International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

A Survey on Properties of Some Power Graphs

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Abstract
Algebraic graph theory explores the relationship between algebraic structures and graph theory. One significant area of study within this field is the investigation of power graphs derived from groups. In recent years, there has been substantial progress in understanding various aspects of power graphs, including their connectivity, spectral properties, isomorphism, automorphism, and characterization in terms of groups. A power graph is a type of graph derived from a mathematical structure, the power graph $G(G)$ of a group G is a simple graph where the vertices represent the elements of the group, and two distinct vertices are adjacent if one is a power of the other. In this paper, our main objective is to provide a comprehensive survey focusing on the properties of enhanced power graphs, reduced power graphs. These variations of power graphs offer refined insights into the underlying group structure and its relationship with graph-theoretic properties.

Keywords: enhanced Power graph, reduced power Graph, connectivity, spectra.

Introduction

The study of reduced power graphs and enhanced power graphs of groups has been a subject of interest for many researchers, leading to significant advancements in group theory and graph theory. Many results on these graphs can be found in the survey paper [15,16].

The concept of directed power graph (G) of a group G, introduced by Kelarev and Quinn [1], is a digraph with vertex set G and for any a, b ∈ G, there is a directed edge from a to b in P(G) if and only if $a^k = b$, where k ∈ N.

Following this, Chakrabarty et al [2] defined the undirected power graph P(G) of a group G as an undirected graph whose vertex set is G and two vertices u; v are adjacent if and only if u = v and $u^m = v$ or $v^m = u$ for some positive integer m. After this, undirected power graph became the main focus of study.

The directed reduced power graph of G**,** denoted by $\overrightarrow{\boldsymbol{P}(\mathrm{G})}$, is a digraph with vertex set G, and for u, v ∈ G, there is an arc from u to v if and only if $u \neq v$, $v = u^n$ for some positive integer n and $\langle v\rangle \neq \langle u\rangle$; which is equivalent to say $u \neq v$ and $\langle v\rangle \subset \langle u\rangle$. **The (undirected) reduced power graph of** G**,** denoted by **P**(G), is the underlying graph of $\overrightarrow{\boldsymbol{P}(\mathrm{G})}$. This means that the set of vertices of **P**(G) is equal to G and two vertices u and v are adjacent if and only if $u \neq v$, $u^n = v$ and $\langle v\rangle \neq \langle u\rangle$ or $v^n = u$ and $\langle v\rangle \neq \langle u\rangle$ for some positive integer n; which is equivalent to say $u \neq v$ and $\langle v\rangle \subset \langle u\rangle$ or $\langle u\rangle \subset \langle v\rangle$. **The term enhanced power graph of a group was introduced by Aalipour et al. [4]** as a graph that includes properties from both the power graph and the commuting graph, where the commuting graph of a group G, denoted by C(G), is the graph whose vertex set is G, and two distinct elements x, y are adjacent if $xy = yx$. The **enhanced power graph** of a group G is denoted by $\mathbf{P}_e(G)$ and is defined as a simple graph with vertex set consisting of all elements of G, where two distinct vertices

x, y are adjacent if and only if $\langle x, y\rangle$ is a cyclic subgroup of G. Example, the power graph and the enhanced power graph of abelian group $Z_2 \times Z_6$ in Fig. 1. **[14]**

