + \frac{af^2-be^2}{ab-d^2} + \frac{af-de}{a(ab-d^2)^2}(2a^2bf - ad^2f - d^3e)$.

Using the expressions for elements in $\boldsymbol{B}$, it may be verified the congruent matrices of the quadratic form matrices that have been encountered earlier are correct.

CONCLUSION

The study's findings demonstrate that investigating the methods of congruence transformation and Lagrangian reduction is crucial for matrix analysis. By employing these techniques, the study effectively achieved its objectives, providing answers to the research questions that enhance our understanding of the literature and will benefit future researchers.

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