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Some New Congruences for Andrews Partition Function

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Abstract
Recently, Andrews introduced partition function $\mathcal{EO}(n)$and $\overline{\mathcal{EO}}(n)$ where the function $\mathcal{EO}(n)$ denotes the number of partitions of n in which every even part is less than each odd part and the function $\overline{\mathcal{EO}}(n)$ denotes the number of partitions enumerated by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times. Pore and Fathima in [2] obtained some congruences modulo 2, 4, 10 and 20 for the partition function $\mathcal{EO}(n)$. In this paper, we prove some conjectures due to Pore and Fathima [2] and also find some new congruences for the partition functions $\overline{\mathcal{EO}}(n)$ and $\mathcal{EO}_e(n)$.

Keywords: Partition, congruences, Rogers-Ramanujan continued fraction.

1. Introduction

A partition of a nonnegative integer n is a representation of n as a sum of a positive integers, called summands or parts of the partition. For example, the partitions of 6 are $6,\ 5+1,\ 4+2,\ 4+1+1,\ 3+3,\ 3+2+1,\ 3+1+1+1,\ 2+2+2,\ 2+2+1+1,\ 2+1+1+1+1,\ 1+1+1+1+1+1.$

$p(n)$ denotes the number of partitions of n. So, $p(6)=11$.

The generating function for $p(n)$, due to Euler is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where, as customary, for any complex number a and $|q|<1$

$$(a;\ q)_\infty := \prod_{i=0}^{\infty}(1-ae^i).$$

Throughout this paper we use $f_k=(q^k;q^k)_\infty$.

Ramanujan [3], [4, pp. 210-213] found three simple congruences satisfied by $p(n)$, namely

$p(5n+4)\equiv 0\ (mod\ 5),$ (1.1)

$p(7n+5)\equiv 0\ (mod\ 7),$ (1.2)

$p(11n+6)\equiv 0\ (mod\ 11)$. (1.3)

Andrews [1] introduced the partition function $\mathcal{EO}(n)$ which counts the number of partitions of n in which every even part is less than each odd part. For example, $\mathcal{EO}(n)$=7. The seven

partitions of 6 it enumerates are $6,\ 5+1,\ 4+2,\ 3+3,\ 3+1+1+1,\ 2+2+2,$ $1+1+1+1+1+1$.
In [1], Andrews shows that the generating function for $\mathcal{EO}(n)$ is

$$\sum_{n=0}^{\infty} \mathcal{EO}(n)q^n = \frac{1}{(1-q)f_2}$$

Andrews also defined the partition function $\overline{\mathcal{EO}}(n)$ which counts the number of partitions enumerated by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times. For example, $\overline{\mathcal{EO}}(6) = 4$. The four partitions of 6 it enumerates are $6,\ 3+3,\ 2+2+2,\ 1+1+1+1+1+1$. Andrews provide the generating function for $\mathcal{EO}(n)$ as

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n)\, q^n = \frac{f_4^3}{f_2^2}\ . \qquad (1.4)$$

Pore and Fathima [2] proved some congruences modulo 2, 4, 5, 10 and 20 for $\overline{\mathcal{EO}}(n)$. For example
$\overline{\mathcal{EO}}(8n+4) \equiv 0 \pmod 2$,
$\overline{\mathcal{EO}}(8n+6) \equiv 0 \pmod 4$,
$\overline{\mathcal{EO}}(10n+8) \equiv 0 \pmod 5$,
$\overline{\mathcal{EO}}(40n+38) \equiv 0 \pmod{20}$,
$\overline{\mathcal{EO}}(20n+18) \equiv 0 \pmod{10}$.

They conclude with a conjecture on $\overline{\mathcal{EO}}(n)$.

Conjecture 1.1.
$\overline{\mathcal{EO}}(10(5n+r)+8) \equiv 0\ (mod\ 20), r = 1(1)4$ (1.5)
They also consider

$$\sum_{n=0}^{\infty} \mathcal{EO}_e(n)q^n = \frac{f_4^2}{f_2^2}\ , \qquad (1.6)$$

where the function $\mathcal{EO}_e(n)$ counts the number of partitions enumerated by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times except when parts are odd and number of parts is even. For example, $\mathcal{EO}_e(6) = 6$. The six partitions of 6 it enumerates are 6, 3 + 3, 2 + 2 + 2, 5 +1, 3 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1.

They obtained some congruences modulo 2 for $\mathcal{EO}_e(n)$, for exa
$\mathcal{EO}_e(4n+2) \equiv 0 \pmod 2$. (1.7)

In this paper, we find exact generating functions for $\overline{\mathcal{EO}}(10n+8)$, $\overline{\mathcal{EO}}(50\mathrm{n}+8)$ and $\mathcal{EO}_e(10n+4)$ and prove some congruences modulo 8 and 20 for these partition functions.

