International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 • Website: www.ijfmr.com • Email: editor@ijfmr.com

Duality Theory in Minimax Multiobjective Fractional Programming Under Generalized (F, Α, Ρ, D)-Convexity

Dr. G. Varalakshmi

Lecturer in Statistics, PRR & VS Government College, Vidavalur

Abstract
This study investigates the duality structure within minimax multiobjective fractional programming models defined under the generalized (F, α, ρ, d)-convexity framework. Although significant progress exists for single-objective and non differentiable minimax problems, the extension to multiobjective cases under such generalized convexity has seen limited attention.
Building on the foundational work of Liang et al. and others, this study develops sufficient optimality conditions and formulates appropriate dual models**.** It establishes weak**,** strong**,** and strict converse duality theorems, thereby extending the theoretical landscape of fractional programming. The results contribute to a more comprehensive understanding of multi objective optimization under relaxed convexity assumptions and open avenues for future research in robust decision-making and multi-criteria analysis.

Keywords: (F, α, ρ, d)-convexity, Weak Duality and Strong Duality

Introduction
Minimax fractional programming has proven essential in modeling real-world systems, including discrete/continuous rational approximation, rational games, and multi-criteria decision-making in engineering and finance. (especially under the Chebyshev norm [8), rational games [14**],** multiobjective optimization [15], engineering design**,** and portfolio selection**,** as highlighted by Bajona-Xandri and Martínez-Legaz [4].
Over the past few decades, minimax programming has attracted considerable scholarly attention [1, 6, 7, 12, 1318,19,20]. The foundational contributions of Schmitendorf [12] provided both necessary and sufficient conditions for optimality in early minimax programming models. Tanimoto [16] extended these results to formulate dual problems and derive duality theorems, which were further generalized to fractional settings by Yadav and Mukherjee [20].
Inspired by the evolution of generalized convexity, Liang et al. [9, 10] introduced a unified framework known as (F,α,ρ,d)-convexity, enabling the derivation of optimality and duality results for both single-objective and multiobjective fractional problems. Subsequent contributions by Liang and Shi [11], Lai and Lee [8], expanded the theory to encompass nondifferentiable and univex functions.
Despite these advancements, the application of multi objective fractional minimax models under generalized convexity remains largely unexplored. This paper aims to address this gap by developing

new duality theorems based on (F,α,ρ,d))-convex functions, thereby enriching the theoretical foundation of multi objective fractional optimization.

2. Formulation:

We now consider the following non-differentiable minimax fractional programming problem.

2.1. Primal Problem:

$$(FP) \quad \min_{x \in R^n} \sup_{y \in Y} \frac{f_i(x,y) + (x^t, Bx)^{\frac{1}{2}}}{f_i(x,y) - (x^t, Bx)^{\frac{1}{2}}}$$

Subject to $g_j(x) \le 0$

Where Y is a compact subset of $R^m, f_i, h_i : R^n \times R^m \to R$ are C^1 on $R^n \times R^m$ and $g_j : R^n \to R^p$ is C^1 on R^n. B and D are n x n positive semi definite matrices.

Let $S = \{x \in X : g(x) \le 0\}$ denote the set of all feasible solutions of (FP). For each $(x, y) \in R^n \times R^m$, we define

$$\phi(x,y) = \frac{f_i(x,y) + (x^t, Bx)^{\frac{1}{2}}}{h_i(x,y) - (x^t, Dx)^{\frac{1}{2}}},$$

Such that for each $(x, y) \in s \times Y, f_i(x,y) + (x^t, Bx)^{\frac{1}{2}} \ge 0$ and $h_i(x,y) - (x^t, Dx)^{\frac{1}{2}} > 0$. For each $x \in s$, we define

$$J(x) = \{ j \in J \;\; g_j(x) = 0 \}. \text{ where } J = \{1, 2, ... p\}$$

$$Y(x) = \left\{ y \in Y : \frac{f_i(x,y) + (x^t, Bx)^{\frac{1}{2}}}{h_i(x,y) - (x^t, Dx)^{\frac{1}{2}}} = \sup_{z \in Y} \frac{f_i(x,z) + (x^t, Bx)^{\frac{1}{2}}}{h_i(x,z) - (x^t, Dx)^{\frac{1}{2}}} \right\},$$

$$k(x) = (s, t_i, \bar{y}) \in N \times R_+^s \times R^{ms} : 1 \le s \le n+1, t = (t_1, t_2 ..... t_s) \in R_+^s$$

$$\text{with } \sum_{i=1}^{s} t_i = 1, \; \bar{y} = (\bar{y}_1, \bar{y}_2 ..... \bar{y}_s) \text{ with } \bar{y}_i \in Y(x) \; (i = 1, 2, ... s)$$