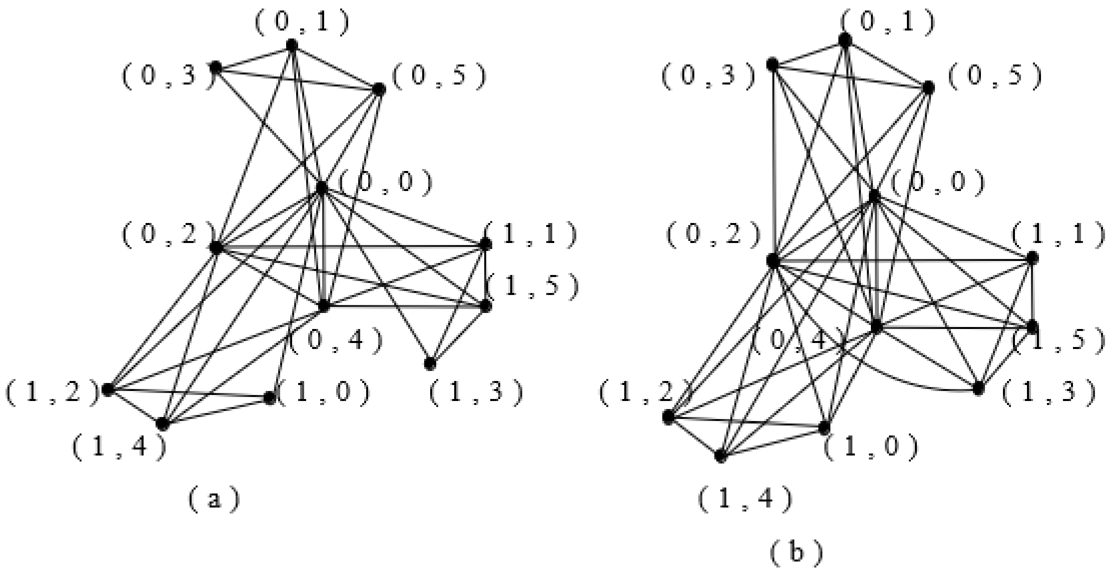

Figure 1. (a) P $(Z_2 \times Z_6)$; (b) $P_e(Z_2 \times Z_6)$.

The enhanced power graph is called **dominatable** if it has a dominating vertex other than identity. The study of the spectral properties of the power graphs has been made for some classes of groups. In [5,6,7,9,10,11] the like *L*-spectra of power graph of finite cyclic groups and dihedral groups, some finite abelian p- groups, A -spectra of power graphs of cyclic groups, dihedral groups whose order is twice a prime power, quaternion groups and also obtained lower and upper bounds for spectral radii of these graphs.

1. Spectrum and connectivity of Reduced Power Graph:

Throughout this paper we consider G as a finite group. For an undirected graph G with vertex set V(G) = $\{v_1, v_2, \ldots, v_n\}$,

Adjacency matrix A(G) of G is the n × n matrix with (i, j) th entry is 1, if v_i and v_j are adjacent, and 0 otherwise. **Degree of the Vertex** D(G)= diag $(d_1, d_2, \ldots, d_n)$ where d_i is the degree of the ver-tex v_i of G, i 1, 2, …, n. **Laplacian matrix** L(G) of G is the matrix D(G)_ A(G). The eigenvalues of *A(G)* and *L(G)* are said to be the **A-spectrum and L-spectrum of G**, respectively. The **vertex connectivity** of a graph G, denoted by κ P(G), is the minimum number of vertices whose removal results in a disconnected or trivial graph.

Theorem 1.1 (Theorem 2.2 [12]): (1) If p is a prime and $m \geq 1$ is an integer, then the L-spectrum of RP $(Z_p m)$ is 0, p^m, $p^m - \phi(p^m)$, $p^m - \phi(p^{m-1})$, $p^m - \phi(p^{m-2})$, …, $p^m - \phi(p)$ with multiplicities 1, m, $\phi(p^m) - 1$, $\phi(p^{m-1}) - 1$, $\phi(p^{m-2}) - 1$, …, $\phi(p) - 1$, respectively.

(2) Let n = pq, where p, q be two distinct primes. Then the L-spectrum of $RP(Z_n)$ is 0, n,$n - \phi(n)$, $\phi(n) + 1$ with multiplicities 1, 2, $\phi(n) - 1$ and $n - \phi(n) - 2$, respectively.

Lemma 1.1 (Lemma 2.1 [12]): If $n \geq 2$ is an integer which is not a prime power, then $\overline{RP(\mathbb{Z}n)}$ is the disjoint union of connected components K_1, $K_{\phi(n)}$ and $\overline{RP(\mathbb{Z}n)} - Sn$.

Lemma 1.2 (Lemma 2.2 [12]): $RP(\mathbb{Z}_n) - S_n$ is connected if and only if $n \neq p^2$ and pq, where p, q are distinct primes.

Theorem 1.2 (Theorem 2.4 [12]): If $n \geq 1$ is an integer and p isa prime, then the L-spectrum of $RP(\mathbb{Z}^n_{p^2})$ is 0, $1, p^{2n}$, $p^{n+1} - p^n + p$, p and $p^{n+1} - p^n + 1$ with multiplicities 1, $t - 1$, 1, t, $t(p^{n+1} - p^n - 1)$ and $t(p - 2)$, respectively, where $t = (p^n - 1)(p - 1)$.