Theorem 1.2. *We have*

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(10n+8)q^n = 5f_{10}f_5^2\frac{f_2^2}{f_1^4}\ . \qquad (1.8)$$

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(50n+8)q^n = 5\frac{f_5^2\,f_5^4}{f_1^6\,f_{10}^2} + 800q\frac{f_2^6 f_{10} f_5^3}{f_1^9} + 14000q^2\frac{f_2^7 f_5^2 f_{10}^4}{f_1^{12}} +$$

$$+80000q^3\frac{f_2^8 f_5 f_{10}^7}{f_1^{15}}+160000q^4\frac{f_2^9 f_{10}^{10}}{f_1^{18}} \qquad (1.9)$$

Corollary 1.3. *Conjecture* 1.1 is true.

Theorem 1.4. *We have*

$$\sum_{n=0}^{\infty}\mathcal{EO}_e(10n+4)q^n=3\frac{f_5^6 f_2^4}{f_{10}^2 f_1^8}+8q\frac{f_5 f_{10}^3 f_2^3}{f_1^7} \qquad (1.10)$$

Corollary 1.5.

$$\mathcal{EO}_e(10(5n+r)+4)\equiv 0\ (mod\ 2^3),\ r=1(1)4\,. \qquad (1.11)$$

2. Preliminaries

Rogers-Ramanujan continued fraction $R(q)$ is defined by

$$R(q):=\frac{q^{1/5}}{1}+\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\cdots}=q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty},|q|<1$$

Lemma 2.1. [6, p. 165] *If* $R=\frac{q}{R(q^5)}$ *then*

$$f_1=f_{25}\left(R-q-\frac{q^2}{R}\right) \qquad (2.1)$$

and

$$\frac{1}{f_1}=\frac{f_{25}^5}{f_5^6}\left(R^4+qR^3+2q^2R^2+3q^3R+5q^4-\frac{3q^5}{R}+\frac{2q^6}{R^2}-\frac{q^7}{R^3}+\frac{1}{R^4}\right)\quad . \qquad (2.2)$$

Lemma 2.2. [5, Equations (2.10)-(2.18)] *If* $x=\frac{q^{1/5}}{R(q)}$ *and* $y=\frac{q^{2/5}}{R(q^2)}$, *then*

$$a_1:=xy^2-\frac{q^2}{xy^2}=K, \qquad (2.3)$$

$$a_2:=\frac{x^2}{y}-\frac{y}{x^2}=\frac{4q}{K}, \qquad (2.4)$$

$$a_3:=\frac{y^3}{x}-q^2\frac{x}{y^3}=K+\frac{4q^2}{K}-2q\ , \qquad (2.5)$$

$$a_4:=x^3y+\frac{q^2}{x^3y}==K+\frac{4q^2}{K}+2q\ , \qquad (2.6)$$

$$a_5:=x^5-\frac{q^2}{x^5}=a_1+a_2a_4\ , \qquad (2.7)$$

$$a_6:=x^{10}-\frac{q^4}{x^{10}}=a_5^2+2q^2\ , \qquad (2.8)$$

$$a_7:=x^8y-\frac{q^4}{x^8y}=a_4a_5-q^2a_2\ , \qquad (2.9)$$

$$a_8:=\frac{x^7}{y}+q^2\frac{y}{x^7}=a_2a_5+a_4\quad , \qquad (2.10)$$

$$a_9:=\frac{x^{12}}{y}-q^4\frac{y}{x^{12}}=a_2a_6+a_7\quad , \qquad (2.11)$$

where $K=\frac{f_2f_5^5}{f_1f_{10}^5}$

Lemma 2.3. *From* [5, Eqs. (2.6), (2.7), (2.29)] *we have*

$$\frac{f_5^5}{f_1^4 f_{10}^3}-4q\frac{f_{10}^2}{f_1^3 f_2}=\frac{f_5}{f_2^2 f_{10}}\ , \qquad (2.12)$$

$$\frac{f_5^5}{f_1 f_{10}^3}+q\frac{f_{10}^2}{f_2}=\frac{f_2^2 f_5^2}{f_1^2 f_{10}^2}\ , \qquad (2.13)$$

$$\frac{f_5^5}{f_1 f_{10}^3}-5q\frac{f_{10}^2}{f_2 f_1^3}=\frac{f_5}{f_{10} f_2^2}\quad . \qquad (2.14)$$

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The following congruences which can be easily established by applying binomial theorem

$$f_1^4 \equiv f_2^2 \pmod 4 \quad (2.15)$$

$$f_1^8 \equiv f_2^4 \pmod 8 \quad (2.16)$$

3. EXACT GENERATING FUNCTION FOR $\overline{\mathcal{EO}}(10n+8)$, $\overline{\mathcal{EO}}(50n+8)$

Proof of Theorem 1.2. From [2, Eqs. (3.7)] we have

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(2n)q^n = \frac{f_2^3}{f_1^2} \quad . \quad (3.1)$$