Since f and h are continuously differentiable and Y is compact in R^m, it follows that for each $x^* \in s, y(x^*) \ne \phi$, and for any $\bar{y}_i \in Y(x^*)$, we have a positive constant

$$k_0 = \phi(x^*, \bar{y}_i) = \frac{f_i(x^*, \bar{y}_i) + (x^{*t}, Bx^*)^{\frac{1}{2}}}{h_i(x^*, \bar{y}_i) - (x^{*t}, Dx^*)^{\frac{1}{2}}}$$

We shall need the following generalized Schwartz inequality.

Let B be a positive semi definite matrix of order n. Then for all $x, w \in R^n$,

$$x^t Bw \le (x^t Bx)^{\frac{1}{2}} (w^t Bw)^{\frac{1}{2}} \qquad (1.1.1)$$

We observe that equality holds if $Bx = \lambda Bw$ for some $\lambda \ge 0$. Evidently, if $(w^t Bw)^{\frac{1}{2}} \le 1$, we have $x^t Bw \le \left(x^t Bw\right)^{\frac{1}{2}}$.

If the functions f_i, g_i and h_i in problem (FP) are continuously differentiable with respect to $x \in R^n$.

2.2. Dual Formulation:

(D) $$\max_{(s,t^*,\bar{y})\in K(z)} \ \sup_{(z,\mu,v,w)\in H(s,t_i^*,\bar{y})} \frac{\sum_{i=1}^{s} t_i^* \left\{ f_i(z,\bar{y}_i) + z^t Bw \right\} + \sum_{j=1}^{p} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i^* \left\{ h_i(z,\bar{y}_i) - z^t DV \right\}}$$

Where $H(s, t_i^*, \bar{y})$ denote the set of all $(z, \mu, v, w) \in R^n \times R_+^p \times R^n \times R^n$

Satisfying

$$\nabla \left(\frac{\sum_{i=1}^{s} t_i^* \left\{ f_i(z,\bar{y}_i) + z^t Bw \right\} + \sum_{j=1}^{p} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i^* \left\{ h_i(z,\bar{y}_i) - z^t DV \right\}} \right) = 0$$

$$\begin{cases} w^t Bw \le 1, & (z^t Bz)^{\frac{1}{2}} = z^t Bw, \\ v^t Dv \le 1, & (z^t Dz)^{\frac{1}{2}} = z^t Dv \end{cases}$$

If the set $H(s, t^*, \bar{y})$ is empty, we define supremum over it to be $-\infty$. For convenience, we use the notation:

$$\psi(\square) = \left[\sum_{i=1}^{s} t_i^* \left\{ h_i(z,\bar{y}_i) - z^t Dv \right\} \right] \left[\sum_{i=1}^{s} t_i^* \left\{ f_i(\cdot,\bar{Y}_i) + (\cdot)^t, Bw \right\} + \sum_{j=1}^{p} \mu_j g_j(\cdot) \right]$$

$$- \left[\sum_{i=1}^{s} t_i^* \left\{ f_i(z,\bar{y}_i) + z^t, Bw) \right\} + \sum_{j=1}^{p} \mu_j g_j(z) \right] \left[\sum_{i=1}^{s} t_i^* \left\{ h_i(\cdot,\bar{Y}_i) - (\cdot)^t Dv \right\} \right]$$

Suppose that

$$\sum_{i=1}^{s} t_i^* \left\{ f_i(z,\bar{y}_i) + z^t Bw \right\} + \sum_{j=1}^{p} \mu_j g_j(z) \ge 0 \text{ and}$$

$$\sum_{i=1}^{s} t_i^* \left\{ h_i(z,\bar{y}_i) - z^t Dv \right\} > 0,$$

for all $(s, t^*, \bar{y}) \in k(z)$, $(z, \mu, w, v) \in H(s, t^*, \bar{y})$.

