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Some Transformations for Hyper geometric Series

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Abstract:
In this paperwork, we have taken certain transformation formulae due to Slater [2]; App. (III) Verma & Jain [1] and making use of known identities, to establish some double hyper geometric series into single series in original research work.

Keywords: Generalized hyper - geometric function / Gauss hyper - geometric function and Ordinary hyper-geometric series; identities, known transformation formulae.

2010AMS Subject Classification: 33C15, 33C20, 33D15.

1. Introduction, Notation and Definitions:

In this paper we shall take the following notation and definition;

$$\text{for any numbers } \alpha \text{ and } q, \text{real or complex and } |q| < 1,$$

Let

$$[a]_n \equiv [a;q]_n = \begin{cases} 1 & ; \; n = 0 \\ (1-a)(1-aq)(1-aq^2)\ \dots\ (1-aq^{n-1}); & n > 0 \end{cases} \tag{1.1}$$

Accordingly,

$$[a;q]_n = \prod_{r=0}^{\infty}(1-aq^r) \tag{1.2}$$

Also,

$$[a_1, a_2, a_3, a_4, \ \dots\ a_r\,; q]_n = [a_1;q]_n[a_2;q]_n[a_3;q]_n[a_4;q]_n \dots [a_r;q]_n. \tag{1.3}$$

We define a basic hyper-geometric series:

$${}_r\emptyset_s\begin{bmatrix} a_1, a_2, \dots, a_r; q; z \\ b_1, b_2, \dots, b_s; q^\lambda \end{bmatrix} = \sum_{n=0}^{\infty} \frac{[a_1, a_2, \dots a_r\,; q]_n z^n q^{\lambda n(n+1)/2}}{[q, b_1, b_2, \dots b_r\,; q]_n}, \quad (|q| < 1). \tag{1.4}$$

We, also, define a truncated series:

$${}_r\emptyset_s\begin{bmatrix} a_1, a_2, \dots, a_r; q; z \\ b_1, b_2, \dots, b_s; q^\lambda \end{bmatrix}_N = \sum_{n=0}^{N} \frac{[a_1, a_2, \dots a_r\,; q]_n z^n q^{\lambda n(n+1)/2}}{[q, b_1, b_2, \dots b_r\,; q]_n}, \tag{1.5}$$

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We shall have the occasion to use the following well known Baily's transformation
If

$$\beta_n = \sum_{r=0}^{n} \alpha_r\, u_{n-r}\, v_{n+r}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} u_{n+r}\, v_{n-r} \delta_r = \sum_{r=0}^{\infty} u_r\, v_{r+2n} \delta_{r+n} \qquad (1.6)$$

***Then under suitable convergence conditions*:**

$$\sum_{r=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n\, \delta_n \qquad (1.7)$$

Where α_n, δ_n, u_r ***and*** v_r ***are any functions r only, such that the series*** γ_n ***exists.***

In order to establish certain transformation and summation formulae for basic hyper-geometric functions, we shall be in need of the following known results.

$$4^{\emptyset_3}\begin{bmatrix} a, & q\sqrt{a}, & -q\sqrt{a}, & e; & q; & \frac{1}{e} \\ & \sqrt{a} & -\sqrt{a} & \frac{aq}{e} & & \end{bmatrix}_N = \frac{[aq,eq;q]_N}{[q,aqe;q]_N\, e^N} \qquad (1.8)$$

[1; App. II(23)]

$$\sum_{k=0}^{n} \frac{(1-ap^k q^k)[a;p]_k[c;q]_k c^k}{(1-a)[q;q]_k[\frac{ap}{c};p]_k} = \frac{[ap;q]_n[cq;q]_n c^n}{[q;q]_n[\frac{ap}{c};p]_n} \qquad (1.9)$$

[8; App. II(34)]

$$\sum_{k=0}^{n} \frac{(1-ap^k q^k)(1-bp^k q^{-k})\,[a,b;p]_k[c,a,bc;q]_k q^k}{(1-a)(1-b)\,[q,\frac{aq}{b};q]_k[\frac{ap}{c},bcp;p]_k} = \frac{[ap,bp;p]_n[cq,aq/bc;q]_n}{[q,aq/b;q]_n[\frac{ap}{c},bcp;p]_n} \qquad (1.10)$$

[8; App. II(35)]

2. Main Results:

In this paper, we shall establish our main results.

