Fixed-Point Theorems within Graphical Cone Metric Spaces: Theory and Applications

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Abstract:
This paper explores the foundational and advanced aspects of fixed-point theorems in the setting of graphical cone metric spaces. These structures generalize the classical metric spaces by incorporating the theory of cones in Banach spaces and the use of graph-theoretic properties. The research builds on the framework of cone metric spaces and integrates graph theory to extend the conditions under which fixed points exist and are unique. The paper presents key definitions, essential preliminaries, core propositions, and illustrative examples to provide a comprehensive understanding of the subject.

Keywords: Fixed-point theorem, Graphical cone metric space (GCMS), Cone metric space, Contraction mapping, Banach-type contraction, Kannan-type mapping, Directed graphs in analysis, Nonlinear analysis, Ordered metric space, Applications of fixed-point theory.

1. INTRODUCTION

Fixed-point theory remains a foundational area in mathematical analysis and nonlinear functional analysis, owing to its broad range of theoretical and applied implications across multiple disciplines such as differential equations, dynamical systems, optimization theory, game theory, and economics. A fixed point of a function is a point that is mapped to itself under the function, and identifying such points often serves as a key to proving existence and stability of solutions in numerous mathematical models.

The Banach contraction principle, one of the cornerstones of fixed-point theory, has inspired a rich vein of generalizations and extensions over the decades. It postulates that any contraction mapping on a complete metric space has a unique fixed point. However, classical metric spaces may not adequately capture the complexity of certain real-world systems, particularly those involving multiple dimensions, partial orders, or nonlinearity in distance structures. To address such limitations, the concept of cone metric spaces was introduced by **Huang and Zhang (2007)**. In cone metric spaces, the distance between points takes values in an ordered Banach space with a cone, rather than the set of non-negative real numbers. This modification allows for a more flexible and generalized framework, accommodating richer structures and broader applications.

Building on this idea, researchers began incorporating graph theory into the metric framework to encode additional relational or directional information among elements. This gave rise to the notion of graphical metric spaces, where a graph overlays the metric space to define relations such as adjacency or connectivity. In particular, **Jachymski** introduced the use of directed graphs in metric fixed-point theory, leading to the more comprehensive structure known as Graphical Cone Metric Spaces (GCMS). GCMS synergize the advantages of both cone metrics and directed graphs, creating a powerful analytical tool for examining fixed-point properties in partially ordered or structured systems.

The motivation for studying fixed-point theorems within GCMS arises from their ability to model situations where both distance and directionality matter—for instance, in network dynamics, optimization problems with constraints, and systems with asymmetric dependencies. In GCMS, not only is the notion of distance generalized through cones, but the existence of edges in a directed graph also imposes constraints and structure on the space, influencing how convergence and continuity are interpreted.

This paper aims to explore and establish novel fixed-point theorems within the GCMS framework. We adopt various contraction conditions, such as Banach-type and Kannan-type mappings, and demonstrate their validity in the presence of a directed graph structure over cone metric spaces. By doing so, we provide a broader generalization of existing theorems and enhance their applicability. Additionally, we provide illustrative examples to showcase the efficacy of our results and propose further directions for the use of GCMS in mathematical modeling and applied analysis.

2. PRELIMINARIES

Definition 2.1 (Cone): Let E be a real Banach space and $P \subset E$ be a nonempty, closed, convex subset. P is called a cone if.

- $P + P \subset P$.
- $\lambda P \subset P$ for all $\lambda \geq 0$.
- $P \cap (-P) = \{0\}$.

Definition 2.2 (Cone Metric Space): A function $d: X \times X \to E$ is a cone metric if for all $x, y, z \in X$:

- $d(x, y) \in P$, and $d(x, y) = 0 \Leftrightarrow x = y$.
- $d(x, y) = d(y, x)$.
- $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 2.3 (Graphical Cone Metric Space): Let (X, d) be a cone metric space and $G = (V, E)$ a directed graph with $V = X$. The space (X, d, G) is called a graphical cone metric space if the graph captures relationships among points such that fixed-point analysis is graph-sensitive.

3. FIXED-POINT THEOREMS IN GCMS:

Proposition 3.1 (Banach-type Contraction): Let (X, d, G) be a complete GCMS, and suppose $T: X \to X$ is a mapping such that for all $x, y \in X$ with $(x, y) \in E(G), d(Tx, Ty) \leq kd(x, y)$ for some $0 < k < 1$. Then T has a unique fixed point.

Proof: Let $x_0 \subset X$ and define a sequence $x_{n+1} = T(x_n)$. Since $(x_n, x_{n+1}) \in G$ and by the contraction condition, $d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \leq kd(x_n, x_{n+1})$ which shows the sequence is Cauchy. Since X is complete, it converges to some point $x^* \in X$. Taking the limit and using continuity of T, we get $T(x^*) = x^*$. Uniqueness follows from the contraction condition.

Proposition 3.2 (Kannan-type Contraction): Let $T: X \to X$ satisfy $d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$ for $x, y \in X, (x, y) \in E(G), 0 < k < \frac{1}{2}$. Then T has a unique fixed point in X.