3. Definition:

Let R^n be the n-dimensional Euclidean space and X and open set in R^n.

Definition (3.1): A functional $F: X \times X \times R^n \to R$ is said to be sublinear (Consider citing a standard source like Avriel (1979) explicitly for the definition of sublinear functional.) if $\forall x, \bar{x} \in X$

(i) $F(x, \bar{x}; a_1 + a_2) \le F(x, \bar{x}; a_1) + F(x, \bar{x}; a_2)\ \forall\, a_1, a_2 \in R^n$

(ii) $F(x, \bar{x}; \beta a) = \beta F(x, \bar{x}; a)\ \forall \beta \in R_+$ and $\forall a \in R^n$.

By (ii) it is clear that $F(x, \bar{x}, 0) = 0$.

Definition (3.2):

[8, 9] given an open set $x \subset R^n$, a number $\rho \in R$, and two functions $\alpha: X \times X \times \to R_+ \setminus \{0\}$ and $d\left(\begin{matrix} . & . \\ & , \end{matrix}\right) X \times X \to R$, a differentiable function ζ over x is said to be (F, α, ρ, d) - convex at $\bar{x}$, if for any $x \in X$, $F: X \times X \times \to R^n \to R$ is sub linear, and $\zeta(x)$ satisfies the following condition.

$$\zeta(x) - \zeta(\bar{x}) \ge F\left(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})\right) + \rho d^2(x, \bar{x})$$

Definition (3.3):

Given an open set $x \subset R^n$, a number $\rho \in R$, and two functions $\alpha: X \times X \times \to R_+ \setminus \{0\}$ and $d\left(\begin{matrix} . & . \\ & , \end{matrix}\right) X \times X \to R$, a differentiable function ζ over x is said to be (F, α, ρ, d)-Pseudo convex at $\bar{x}$, if for any $x \in X$, there exists a sub linear functional $F: X \times X \times R^n \to R$ such that

$$\mu(x) < \mu(\bar{x}) \Rightarrow F\left(x, \bar{x}; \alpha(x, \bar{x}) \nabla G(\bar{x})\right) - \rho^2 d^2(x, \bar{x})$$

Further μ is said to be strictly $F(\alpha, \rho, d)$-Pseudo convexity at $\bar{x}$ if for any $x \in X$, there exists a sub linear functional $FiX \times X \times R^n \to R$

Such that

$$F\left(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})\right) \ge -\rho d^2(x, \bar{x}) \Rightarrow \zeta(x) > \zeta(\bar{x}).$$

4. Necessary optimality condition:

If x^* is a solution of problem (FP). Assuming $z_{\bar{y}}(x^*)$ to be empty, there exist $(s, t^*, \bar{y}) \in k(x^*)$, $w, v \in R^n$ and $\mu^* \in R_+^p$ satisfying

$$\nabla \left(\frac{\sum_{i=1}^{s} t_i^* \left\{ f(x^*, \bar{y}_i) + x^{*t} Bw \right\} + \sum_{j=1}^{p} \mu_j^* g_j(x^*)}{\sum_{i=1}^{s} t_i^* \left\{ h(x^*, \bar{y}_i) - x^{*t} DV \right\}} \right) = 0$$

$$\sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0$$

$$t_i^* \in R_+^s\ (i = 1, 2, \ldots.. s),\ \sum_{i=1}^{s} t_i^* = 1$$

$$w^t Bw \le 1, \quad (x^{*t} Bx^*)^{\frac{1}{2}} = x^{*t} Bw,$$

$$v^t Dv \le 1, \quad (x^{*t} Dx^*)^{\frac{1}{2}} = x^{*t} Dv$$

5. Duality Theorems:

5.1. Theorem (Weak Duality): Let x be a feasible point of the primal problem and y satisfy the feasibility conditions of the dual. Assuming ψ(.) satisfies the (F,α,ρ,d)-pseudo convexity condition at the reference point z, and that inequality constraints hold, we assert that the dual objective value does not exceed the primal.. Also assume that Ψ is $[F, \alpha, \rho, d]$-Pseudo Convex at z and the inequality $\frac{\rho}{\alpha(x,z)} \ge 0$ holds, then