Setting $u_r = v_r = 1$

And

$$\delta_r = \frac{[a;q]_r[a;q]_r\, q^r}{[e;q]_r\left[\frac{abq^2}{e};q\right]_r}$$

In (1.1.2) we get

$$\gamma_n = \sum_{r=n}^{\infty} u_r\, v_{2n+r}\delta_{r+n}$$

$$= \frac{[a;q]_r[a;q]_r\, q^r}{[e;q]_r\left[\frac{abq^2}{e};q\right]_r}\, 3^{\emptyset_2}\begin{bmatrix} aq^n, bq^n, q; q; z \\ eq^n, abq^{2+n} \end{bmatrix}, \qquad (2.1)$$

Now using

$n \to \infty$ ***case of*** (1.4) ***to sum the*** $3^{\emptyset_2}$ ***series on the right hand side of*** (2.1) ***we get*:**

$$\gamma_n = \frac{(1-\frac{q}{e})\;(1-\frac{ab}{e}q)}{(1-\frac{aq}{e})\;(1-\frac{bq}{e})}\left\{\frac{[e;q]_n\left[\frac{abq^2}{e};q\right]_n}{q^n\left[\frac{e}{q};q\right]_n\left[\frac{abq}{e};q\right]_n} - \frac{[a,b;q]_\infty\left[e,\frac{abq^2}{e};q\right]_n}{\left[\frac{e}{q},\frac{abq}{e};q\right]_n[a,b;q]_n q^n}\right\} \qquad (2.2)$$

Subsituting these values of γ_n **and** δ_n **in** (1.7) **we get the following:**

Master Results:

$$\sum_{n=0}^{\infty} \alpha_r \left\{ \frac{[e;q]_n\left[\frac{abq^2}{e};q\right]_n}{q^n\left[\frac{e}{q};q\right]_n\left[\frac{abq}{e};q\right]_n} - \frac{[a,b;q]_\infty\left[e,\ \frac{abq^2}{e};q\right]_n}{\left[\frac{e}{q},\ \frac{abq}{e};q\right]_n [a,b;q]_n q^n} \right\} = \frac{(1-\frac{aq}{e})\ (1-\frac{bq}{e})}{(1-\frac{q}{e})\ (1-\frac{ab}{e}q)} \sum_{n=0}^{\infty} \frac{[a;q]_n[b;q]_n q^n}{[e;q]_n\left[\frac{abq^2}{e};q\right]_n}, \qquad (2.3)$$

(i) Now taking

$$\alpha_r = \frac{[X;p]_r[p\sqrt{X};p]_r[-p\sqrt{X};p]_r[Y;p]_r}{[p;p]_r[\sqrt{X};p]_r[-\sqrt{X};p]_r\left[\frac{Xp}{Y};p\right]_r Y^r}$$
$$\text{and } u_r = v_r = 1$$

In (1.1.1) we get:

$$\beta_n = \sum_{r=0}^{n} \frac{[X;p]_r[p\sqrt{X};p]_r[-p\sqrt{X};p]_r[Y;p]_r}{[p;p]_r[\sqrt{X};p]_r[-\sqrt{X};p]_r\left[\frac{Xp}{Y};p\right]_r Y^r}$$
$$= {}_4\emptyset_3\begin{bmatrix} X, & Yp; & a, & b; & p;q; & \frac{q}{Y} \\ & \frac{Xp}{Y}; & e, & \frac{abq^2}{e} & & \end{bmatrix}_n$$

$$\text{Now, summing } {}_4\emptyset_3 \text{ series with the help of } (1.8) \text{ we get:}$$

$$\beta_n = \frac{[Xp;Yp;p]_n}{[p,\frac{Xp}{Y}]_n Y^n} \qquad (2.4)$$

Substituting these values of α_n and β_n in Master Result (2.3) we get:

$${}_6\emptyset_5\begin{bmatrix} X, & p\sqrt{X}, & -p\sqrt{X}, & Y; & e; & \frac{abq^2}{e}; & p, & q; & \frac{1}{Yq} \\ & \sqrt{X}, & -\sqrt{X}, & \frac{Xp}{Y}; & \frac{e}{q}, & \frac{abq}{e} & & & \end{bmatrix} -$$
$$\frac{[a,b;q]_\infty}{[\frac{e}{q},\frac{abq}{p};q]_\infty}\ {}_6\emptyset_5\begin{bmatrix} X, & p\sqrt{X}, & -p\sqrt{X}, & Y; & e; & \frac{abq^2}{e}; & p, & q; & \frac{1}{Yq} \\ & \sqrt{X}, & -\sqrt{X}, & \frac{Xp}{Y}; & a, & b & & & \end{bmatrix} =$$
$$\frac{\left(1-\frac{aq}{e}\right)\left(1-\frac{bq}{e}\right)}{\left(1-\frac{q}{e}\right)\left(1-\frac{abq}{e}\right)}\ {}_4\emptyset_3\begin{bmatrix} X, & Yp; & a, & b; & p;q; & \frac{q}{Y} \\ & \frac{Xp}{Y}; & e, & \frac{abq^2}{e} & & \end{bmatrix} \qquad (2.5)$$

