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Laplacian Energy of $\mathbb{Z}_n$-Graph

Jimly Manuel

Assistant Professor, Department of Mathematics, Mahatma Gandhi College, Iritty, Kerala, India.

Abstract

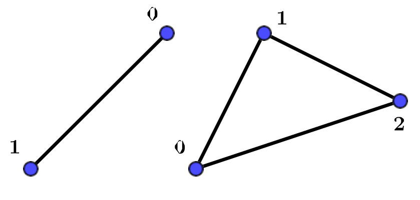
In this paper, we investigate the structural and spectral properties of a graph associated with the finite cyclic group $\mathbb{Z}_n$, referred to as the $\mathbb{Z}_n - graph$. This graph is defined as the underlying simple graph of the digraph constructed using the multiplicative structure of the power digraph of $\mathbb{Z}_n$. We explore various matrices associated with this graph, including the distance matrix, adjacency matrix, and Laplacian matrix. Special attention is given to the Laplacian spectrum and the Laplacian energy of the $\mathbb{Z}_n - graph$. Explicit results are derived for the Laplacian energy of $\mathbb{Z}_{p^r} - graph$ and $\mathbb{Z}_{2q} - graph$, where p and q are primes. Our findings highlight interesting relationships between the algebraic properties of $\mathbb{Z}_n$ and the graph-theoretic properties of its associated $\mathbb{Z}_n - graph$.

Keywords: Distance Matrix. Laplacian Matrix, Laplacian Spectrum, Laplacian Energy

2000 Mathematics Subject Classification 05C25

1. INTRODUCTION

Let $\mathbb{Z}_n$ be the cyclic group of order n. Manuel et al. defined in [4] a digraph called $\mathbb{Z}_n$-digraph with vertex set $\mathbb{Z}_n$ and two distinct vertices $u, v \in \mathbb{Z}_n$ by a directed edge or arc from u to v if and only if there exists a positive integer r such that $v \equiv ru \pmod n$. In other words, two distinct vertices are joined by a directed edge from u to v if and only if $v \in \langle u \rangle$ the cyclic subgroup generated by u. The graph $\mathbb{Z}_n - graph$ is a simple graph with vertex set $V = \{0, 1, 2 ..., n\}$ and two vertices u to v are joined by an edge if and only if either $v \in \langle u \rangle$, the cyclic subgroup generated by u or $u \in \langle v \rangle$, the cyclic subgroup generated by v. The graphs $\mathbb{Z}_2 - graph$ and $\mathbb{Z}_3 - graph$ are shown in figure 1.



2. Matrices related to $\mathbb{Z}_n - graph$

In this section, we discussed some matrices related to $\mathbb{Z}_n - graph$, especially the distance matrix, Laplacian matrix, etc.

2.0.1 Distance Matrix of $\mathbb{Z}_n - graph$

Definition 2.1. [1] Let G be a graph with n vertices, listed as $v_1, v_2, ..., v_n$. The distance matrix of G, with respect to this particular listing of the n vertices of G, is the $n \times n$ matrix $Dis(G) = d_{ij}$ where the $(i, j)^{th}$ entry d_{ij} is the distance between the vertices v_i to the vertex v_j.

Remark 2.2. Let u and v be two vertices of $\square_n - graph$. Since there is a path $u0v$ from u to v of length two, the maximum distance between u and v is 2. That is, $d(u,v) \leq 2$.

The following are some immediate observations on the Distance Matrix, $Dis(\square_n - graph)$ of $\square_n - graph$

a) The only entries of $Dis(\square_n - graph)$ are either $0, 1,$ or, 2.

b) All the diagonal entries of $Dis(\square_n - graph)$) are zero.

c) The rows and columns of $Dis(\square_n - graph)$ corresponding to the generators of $\square_n$ are 1 except the diagonal entry.

d) Let p be a prime number and r be a positive integer, then of $Dis(\square_{p^r} - graph)$ is a p^r- square matrix whose diagonal entries are 0 and all other entries are 1 (since $\square_{p^r} - graph$ is a complete graph).

e) The $(i,j)^{th}$ entry of distance matrix of $\square_n$ −graph is one if and only if $i \neq j$ and either $O(v_i)/O(v_j)$ or $O(v_j)/O(v_i)$.

f) The $(i,j)^{th}$ entry of the distance matrix of $\square_n$ − graph is two if and only if $i \neq j$ and either $O(v_i)$ does not divide $O(v_j)$ or $O(v_j)$ does not divide $O(v_i)$.