$$\sup_{y \in Y} \frac{f_i(x, y) + (x^t, Bx)^{\frac{1}{2}}}{h_i(x, y_i) - (x^*, Dx)^{\frac{1}{2}}} \ge \frac{\sum_{i=1}^{s} t_i^* \left\{ f_i(z, \bar{y}_i) + z^t Bw \right\} + \sum_{j=1}^{p} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i^* \left\{ h_i(z, \bar{y}_i) - z^t DV \right\}}$$

Proof: Assume contrariwise, suppose that

$$\sup_{y \in Y} \frac{f_i(x, y) + (x^t Bx)^{\frac{1}{2}}}{h_i(x, y) - (x^t Dx)^{\frac{1}{2}}} < \frac{\sum_{i=1}^{s} t_i \left\{ f_i(z, \bar{y}_i) + z^t Bw \right\} + \sum_{j=1}^{p} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i \left\{ h_i(z, \bar{y}_i) - z^t Dv \right\}}$$

for $y \in Y$. If we replace y by $\bar{y}_i$ in the above in equality and sum up after multiplying by t_i, then we have

$$\left[\sum_{i=1}^{s} t_i \left\{ f_i(x, \bar{y}_i) + (x^t Bx)^{\frac{1}{2}} \right\} \right] \left[\sum_{i=1}^{s} t_i \left\{ h_i(z, \bar{y}_i) - z^t Dv \right\} \right]$$

$$< \left[\sum_{i=1}^{s} t_i \left\{ f_i(z, \bar{y}_i) + z^t Bw \right\} + \sum_{j=1}^{p} \mu_j g_i(z) \right] \left[\sum_{i=1}^{s} t_i \left\{ h_i(x, \bar{y}_i) - (x^t Dx)^{\frac{1}{2}} \right\} \right]$$

Using the generalized Schwartz inequality and (2.2) we get

$$\phi(x) \le \left[\sum_{i=1}^{s} t_i \left\{ h_i(z, \bar{y}_i) - z^t Dv \right\} \right] \left[\sum_{i=1}^{s} t_i \left\{ f_i(x, \bar{y}_i) + (x^t Bx)^{\frac{1}{2}} + \sum_{j=1}^{p} \mu_j g_j(x) \right\} \right]$$

$$- \left[\sum_{i=1}^{s} t_i \left\{ f_i(z, \bar{y}_i) + z^t Bw \right\} + \sum_{j=1}^{p} \mu_j g_j(x) \right] \left[\sum_{i=1}^{s} t_i \left\{ h_i(x, \bar{y}_i) - (x^t Dx)^{\frac{1}{2}} \right\} \right]$$

$$< \sum_{j=1}^{p} \mu_j g_j(x) \sum_{i=1}^{s} t_i \left\{ h_i(z, \bar{y}_i) - z^t Dv \right\}$$

Since $\sum_{i=1}^{s} t_i \left\{h_i(z, \bar{y}_i) - z^t Dv\right\} > 0 \ \sum_{j=1}^{p} \mu_j g_j(x) \le 0,$

it follows that $\psi(x) < 0 = \psi(z)$

As $\psi(.)$ is (F, α, ρ, d)-Pseudo Convex at z. Therefore $F\left(x, z; \alpha(x, z)\nabla\psi(z)\right) < -\rho d^2(x, z),$ that is

$$F(x, z; \alpha(x, z)\left[\sum_{i=1}^{s} t_i \left\{h_i(z, \bar{y}_i) - z^t Dv\right\}\right] \nabla \left[\sum_{i=1}^{s} t_i \left\{f_i(z, \bar{y}_i) + z^t Bw + \sum_{j=1}^{p} \mu_j g_j(z)\right\}\right]$$

$$-\left[\sum_{i=1}^{s} t_i \left\{f_i(z, \bar{y}_i) \times z^t Bw\right\} + \sum_{j=1}^{p} \mu_j g_j(z)\right] \nabla \left[\sum_{i=1}^{s} t_i \left\{h_i(z, \bar{y}_i) - z^t Dv\right\}\right] < -\rho d^2(x, z)$$