Corollary 1.1 (Corollary 2.2 [12]): If p is a prime, then the L-spectrum of $RP(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ is 0, 1, p^3, $p^3 - p^2 +$
p, p and $p^3 - p^2 + 1$ with multiplicities 1, $p(p - 1)$, 1, 1, $p^3 - p^2 - 1$ and p–2, respectively.

Corollary 1.2 (Corollary 2.3[12]: Let G be a group of order n such that it is isomorphic to a p-group with exponent p or non-nilpotent group of order $p^m q$ with all non-trivial elements are of order p or q,where p,q are distinct primes. Then the L-spectrum of RP(G) is 0,1 and n with multiplicities 1, n−2 and 1, respectively.

Theorem 1.3 (Theorem 3.1[12]): For any integer $n \geq 3$, the L-spectrum of $RP(D_{2n})$ is given by

$$\lambda_i(RP(D_{2n})) = \begin{cases} 2n & for\ i = 1; \\ n & for\ i = 2; \\ \lambda_i(RP(\mathbb{Z}_n)) & for\ 3 \leq i \leq n-1; \\ 1 & for\ n \leq i \leq 2n-1; \\ 0 & for\ i = 2n. \end{cases}$$

In particular, we have the following:

(i) If $n = p^m$, where p is a prime and $m \geq 1$, n /= 2, then the L-spectrum of $RP(D_{2n})$ is0, 2n, n, $n - \phi(p^m)$, $n - \phi(p^{m-1})$, …, $n - \phi(p)$ and 1 with multiplicities 1, 1, $m - 1$, $\phi(n) - 1$, $\phi(p^{m-1}) - 1$, …, $\phi(p) - 1$ and n, respectively.

(ii) If n = pq, where p and q are distinct primes, then the L-spectrum of $RP(D_{2n})$ is 2n, n, $n - \phi(n)$, $\phi(n)+1$, 1 and 0 with multiplicities 1, 1, $\phi(n) - 1$, $n - \phi(n) - 2$, n and 1,respectively.

Corollary 1.3 (Corollary 3.1[12]): (1) The L-spectrum of $RP(Q_{2^\alpha})$, where $\alpha \geq 4$ is 0, 2, $2^{\alpha-1} - \phi(2^2)$ $2^{\alpha-1} - \phi(2^3)$, … , $2^{\alpha-1} - \phi(2^{\alpha-1})$, $2^{\alpha-1}$, and 2^α with multiplicities 1, $2^{\alpha-1}$, $\phi(2^2) - 1$, $\phi(2^3) - 1$, … , $\phi(2^{\alpha-1}) - 1$, $\alpha - 3$ and 2, respectively.

(2) The L-spectrum of $RP(Q_8)$ is 0, 2 and 8 with multiplicities 1, 5 and 2, respectively.

Corollary 1.4 (Corollary 3.2 [12]): The L-spectrum of $RP(SD_{2^\alpha})$, where $\alpha \geq 4$ is 0, 1, 2, $2^{\alpha-1} - \phi(2^2)$, $2^{\alpha-1} - \phi(2^3)$, … , $2^{\alpha-1} - \phi(2^{\alpha-1})$, $2^{\alpha-1}$,$3 \cdot 2^{\alpha-2}$, and 2^α with multiplicities 1, $2^{\alpha-2}$, $2^{\alpha-2}$, $\phi(2^2) - 1$, $\phi(2^3) - 1$, … , $\phi(2^{\alpha-1}) - 1$, $\alpha - 3$, 1 and 1, respectively.

In these Results, author proves the L- Spectrum of reduced power graphs like dihedral groups, Quaternion groups, Semi-dihedral groups.

Conjectures 1 Akbari and Ashrafi, **[11]**

The reduced power graph of a non-abelian simple group G is connected only if G is isomorphic to some alternating group A_n.

Theorem 1.4 [3]: PGL_n (F2) has a disconnected reduced power graph if and only if at least one of the following three situations occurs.

1. n or n−1 is prime.
2. n has a prime factor p_1 such that $p_0 = 2^{p_1} - 1$ is also prime and $n \leq 2^{p_1} - 2$.

3. n−1 has a prime factor p_1 such that $p_0 = 2^{p_1} - 1$ is also prime and $n \le 2^{p_1} - 1$.

By this theorem, author disprove the above given conjecture 1 for connectedness of reduced power graph.