(ii) Further , setting

$$\alpha_r = \frac{(1-AP^rQ^r)[A;P]_r[C;Q]_r C^{-r}}{(1-A)[Q;Q]_r[\frac{AP}{C};P]_r}$$
$$\text{and } u_r = v_r = 1 \quad \text{in } (4.1.1) \text{ we get:}$$

$$\beta_n = \sum_{r=0}^{n} \frac{(1-AP^rQ^r)[A;P]_r[C;Q]_r C^{-r}}{(1-A)[Q;Q]_r[\frac{AP}{C};P]_r}$$

$$\text{Now, using } (1.9) \text{ to sum the above series on the right hand side, we get:}$$

$$\beta_n = \frac{[AP;P]_n[C;Q]_n C^{-n}}{[Q;Q]_n[\frac{AP}{C};P]_n}$$

$$\textit{Substituting these values of } \alpha_n \textit{ and } \beta_n \textit{ in Master Result } (2.3) \textit{we get}:$$

$$5^{\emptyset_4}\begin{bmatrix} C: & A: & APQ: & e; & \frac{abq^2}{e}; & Q, P, & q; & \frac{1}{Yq} \\ & \frac{AP}{C}: & A: & \frac{Xp}{Y}; & \frac{e}{q}, & \frac{abq}{e} & & \end{bmatrix} -$$

$$\frac{[a,b;q]_\infty}{[\frac{e}{q},\frac{abq}{p};q]_\infty} 5^{\emptyset_4}\begin{bmatrix} C: & A: & APQ: & e; & \frac{abq^2}{e}; & Q, & P; & PQ: & q; & \frac{1}{Cq} \\ & & \sqrt{X}, & -\sqrt{X}, & \frac{Xp}{Y}; & a, & b & & & \end{bmatrix} =$$

$$\frac{\left(1-\frac{aq}{e}\right)\left(1-\frac{bq}{e}\right)}{\left(1-\frac{q}{e}\right)\left(1-\frac{abq}{e}\right)} 4^{\emptyset_3}\begin{bmatrix} CQ: & AP: & a, & b; & Q, & P, & q; & \frac{Q}{c} \\ & \frac{AP}{C}; & e, & \frac{abq^2}{e} & & & & \end{bmatrix} \quad \textbf{(2.6)}$$

(iii) **Now, setting**

$$\alpha_r = \frac{(1-Ap^r{q_1}^r)(1-Bp^r{q_1}^r)\,[A,B;P]_r[C,\frac{A}{BC};q_1]_r{q_1}^r}{(1-A)(1-B)[q_1,\frac{Aq_1}{B};q_1]_r[\frac{Ap}{C},BCp;p]_r}$$

$$\textit{and } \ u_r = v_r = 1 \quad \textit{in } (1.1) \textit{we get}:$$

$$\beta_n = \sum_{r=0}^{n} \frac{(1-Ap^r{q_1}^r)(1-Bp^r{q_1}^r)\,[A,B;P]_r[C,\frac{A}{BC};q_1]_r{q_1}^r}{(1-A)(1-B)[q_1,\frac{Aq_1}{B};q_1]_r[\frac{Ap}{C},BCp;p]_r}$$

$$\textit{Now, using } (1.10) \textit{to sum the above series on the right hand side, we get}:$$

$$\beta_n = \frac{[Ap,Bp;p]_n[Cq_1,\frac{Aq_1}{BC};q_1]_n}{[q_1,\frac{Aq_1}{B};q_1]_n[\frac{Ap}{C};BCp;p]_n} \quad \textbf{(2.7)}$$

$$\textit{Substituting these values of } \alpha_n \textit{ and } \beta_n \textit{ in Master Result } (2.3) \textit{we get}:$$

$$8^{\emptyset_7}\begin{bmatrix} C, & \frac{A}{BC}: & A, & B: & Apq_1: & \frac{Bp}{q_1}; & e, & \frac{abq^2}{e}; & q_1, & p, & pq_1; & \frac{p}{q_1}; & \frac{q_1}{q} \\ & \frac{Aq_1}{C}: & \frac{Ap}{C}, & BCp: & A: & B: & \frac{e}{q}, & \frac{abq}{e} & & & & & \end{bmatrix} -$$

$$\frac{[a,b;q]_\infty}{[\frac{e}{q},\frac{abq}{p};q]_\infty} 8^{\emptyset_7}\begin{bmatrix} C, & \frac{A}{BC}: & A, & B: & Apq_1: & \frac{Bp}{q_1}; & e, & \frac{abq^2}{e}; & q_1, & p, & pq_1; & \frac{p}{q_1}; & \frac{q_1}{q} \\ & \frac{Aq_1}{C}: & \frac{Ap}{C}, & BCp: & A: & B: & a, & b & & & & & \end{bmatrix} =$$

$$\frac{\left(1-\frac{aq}{e}\right)\left(1-\frac{bq}{e}\right)}{\left(1-\frac{q}{e}\right)\left(1-\frac{abq}{e}\right)} 6^{\emptyset_5}\begin{bmatrix} Cq_1, & \frac{Aq_1}{BC}: & Ap: & Bp: & a, & b; & q_1, & p, & q; & q \\ & \frac{Aq_1}{B}: & \frac{Ap}{C}, & BCp: & e, & \frac{abq^2}{e} & & & & \end{bmatrix} \quad \textbf{(2.8)}$$

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