Theorem 2.3. Let p be an odd prime, then the number of the entry 2 in $Dis(\square_{2p} - graph)$ is $2(p-1)$.

Proof. Let i be an element of $\square_{2p}$, then $O(i) = 1, 2, p,$ or $2p$. If $O(i) = 1$ or $2p$, then in $Dis(\square_{2p} - graph)$ the $(i,j)^{th}$ entry and the $(j,i)^{th}$ entry are 1 if $i \neq j$. If $O(i) = O(j)$, then also the $(i,j)^{th}$ entry is 1. Now suppose $O(i) = p$ and $O(j) = 2$ or $O(i) = 2$ and $O(j) = p$, then the $(i,j)^{th}$ entry is 2, since 2 does not divide p or p does not divide 2. Now the element has order 2 is p only and the elements having order p are $2, 4, ..., 2(p-1)$. Hence, the number of the entry 2 in $Dis(\square_{2p} - graph)$ is $2(p-1)$.

Theorem 2.4. Let p and q be two distinct primes, then the number of the entry 2 in $Dis(\square_{pq} - graph)$ is $2(p-1)(q-1)$.

Proof. Let i be an element of $\square_{pq}$, then $O(i) = 1\, p, q$ or pq. If $O(i) = 1$ or pq, then in $Dis(\square_{pq} - graph)$ the $(i,j)^{th}$ entry and $(j,i)^{th}$ entry are 1 if $i \neq j$. If $O(i) = O(j)$, then also the $(i,j)^{th}$ entry is 1. Now suppose $O(i) = p$ and $O(j) = q$ or $O(i) = q$ and $O(j) = p$, then the $(i,j)^{th}$ entry is 2, p does not divide q or q does not divide p. Now the elements having order p are $p, 2p, ..., p(q-1)$ only and the elements having order q are $q, 2q, ..., q(p-1)$. Hence the number of the entry 2 in $Dis(\square_{pq} - graph)$ is $2(p-1)(q-1)$.

2.0.2 Adjacency Matrix of $\square_n - graph$

Definition 2.5. [1] Let G be a graph with n vertices, listed as $v_1, v_2, ..., v_n$ The adjacency matrix of G, with respect to this particular listing of the n vertices of G, is the $n \times n$ matrix $A(G) = a_{ij}$ where the $(j,i)^{th}$ entry a_{ij} is the number of edges joining the vertices v_i to the vertex v_j.

Note: The adjacency Matrix and Distance Matrix of $\square_{p^r} - graph$ are the same.

Remark 2.6. The Adjacency Matrix $A(\square_n - graph)$ is obtained from $Dis(\square_n - graph)$ simply by replacing all the $2'$ by $'0'$.

2.0.3 Degree Matrix of $\square_n - graph$

Definition 2.7. [5] For a labeled graph G of n vertices, define an $n \times n$ diagonal matrix D called the degree matrix of G such that the i^{th} diagonal entry in D equals the degree of the i^{th} vertex in G

2.0.4 Laplacian Matrix of $\square_n - graph$

Definition 2.8. [6] The Laplacian matrix L of a graph G is $D - A$, where D is the diagonal matrix of degrees and A is the adjacency matrix. The Laplacian spectrum is the list of eigenvalues of L. The smallest Eigenvalue of L is 0. If G is connected, then eigenvalue 0 has multiplicity 1.

3. Laplacian Energy of $\square_n - graph$

In this section, we find out the Laplacian spectrum and Laplacian Energy of some $\square_n - graph$.

