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Fixed-Point Theorems in Vector-Valued Fuzzy Metric Spaces

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Abstract
This paper presents a comprehensive study on fixed-point theorems within the framework of vector-valued fuzzy metric spaces, extending classical notions of both metric and fuzzy metric spaces. In this generalized setting, the metric function takes values in a partially ordered Banach space, providing a richer structure for analyzing convergence in multi-dimensional and uncertain environments. We establish sufficient conditions for the existence of fixed points for self-mappings in these spaces. The theoretical results are supported by illustrative examples and are positioned to contribute meaningfully to both pure and applied mathematical disciplines, particularly in systems involving fuzzy logic, uncertainty modeling, and vector-valued analysis.

Keywords: Fixed-point theorem, Fuzzy metric space, Vector-valued metric space, Fuzzy set theory, Banach contraction principle, Vector-valued fuzzy distance, Continuity in fuzzy spaces, Mappings in fuzzy metric spaces, Common fixed point, Contraction mapping

1. Introduction

Fixed-point theory is a central pillar of nonlinear analysis, with applications ranging from differential equations to economics and machine learning. Since Banach's contraction principle, numerous generalizations have been proposed to accommodate broader classes of spaces and mappings. Among these, fuzzy metric spaces have garnered significant attention for modeling situations with uncertainty.

In recent years, researchers have extended fuzzy metric spaces by introducing **vector-valued metrics**, wherein the distance function assumes values in ordered Banach spaces rather than real numbers. This generalization enables the modeling of more complex systems characterized by multi-dimensional uncertainty.

This paper investigates fixed-point results in **vector-valued fuzzy metric spaces (VVFMS)**, generalizing the traditional scalar-valued setting. The results presented here unify and extend several known fixed-point theorems and introduce new methods for proving the existence and uniqueness of fixed points in these enriched structures.

2. Preliminaries

We begin by recalling essential definitions and concepts relevant to this study.

2.1 Vector-Valued Fuzzy Metric Space

Let E be a real Banach space with a cone $P \subset E$, inducing a partial ordering $\leq$. A vector-valued fuzzy metric space is defined as a triple $(X, M, *)$, where:

- X is a non-empty set,
- $M: X \times X \times (0, \infty) \to E$ is a mapping satisfying the following:
- $M(x, y, t) > \theta$ for all $x \neq y \in X, t > 0$,
- $M(x, y, t) = M(y, x, t)$,
- $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
- $\lim_{t \to 0^+} M(x, y, t) = \theta$ if and only if $x = y$.

Here, * denotes a continuous t-norm on E, and θ is the zero vector in E.

2.2 Contraction Mappings

Let $T: X \to X$. We say T is a vector-valued fuzzy contraction if there exists a constant $0 < k < 1$ such that for all $x, y \in X, t > 0$,

$$M(Tx, Ty, t) \geq k * M(x, y, t)$$

3. Main Results

We now present our central fixed-point theorems in vector-valued fuzzy metric spaces.

Theorem 3.1 (Banach-type Fixed Point Theorem)

Let $(X, M, *)$ be a complete vector-valued fuzzy metric space, and let $T: X \to X$ be a vector-valued fuzzy contraction. Then T has a unique fixed point in X.

Proof Sketch:

Construct a sequence $\{x_n\}$ by choosing $x_0 \in X$ and setting $x_{n+1} = Tx_n$. Using the contraction condition and properties of M, one shows that $\{x_n\}$ is Cauchy in the sense of the vector-valued fuzzy metric. Completeness of X ensures convergence to some x^*, and continuity of T yields $Tx^* = x^*$.

Theorem 3.2 (Common Fixed Point)

Let $T_1, T_2: X \to X$ be two self-maps such that:

- T_1 and T_2 are weakly compatible,
- Each satisfies a generalized contractive condition:

$$M(T_1 x, T_2 y, t) \geq k * M(x, y, t),$$

for some $0 < k < 1$ and all $x, y \in X$.

Then T_1 and T_2 have a unique common fixed point.

4. Examples and Applications

Example 4.1

Let $E = \mathbb{R}^2$ with the usual norm and cone $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$. Define $M(x, y, t) = (\mu_1(x, y, t), \mu_2(x, y, t))$ where:

$$\mu_i(x, y, t) = \frac{t}{t + d_i(x, y)}, i = 1,2$$

and d_1, d_2 are ordinary metrics. Then $(X, M, *)$ forms a vector-valued fuzzy metric space suitable for modeling bivariate uncertainties.

5. Conclusion

This study lays a theoretical foundation for fixed-point results within the framework of vector-valued fuzzy metric spaces, thereby extending classical fuzzy metric and vector-valued metric constructs. The fixed-point theorems developed herein offer a generalized and flexible approach suitable for addressing problems characterized by multi-dimensional uncertainty. These results have significant potential for application in various domains, including multi-criteria decision-making, systems theory, and the analysis of fuzzy differential equations, where the interaction between fuzziness and vector-valued structures is essential.

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