on multiplying the above inequality by

$\dfrac{1}{\alpha(x, z)\left[\sum_{i=1}^{s} t_i \left\{h_i(z, \bar{y}_i) - z^t Dv\right\}\right]^2}$ and using the sub linearity of F, we have

$$F(x, z; \nabla \left(\frac{\sum_{i=1}^{s} t_i \left\{f_i(z, \bar{y}_i) + z^t Bw\right\} + \sum_{j=1}^{p} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i \left\{h_i(z, \bar{y}_i) - z^t Dv\right\}}\right) < \frac{-\rho d^2(x, z)}{\alpha(x, z)\left[\sum_{i=1}^{s} t_i \left\{h_i(z, \bar{y}_i) - z^t Dv\right\}\right]^2}$$

Using the fact that $\dfrac{\rho}{\alpha(x, z)} \ge 0,$

We have

$$F\left(x, z; \nabla \left(\frac{\sum_{i=1}^{s} t_i \left\{f_i(z, \bar{y}_i) + z^t Bw\right\} + \sum_{j=1}^{p} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i \left\{h_i(z, \bar{y}_i) - z^t Dv\right\}}\right)\right) < 0 \qquad (1.1)$$

In the right of (2.1) in equality (1.1) contradicts F(x, z; 0)=0.

Theorem (2) (Strong Duality):

Suppose that $\bar{x}$ is optimal for (FP) and $\nabla g_j(\bar{x}), j \in J(\bar{x})$ is linearly independent. Then there exist $(\bar{s}, \bar{t}, \bar{y}^*) \in k(\bar{x})$ and $(\bar{x}, \bar{\mu}, \bar{v}, \bar{w}) \in H(\bar{s}, \bar{t}, \bar{y}^*)$ such that $(\bar{x}, \bar{\mu}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ is feasible for (D). Further, if the weak duality (theorem 1) holds for all feasible $(z, \mu, v, w, s, t, \bar{y})$ of (D), then $(\bar{x}, \bar{\mu}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ is optimal for (D) and the two objectives have the same extreme values.

Proof: Since $\bar{x}$ is optimal for (FP) and $\nabla g_j(\bar{x}), j \in J(\bar{x})$ is linearly independent. Then by necessary condition there exist $(\bar{s}, \bar{t}, \bar{y}^*) \in k(\bar{x})$ and $(\bar{x}, \bar{\mu}, \bar{v}, \bar{w}) \in H(\bar{s}, \bar{t}, \bar{y}^*)$ such that $(\bar{x}, \bar{\mu}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ is feasible for (D) and the two objective values are equal. The optimality of $(\bar{x}, \bar{\mu}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ for (FP) thus follows from weak duality (Theorem 1).

Theorem (3) (Strict Converse Duality):

Let $\bar{x}$ and $(\bar{z}, \bar{\mu}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ be optimal solutions for (FP) and (D) respectively. Also suppose that ψ (.) is strictly $[F, \alpha, \rho, d]$-Pseudo Convex, for all $(\bar{s}, \bar{t}, \bar{y}^*) \in k(\bar{x})$, $(\bar{z}, \bar{\mu}, \bar{w}, \bar{v}) \in H(\bar{s}, \bar{t}, \bar{y}^*)$, and the inequality $\frac{\rho}{\alpha(x, z)} \geq 0$ holds, and $\nabla g_j(\bar{x})$, $j \in J(\bar{x})$ is linearly independent. Then $\bar{z} = \bar{x}$; that is, $\bar{z}$ is optimal for (FP).