Proposition 1 (Proposition 5.1. [15]) Let n be a positive integer and let $\phi(n)$ denotes its Eulers totient function. Then (1) $\kappa(\mathrm{RP}(Z_n)) = n - \phi(n)$, if $2\phi(n)+1 \ge n$;

(2) $\kappa(\mathrm{RP}(Z_n)) \ge \phi(n) + 1$, if $2\phi(n)+1 < n$. The equality holds for n = 2p, where p is a prime.

Corollary 1.5 (Corollary 5.1. [15]) Let $n \ge 2$ be an integer. Then we have the following:

(1) $\kappa(\mathrm{RP}^*(Z_n)) = n - \phi(n) - 1$ *if* $2\phi(n) + 1 \ge n$;

(2) $\kappa(\mathrm{RP}^*(Z_n)) \ge \phi(n)$ *if* $2\phi(n)+1 < n$. The equality holds for n = 2p, where p is a prime.

Corollary 1.6 (Corollary 5.2.[15]) $\kappa(\mathrm{RP}\ (\mathbb{Z}_{p^m})) = p^{m-1}$ and $\kappa(\mathrm{RP}^*(\mathbb{Z}_{p^m})) = p^{m-1}$-1, where p is a prime and m is a positive integer.

In these results, author proves the vertex connectivity for cyclic groups.

Proposition 2(Proposition 5.4. [15]) For an integer $n \ge 3$,

(1) $\kappa(\mathrm{RP}(D_{2n})) = 1$ *and* $\kappa(\mathrm{RP}^*(D_{2n})) = \kappa(\mathrm{RP}^*(Z_n))$;

Proposition 3 (Proposition 5.5.[15]) For an integer $n \ge 2$,

(1) $\kappa(\mathrm{RP}(Q_{4n})) = 2$ *and* $\kappa(\mathrm{RP}^*(Q_{4n})) = 1$;

Proposition 4 (Proposition 5.6.[15]) For an integer $n \ge 3$,

(1) $\kappa(\mathrm{RP}(SD_{8n})) = 1 = \kappa(\mathrm{RP}^*(SD_{8n}))$;

Corollary 1.7 (Corollary 5.4.[15]) Let G be a finite p-group, where p is a prime. The

$$\kappa\big(\mathrm{RP}(G)\big) = \begin{cases} p^m - 1 & \text{if } G \cong \mathbb{Z}_{p^m}, m \ge 1; \\ 2 & \text{if } G \cong Q_{2^\alpha}, \alpha \ge 3; \\ 1 & otherwise. \end{cases}$$

In these results, author proves the vertex connectivity of reduced power graphs like dihedral groups, Quaternion groups, Semi-dihedral groups.

2. Connectivity of Enhanced Power Graph

Theorem 2.1 (Theorem 1.1 [14]) Let G be a finite p-group such that G is neither cyclic nor generalized quaternion group. Then $\kappa\big(\mathcal{G}_e(G)\big) = 1$.

Theorem 2.2 (Theorem 1.2 [14]) Let G be a finite non-cyclic abelian group. Then is equal to $\kappa\big(\mathcal{G}_e(G)\big)$ =1 if and only if G is a p-group.

Theorem 2.3 (Theorem 1.5 [14]) Let G be a non-cyclic abelian non p-group such that $G \cong G_1 \times Z_n$; g.c.d ($|G_1|$, n) =1 and G_1 has no cyclic sylow subgroup. Then $\mathcal{G}_e^{**}(G)$ is disconnected if and only if G_1 is a p-group

Theorem 2.4 (Theorem 1.7 [14]) Let G be a non-cyclic abelian non-p-group such that $G \cong G_1 \times Z_n$; g.c.d ($|G_1|$, n) = 1 and G_1 is a p-group with no cyclic sylow subgroup. Then $\kappa\big(\mathcal{G}_e(G)\big) = n$.

Lemma 1. (Lemma 2.5 [14]) Let G be a finite group and x, y ϵ G \ {e} be such that gcd (o(x), o(y)) = 1 and xy = yx. Then, x ~ y in $\mathcal{G}_e^*(G)$.

Lemma 2. (Lemma 2.7 [14]) Let G be any non-cyclic group. For any dominating vertex v ($\neq$ e) of G there exists a prime p dividing o(v) such that G has a unique subgroup of order p.

Theorem 2.5 (Theorem 2.8 [14]) For any group G, the graph $P^*(G)$ is connected if and only if the graph $\mathcal{G}_e^*(G)$ is connected.