If n and m are the number of vertices and edges of G respectively, and $\mu_1, ..., \mu_n$ are the eigenvalues of L, then Laplacian Energy of a graph G is defined as $LE(G) = \sum_{i=1}^{n} \left| \mu_i - 2\frac{m}{n} \right|$.

Theorem 3.1. Let p be a prime number, then for any positive integer r, the Laplacian Energy of $\square_{p^r} - graph$ is $2(p^r - 1)$.

Proof. Let $n = p$. The Laplacian Spectrum of $\square_{p^r} - graph$ is, $S_L(\square_{p^r} - graph) = \{0, p^r\}$, where the multiplicity of p is $p = n - 1$.

Since $n = p^r$, $m = \frac{p^r(p^r - 1)}{2}$

$\therefore \frac{2m}{n} = p^r - 1$

$$LE(\square_{p^r} - graph) = \left|0 - (p^r - 1)\right| + (p^r - 1)\left|p^r - (p^r - 1)\right|$$
$$= 2(p^r - 1)$$

Illustration 3.2

Consider $\square_2 - graph$ in Figure 1, then $L(\square_2 - graph) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Here, the characteristic polynomial of $L(\square_2 - graph) = x(x-2)$ and its Laplacian spectrum is $\{0, 2\}$ and $\frac{2m}{n} = 1$.

$$LE(\square_2 - graph) = |0 - 1| + |2 - 1| = 2$$

Now consider $\square_3 - graph$ in Figure 1, then $L(\square_3 - graph) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Here, the characteristic polynomial of $L(\square_3 - graph) = -x(x-3)^2$ and its Laplacian spectrum is $\{0, 3, 3\}$ and $\frac{2m}{n} = 2$.

$$LE(\square_3 - graph) = |0 - 2| + 2|3 - 2| = 4$$

Remark 3.3. The number of edges in $\square_n - graph$ is the same as the half of trace of $L(\square_n - graph)$

Theorem 3.4. Let q be a prime number, then the Laplacian Energy of $\square_{2q}-graph$ is $\frac{6q^2-8q+4}{q}$.

Proof. Let q be a prime number, then the diagonal elements in $L(\square_{2q}-graph)$ are $2q-1$ with multiplicity q; $2q-2$ with multiplicity $q-1$ and q with multiplicity 1. So the number of edges m in $\square_{2q}-graph$ is

$$m=\frac{q(2q-1)-(q-1)(2q-2)+q}{2}$$

$$=2q^2-2q+1$$

The eigenvalues of $L(\square_{2q}-graph)$ are 0 of multiplicity 1, q of multiplicity 1, $2q-1$ of multiplicity $q-2$ and $2q$ of multiplicity q. Now

$$LE(\square_{2q}-graph)=\sum_{i=1}^{n}\left|\mu_i-2\frac{m}{n}\right| \text{ and } \frac{2m}{n}=\frac{4q^2-4q+2}{2q}=\frac{2q^2-2q+1}{q}$$

$$LE(\square_{2q}-graph)=\left|\frac{2q^2-2q+1}{q}\right|+\left|q-\frac{2q^2-2q+1}{q}\right|+(q-2)\left|(2q-1)-\frac{2q^2-2q+1}{q}\right|+q\left|2q-\frac{2q^2-2q+1}{q}\right|$$

$$=\left|\frac{2q^2-2q+1}{q}\right|+\left|\frac{q^2-2q^2+2q-1}{q}\right|+(q-2)\left|\frac{2q^2-q-2q^2+2q-1}{q}\right|+q\left|\frac{2q^2-2q^2+2q-1}{q}\right|$$

$$=\frac{2q^2-2q+1}{q}+\frac{1}{q}\left|-\{q^2-2q+1\}\right|+\frac{q-2}{q}[q-1]+\frac{q}{q}[2q-1]$$

$$=\frac{5q^2-6q+3+\left|q^2-2q+1\right|}{q}$$

$$=\frac{6q^2-8q+4}{q}$$

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