Proof: We shall assume that $\bar{z} \neq \bar{x}$; and exhibit a contradiction. Since $(\bar{z}, \bar{\mu}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ is feasible for (D), it follows that

$$\nabla\left(\frac{\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{f_i(\bar{z}, \bar{y}_i^*) + \bar{z}^t B\bar{w}\right\} + \sum_{j=1}^{p} \bar{\mu}_j g_j(\bar{z})}{\sum_{i=1}^{s} \bar{t}_i \left\{h_i(\bar{z}, \bar{y}_i^*) - \bar{z}^t D\bar{v}\right\}}\right) = 0$$

The above inequality along with the sub linearity of F and $\frac{\rho}{\alpha(\bar{x}, \bar{z})} \geq 0$

$$F(\bar{x}, \bar{z}; \nabla\left[\frac{\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{f_i(\bar{z}, \bar{y}_i^*) + \bar{z}^t B\bar{w}\right\} + \sum_{j=1}^{p} \bar{\mu}_j g_j(\bar{z})}{\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{h_i(\bar{z}, \bar{y}_i^*) - \bar{z}^t D\bar{v}\right\}}\right] = 0 \geq \frac{-\rho d^2(\bar{x}, \bar{z})}{\alpha(\bar{x}, \bar{z})}$$

Which together with the sub linearity of F and $\alpha(\bar{x}, \bar{z}) > 0$ yields

$$F\left(\bar{x}, \bar{z}; \alpha(\bar{x}, \bar{z})\right) \nabla\left[\frac{\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{f_i(\bar{z}, \bar{y}_i^*) + \bar{z}^t B\bar{w}\right\} + \sum_{j=1}^{p} \bar{\mu}_j g_j(\bar{z})}{\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{h_i(\bar{z}, \bar{y}_i^*) - \bar{z}^t D\bar{v}\right\}}\right] \geq -\rho d^2(\bar{x}, \bar{z})$$

Using strict $[F, \alpha, \rho, d]$- Pseudo convexity of ψ (.), we obtain $\psi(\bar{x}) > \psi(\bar{z})$

Since $\psi(\bar{z})$=0, then we have $\psi(\bar{x}) > 0$, that is

$$\left[\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{h_i(\bar{z}, \bar{y}_i^*) - \bar{z}^t D\bar{v}\right\}\right]\left[\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{f_i(\bar{x}, \bar{y}_i^*) + (\bar{x}^t B\bar{w})\right\} + \sum_{j=1}^{p} \bar{\mu}_j g_j(\bar{x})\right]$$

$$> \left[\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{f_i(\bar{z}, \bar{y}_i^*) + \bar{z}^t B\bar{w}\right\} + \sum_{j=1}^{p} \bar{\mu}_j g_j(\bar{z})\right]\left[\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{f_i(\bar{z}, \bar{y}_i^*) + (\bar{z}^t B\bar{w})\right\} + \sum_{j=1}^{p} \bar{\mu}_j g_j(\bar{z})\right]$$

$$\left[\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{h_i(\bar{x}, \bar{y}_i^*) - (\bar{x}^t D\bar{v})\right\}\right] \qquad (3.1)$$

The relations (1), (2.2), (3.1) and

$\sum_{j=1}^{p} \bar{\mu}_j g_j(\bar{x}) \leq 0$ imply

$$\sup_{y\in Y} \frac{f_i(\bar{x}, y) + (\bar{x}^t \bar{B}x)^{\frac{1}{2}}}{h_i(\bar{x}, y) - (\bar{x}^t D\bar{x})^{\frac{1}{2}}} > \frac{\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{ f_i(\bar{z}, \bar{y}_i^*) + \bar{z}^t B\bar{w} \right\} + \sum_{j=1}^{p} \mu_j g_j(\bar{z})}{\sum_{i=1}^{s} \bar{t}_i \left\{ h_i(\bar{z}, \bar{y}_i^*) - \bar{z}^t D\bar{v} \right\}}$$

(3.2)

Since $\bar{x}$ is optimal for (FP) and $g_j(\bar{x}), j \in J(\bar{x})$ is linearly independent, by strong duality (Theorem 2), there exist $(\bar{s}, \bar{t}, \bar{y}^*) \in k(\bar{x})$ and $(\bar{x}, \bar{\mu}, \bar{v}, \bar{w}) \in H(\bar{s}, \bar{t}, \bar{y}^*)$ such that $(\bar{x}, \bar{\mu}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ turns to be an optimal solution of (D) and

$$\sup_{y\in Y} \frac{f_i(\bar{x}, y) + (\bar{x}^t \bar{B}x)^{\frac{1}{2}}}{h_i(\bar{x}, y) - (\bar{x}^t D\bar{x})^{\frac{1}{2}}} = \frac{\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{ f_i(\bar{z}, \bar{y}_i^*) + \bar{z}^t B\bar{w} \right\} + \sum_{j=1}^{p} \bar{\mu}_j g_j(\bar{z})}{\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{ h_i(\bar{z}, \bar{y}_i^*) - \bar{z}^t D\bar{v} \right\}}$$

Which contradicts the fact of (3.2). Hence $\bar{x} = \bar{z}$.