Employing (2.1) and (2.2) in the above, extracting the terms involving q^{5n+4}, dividing both sides of the resulting identity by q^4, and then replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(10n+8)q^n = \frac{f_{10}^3 f_5^{10}}{f_1^{12}}[75q^2 + 50q\left(x^5 - \frac{q^2}{x^5}\right) + 20\left(x^4y^3 - \frac{q^4}{x^2y^3}\right)$$

$$-20q\left(\frac{y^3}{x} - q^2\frac{x}{y^3}\right) - 5q^2\left(\frac{x^6}{y^3} - \frac{y^3}{x^6}\right) - 15\left(x^6y^2 + \frac{q^4}{x^6y^2}\right)$$

$$-60q\left(xy^2 - \frac{q^2}{xy^2}\right) - 60q^2\left(\frac{x^4}{y^2} + \frac{y^2}{x^4}\right)] \ , \quad (3.2)$$

where x and y are as defined in Lemma 2.2.
Using Lemma 2.2 we have

$$a_{10} := \frac{x^4}{y^2} + \frac{y^2}{x^4} = a_2^2 + 2 \quad (3.3)$$

$$a_{11} := \frac{x^6}{y^3} - \frac{y^3}{x^6} = a_2^3 + 3a_2 \quad (3.4)$$

$$a_{12} := x^6y^2 + \frac{q^4}{x^6y^2} = a_4^2 - 2q^2 \quad (3.5)$$

$$a_{13} := x^4y^3 - \frac{q^4}{x^2y^3} = a_1a_4 - q^2a_2 \quad (3.6)$$

Employing (2.7), (2.5), (2.3), (3.3), (3.4), (3.5), (3.6) in (3.2), we arrive at

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(10n+8)q^n = 5\frac{f_{10}^{18}}{f_1^9 f_5^5 f_2^3}\left(\frac{f_2 f_5^5}{f_1 f_{10}^5} - 4q\right)^3\left(\frac{f_2 f_5^5}{f_1 f_{10}^5} + q\right)^2. \quad (3.7)$$

Using (2.12) and (2.13) in (3.7), we arrive at (1.8).
Applying [5, Eqs (1.7)], (1.8) can be written as

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(10n+8)q^n = 5\frac{f_{10}^2}{f_5}\sum_{n=0}^{\infty} Q(5n+1)\, q^n \ , \quad (3.8)$$

where $Q(n)$ denote the number of partitions of a nonnegative integer into distinct parts.
Now extracting the terms involving q^{5n} in (3.8) and then replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(50n+8)q^n = 5\frac{f_2^2}{f_1}\sum_{n=0}^{\infty} Q(25n+1)\, q^n \quad (3.9)$$

Applying [5, Eqs (1.8)] in the above we arrive at (1.9).
Now taking modulo 4 in both sides of (1.8) and applying (2.15), we arrived at

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(50n+8)q^n \equiv 5f_{10}f_5^2 \pmod 4. \quad (3.10)$$

Equating coefficients of q^{5n+r} , $r = 1(1)4$ from both sides of the above equation we can prove (1.9).

4. EXACT GENERATING FUNCTION FOR $\mathcal{EO}_e(10n+4)q^n$

Proof of Theorem 1.4. From (1.6), we have

$$\sum_{n=0}^{\infty} \mathcal{EO}_e(2n)q^n = \frac{f_2^2}{f_1^2} \qquad (4.1)$$

Employing (2.1) and (2.2) in the above and extracting the terms involving q^{5n+2}, dividing both sides of the resulting identity by q^2, and then replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty} \mathcal{EO}_e(10n+4)q^n = \frac{f_{10}^3 f_5^{10}}{f_1^{12}}\Big[-15q^2 - 10q\left(x^5 - \frac{q^2}{x^5}\right) + 5\left(x^6y^2 + \frac{q^4}{x^6y^2}\right) + 20q\left(xy^2 - \frac{q^2}{xy^2}\right) + 20q^2\left(\frac{x^4}{y^2} + \frac{y^2}{x^4}\right) - 2\left(x^8y - \frac{q^4}{x^8y}\right) - 32q\left(x^3y + \frac{q^2}{x^3y}\right) + 54q^2\left(\frac{x^2}{y} - \frac{y}{x^2}\right) + 4q\left(\frac{x^7}{y} + q^2\frac{y}{x^7}\right)\Big] \quad (4.2)$$

where x and y are as defined in Lemma 2.2.

Using (2.3), (2.4), (2.6), (2.7), (2.9), (2.10) and (3.3) in (4.2), we arrive at

$$\sum_{n=0}^{\infty} \mathcal{EO}_e(10n+8)q^n = \frac{f_{10}^{17}}{f_5^5 f_1^9 f_2^3}\left(\frac{f_2 f_5^5}{f_1 f_{10}^5} - 4q\right)^2 \left(\frac{f_2 f_5^5}{f_1 f_{10}^5} + q\right)\Big)^2 \left(3\frac{f_2 f_5^5}{f_1 f_{10}^5} + 8q\right) \qquad (4.3)$$

Invoking (2.12) and (2.13) in (4.3), we arrive at (1.10).
Now taking modulo 8 in both sides of (1.10) and applying (2.16) we get

$$\sum_{n=0}^{\infty} \mathcal{EO}_e(10n+4)q^n \equiv 3\frac{f_5^6}{f_{10}^2} \pmod 8 \qquad (4.4)$$

Equating coefficients of $q^{5n+r}, r = 1(1)4$ from both sides of the above equation we can prove (1.11).

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