Corollary 1 (Corollary 1 [14]) Let G be a finite non-cyclic abelian group. Then $\kappa\big(\mathrm{P}(G)\big)$ is equal to1 if and only if G is a p-group. $\mathbb{Z}_{{p_1}^{t_{11}}}$

Corollary2 (Corollary 2 [14]): Let G be a non-cyclic abelian group such that
G $\cong \mathbb{Z}_{{p_1}^{t_{11}}} \times \mathbb{Z}_{{p_1}^{t_{12}}} \times \mathbb{Z}_{{p_1}^{t_{13}}} \times \ldots . \times \mathbb{Z}_{{p_1}^{t_{1k_1}}} \times \mathbb{Z}_{{p_2}^{t_{21}}} \times \mathbb{Z}_{{p_2}^{t_{22}}} \times \ldots . \times \mathbb{Z}_{{p_2}^{t_{2k_2}}} \times \ldots \times \mathbb{Z}_{{p_r}^{t_{r1}}} \times \mathbb{Z}_{{p_r}^{t_{r2}}} \ldots \times$ $\mathbb{Z}_{{p_r}^{t_{rk_r}}}$,*where* $k_i \geq 1$ *and* $1 \leq t_{i1} \leq t_{i2} \leq \cdots \leq t_{ik_i}$, *for all* $i \in [r]$.
Then, $\kappa(P(G)) \leq {p_1}^{t_{11}}{p_2}^{t_{21}} \ldots . . {p_r}^{t_{r1}} - \phi({p_1}^{t_{11}}{p_2}^{t_{21}} \ldots . {p_r}^{t_{r1}})$.

By these Above theorems, author shows the vertex connectivity of non-cyclic abelian groups.

Theorem 2.6 (Theorem 4.1 [14]): Let G be the dihedral group of order 2n. Then to $\kappa\big(\mathcal{G}_e(G)\big)$ is equal to 1.
Theorem 2.7(Theorem 4.2 [14]): For n ≥ 3; let Q_2n be the generalized quaternion group. Then the vertex connectivity of to $\big(\mathcal{G}_e(Q_{2^n})\big)$ is 2.
Corollary 3 (Corollary 3 [14]): Let Q_{2^n}be the generalized quaternion group. Then the enhanced power graph ${\mathcal{G}_e}^*(Q_{2^n})$ is connected but the proper enhanced power graph Then ${\mathcal{G}_e}^{**}(Q_{2^n})$ is disconnected.

Theorem 2.8(Theorem 4.3 [14]): Let G be a symmetric group with n ≥ 3: Then
(1) If n ≥ 3 and neither n nor n — 1 is a prime, then $P^*(G)$ is connected.
(2) If n is such that either n or n — 1 is a prime, then $P^*(G)$ is disconnected.

Theorem 2.9(Theorem 4.6 [14]): Let G = A_n be the alternating group and n ≥ 4. Then
(1) If n, n — 1, n — 2, n=2, (n — 1)/2, (n — 2)/2 are not primes, then $P^*(G)$ is connected.
(2) If n is such that any one of n, n — 1, n — 2, n/2, (n — 1)/2, (n — 2)/2 is prime,then $P^*(G)$ is not connected.
In these results, author find the vertex connectivity of some non-abelian groups (dihedral groups, quaternion groups, alternating groups).

Conclusion:
Algebraic graph theory emerged at the crossroads of algebra and graph theory, leveraging abstract algebraic concepts to study graphs while also applying graph theory to understand properties of algebraic structures. In recent years, Power graph is one such major graph representation of semigroups, groups, many results on the power graphs have been obtained Our review deals with recent advancements, focusing on some properties of reduced and enhanced power graphs of groups. We want to conclude by listing a few fundamental open problems. we also redirect the interested reader to the survey article [8,17] for open questions that are still unsolved. While our list is not exhaustive and reflects our own interests and experiences, we encourage readers to develop deeper into the subject and explore the rich landscape of unsolved questions.
Problem 1. [8] *Is there a simple algorithm for constructing the directed power graph or the enhanced power graph from the power graph, or the directed power graph from the enhanced power graph?*
Problem 2. [8] *What can be said about groups G for which* DC(*G*) = P_e(*G*)*?*
Problem 3. (Problem 6 [8]) (a) *What is the smallest group for which a given graph is embeddable in the enhanced power graph of the group?*
(b) *What is the smallest group in which every graph on n vertices can be embedded the enhanced power graph of the group?*

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