References

1. Ahmad, I. (2003). Optimality conditions and duality in-fractional minimax programming involving generalized ρ-invexity, Inter. j. manag. sys. 19: 165-180.
2. Antczak, T. (2008). Generalized fractional minimax programming with B-(p, r)-invexity. Comput. Math. Appl. 56: 1505-1525.
3. Avriel, M. (1979). Non Linear Programming; Analysis and method. Printice Hall, Englewood cliffs, Newjersy
4. Bajona–Xandri, C and Martinez–Legaz, J.E. (1998). Lower sub differentiability in minimax fractional programming, optimization.
5. Barrodale, I. (1973). Best rational approximation and strict quasi convexity, SIAM J. NUMER. Anal. Loc, 8-12.
6. Bector, C.R and Bhatia, B.L. (1985). Sufficient optimality condition and duality for minimax problems. Utilitas Mathematical, 27: 229 – 247.
7. Chandra, S. and Kumar, V. (1995). Duality in fractional minimax programming. J. Aust. Math. Soc. Ser. A 58: 376 – 386.
8. Lai, H.C and Lee, J.C. (2002). On duality theorems for a nondifferentiable minimax fractional programming. Journal of computational and applied mathematics. 146:115-126.
9. Liang, Z.A and Shi, Z.W. (2003). Optimality conditions and duality for a minimax fractional programming with generalized convexity, J. math. Anal. Appl. 227: 474-488.
10. Liang, Z.A., Huang, H.X and Pardalas, P.M. (2001). Optimality conditions and duality for a class of nonlinear fractional programming problems, J. Optim. Theory Appl. 110: 611-619.
11. Liang, Z.A., Huang, H.X. and Pardalas, P.M. (2003). Efficiency conditions and duality for a class of multi-objective programming problems J. Global optim. 27: 1-25.
12. Liu, J.C and Wu, C.S. (1998). On minimax fractional optimality conditions with invexity. Journal of Mathematical Analysis and Applications, Vol.219 (1), pp.21-35.
13. Schmitendorf, W.E. (1977). Necessary conditions and sufficient conditions for static minimax problems, J. Math. Anal. Appl. 57: 683 – 693.

14. Schroeder, R.G. (1970). Linear programming solutions to ratio games, Oper. Res. 18, 300-305.
15. Soyster, A.L., Lev, B and Loof, D. (1977). Conservative linear programming with mixed multiple objectives, Omega 5: 193-205.
16. Tanimoto, S. (1981). Duality for a class of non differentiable mathematical programming problems. J. Math. Anal. Appl. 79: 283 – 294.
17. Varalakshmi, G and Reddy, P.R.S (2007). Multi-objective fractional minimax problem involving locally lipschitz functions v–invexty. International conference on statistical science, OR & IT, Tirupati; OR: 47
18. Weir, T. (1992). Pseudo convex minimax programming, utiliats Math. 42: 234–240.
19. Yadav, S.R. and Mukhrjee, R.N. (1990). Duality for fractional minimax programming problems. J. Aust. Math. Soc. Ser. B 31: 484 – 492.
20. Zalmai, G.J. (1987). Optimality criteria and duality for a class of minimax programming problems with generalized invexity conditions. Utilitas math. 32: 35 – 57.
21. Indira p. Debnath, x. Qin (2021). Robust optimality and duality for minimax fractional programming problems with support functions J. Nonlinear Funct. Anal. 2021 (2021), 1-22
22. E.K. Kervin, H.L.M. Kerivin and M. M. Wiecek, Robust multiobjective optimization problem with applica tion to internet routing, Ann. Oper. Res. 271 (2018), 487-525.