Proof: Let $x_0 \in X$ and define $x_{n+1} = T(x_n)$. Then $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq k[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] = k[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$ Solving this recursive inequality shows that $d(x_n, x_{n+1}) \to 0$, so $\{x_n\}$ is Cauchy. Completeness implies convergence to a fixed point x^*. Uniqueness is shown similarly.

- **Example 3.1:** Let $X = \mathbb{R}, E = \mathbb{R}, P = [0, \infty)$, and $d(x, y) = |x - y|$.

Graph G has edges between x, y with $|x - y| < 1$.
Define $T(x) = 0.5x$.
Then $d(Tx, Ty) = |0.5x - 0.5y| = 0.5|x - y| \leq kd(x, y)$ with $k = 0.5 < 1 \Rightarrow$ Banach-type condition satisfied.
The fixed point is $x = 0$.

- **Example 3.2:** Let $X = [0,1], T(x) = \frac{1}{2}(x + \sqrt{x})$. Let G be a graph over rational points with edges between close points. For small enough $x, y, d(Tx, Ty) = \left|\frac{1}{2}(x + \sqrt{x}) - \frac{1}{2}(y + \sqrt{y})\right| \leq \frac{1}{2}\left| x - y \right| + \frac{1}{2}\left|\sqrt{x} - \sqrt{y}\right| \leq k|x - y|$ for some $k < 1$.

Thus, Banach contraction holds and fixed point exists (numerically, near 0.618).

- **Example 3.3:** Let $X = \mathbb{R}^2, d\big((x_1, y_1), (x_2, y_2)\big) = (|x_1 - x_2| + |y_1 - y_2|)$ e, where $e \in \mathbb{R}^2, e = (1,1)$. Define $T(x, y) = (0.5x, 0.5y)$.

Then $d\big(T(x_1, y_1), T(x_2, y_2)\big) = 0.5d\big((x_1, y_1), (x_2, y_2)\big) \Rightarrow$ Banach contraction holds. Fixed point is (0,0).

1. GRAPHICAL CONE METRIC SPACE: THEORETICAL DEVELOPMENTS

Proposition 4.1: Let (X, d, G) be a GCMS and suppose $T: X \to X$ is such that for all $(x, y) \in E(G)$, $d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$ for some continuous, non-decreasing function $\phi: E \to P$ with $\phi(t) > 0$ for $t > 0$. Then T has a unique fixed point.

Proof: From the inequality and properties of ϕ, one can construct a decreasing sequence of distances $d(x_n, x_{n+1})$ with a lower bound of 0. This ensures $\{x_n\}$ is Cauchy and converges to some x^*. Using the continuity of T, it follows that $T(x^*) = x^*$.

Proposition 4.2: In a GCMS, if $T: X \to X$ is such that $d(Tx, Ty) \leq kd(x, y) + c$ where $0 < k < 1$ and $c \in P$, then T has at least one fixed point in X.

Proof: The inequality implies eventual convergence of the sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ because the distance is reduced at each step, allowing the series to converge due to boundedness and completeness.

Example 4.1: Let $X = [0,1], d(x, y) = |x - y|$, and $\phi(t) = \frac{t}{2}$. Define $T(x) = \frac{x}{2}$. Then $d(Tx, Ty) = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x - y| = d(x, y) - \phi(d(x, y)) \Rightarrow$ conditions of Prop. 4.1 are satisfied. Fixed point is $x = 0$.

Example 4.2: Let $X = \mathbb{R}, d(x, y) = |x - y|$, and $T(x) = 0.8x + 0.1$. Then $d(Tx, Ty) = | 0.8x + 0.1 - (0.8y + 0.1)| = 0.8|x - y | = kd(x, y)$ with $k = 0.8 < 1 \Rightarrow$ condition of Prop. 4.2 met. Fixed point is $x = \frac{0.1}{1-0.8} = 0.5$.

Example 4.3: Let $X = \mathbb{R}^+, d(x, y) = |\ln x - \ln y|$, and $T(x) = \sqrt{x}$. Then $d(Tx, Ty) = | \ln \sqrt{x} - \ln \sqrt{y} \,| = \frac{1}{2}\left| \ln x - \ln y \right| = \frac{1}{2} d(x, y) \Rightarrow$ contraction condition satisfied. Fixed point occurs when $x = \sqrt{x} \Rightarrow x = 1$.

2. CONCLUSION

In this research, we have explored the foundational and advanced aspects of fixed-point theorems within the framework of Graphical Cone Metric Spaces (GCMS). By integrating the algebraic structure of cones in Banach spaces with the relational features of directed graphs, GCMS provides a fertile ground for generalizing classical fixed-point results. We formulated and proved variants of Banach-type and Kannan-type contraction principles adapted to this enriched structure.

Furthermore, through well-constructed examples and propositions, we demonstrated how these theorems apply under varying contractive conditions. Our study highlights the flexibility and power of GCMS in addressing fixed-point problems that arise in more complex and structured spaces, making them particularly relevant for applications in optimization, dynamic programming, and mathematical modeling with networked or ordered relationships.

Future research may explore multivalued mappings, hybrid fixed-point models, or extend the setting to probabilistic or fuzzy cone metric spaces with underlying graph structures. This direction will not only broaden the theoretical scope but also enhance the applicability of GCMS in practical problem-solving domains.

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