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Common Fixed Point Theorems For Four Fuzzy Mappings

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ABSTRACT: In this paper we introduce a contractive inequality for four fuzzy mappings through a 4-variable generalization of altering distance function and then prove that the two fuzzy mappings defined on a complete ordered metric linear space satisfying such inequality have a common fixed point. We have discussed some specific results, which are obtainable under special choices of the generalized altering distance function. We also show that a more general result in the fixed point theory of multi-valued mappings can be established and the result we obtained for fuzzy mappings can be deduced from the general theorem.

1. INTRODUCTION

In 1965, the theory of fuzzy sets was investigated by Zadeh [23]. In 1981, Heilpern [11] first introduced the concept of fuzzy contractive mappings and proved a fixed point theorem for these mappings in metric linear spaces. Es- truch and Vidal [10] proved a fixed point theorem for fuzzy contraction mappings in a complete metric spaces which in turn generalized Heilpern fixed point the- orem. Afterwards a number of works appeared in which fixed points of fuzzy mappings satisfying contractive inequalities have been studied (see [9])

A new category of contractive fixed point problems was addressed by M.S. Khan et. al [13]. There they introduced Altering Distance Function, which is a control function that alters distance between two points in a metric space.

In this paper we introduce a contractive inequality for four fuzzy mappings through a 4-variable generalization of altering distance function and then prove that the two fuzzy mappings defined on a complete ordered metric linear space satisfying such inequality have a common fixed point. We have discussed some specific results, which are obtainable under special choices of the generalized alter- ing distance function. We also show that a more general result in the fixed point theory of multi-valued mappings can be established and the result we obtained for fuzzy mappings can be deduced from the general theorem.

2. PRELIMINARIES

Throughout the rest of the paper unless otherwise stated (X, d) stands for a complete metric space. A fuzzy set in X is a function with domain X and values in $[0, 1]$. If A is a fuzzy set on X and $x \in X$ then the functional value Ax is

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called the grade of membership of x in A. The $\alpha-$level set of A, denoted by A_α, is defined by $A_\alpha = \{x : Ax \geq \alpha\}$, if $\alpha \in (0, 1]$, $A_0 = \{x : Ax \geq 0\}$, where $\overline{B}$ denoted the closure of the set B. For any two subsets A and B of X we denote by $H(A, B)$ the Hausdroff distance. For any two subsets A and B of X we write $\delta(A, B) = \sup_{\alpha\in A,\beta\in B} d(a, b)$.

Definition 2.1. A function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if and only if

(i) ψ is continuous,
(ii) ψ is non-decreasing,
(iii) $\psi(t) = 0 \Leftarrow\Rightarrow t = 0$.

Choudhury [9] introduced the concept of a generalized altering distance function for three variables.

Definition 2.2. A function $\psi : [0, +\infty) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is called an altering distance function if and only if

(i) ψ is continuous,
(ii) ψ is non-decreasing in all three variables,
(iii) $\psi(x, y, z) = 0 \Longleftrightarrow x = y = z = 0$.

Rao et al. [18] introduced the concept of a generalized altering distance function for four variables.

Definition 2.3. A function $\psi : [0, +\infty) \times [0, +\infty) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is called an altering distance function if and only if

(i) ψ is continuous,
(ii) ψ is non-decreasing in all three variables,
(iii) $\psi(x, y, z, w) = 0 \Longleftrightarrow x = y = z = w = 0$.

Definition 2.4. Let (X, d) be a metric space and $f, g : X \to X$. If $w = fx = gx$, for some $x \in X$, then x is called a coincidence point of f and g, and w is called a coincidence point of f and g. If $x = w$, then x is a common fixed point of f and g. The pair $\{f, g\}$ is said to be comparable if and only if $\lim_{n\to+\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$ for some $t \in X$.

Definition 2.5. Let f and g be two self mappings defined on a set X. Then f and g are said to be weakly comparable if they commute at every coincidence point.

Definition 2.6. Let X be a nonempty set. Then $(X, d, \preceq)$ is called an ordered metric linear space iff

(i) (X, d) is a metric linear space,
(ii) $(X, \leq)$ is a partial order.

Definition 2.7. Let $(X, \leq)$ be a partially ordered set. Then $x, y \in X$ are comparable if $x \leq y$ or $y \leq x$ holds.

Definition 2.8. Let $(X, \leq)$ be a partially ordered set. A pair (f, g) of self maps of X is said to be weakly increasing if $gx \leq gfx$ and $gx \leq fgx$ for all $x \in X$.

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The notion of partially weakly increasing of pair of mappings is introduced by Abbas et al [1].

Definition 2.9. Let $(X, \leq)$ be a partially ordered set and f and g be two self maps on X. An ordered pair (f, g) is said to be partially weakly increasing if $gx \leq gfx$ and $gx \leq fgx$ for all $x \in X$.

Note that a pair (f, g) is weakly increasing if and only if ordered pair (f, g) and (g, f) are partially weakly increasing. In the following, an example of an ordered pair (f, g) of self-maps f and g which is partially weakly increasing but not weakly increasing.

Example 2.10. Let $X = [0, 1]$ be endowed with a usual ordering and $f, g : X \to X$ be defined by $fx = x^2$ and $gx = \sqrt{x}$. Clearly, (f, g) is partially weakly increasing but (g, f) is not partially weakly increasing.

Definition 2.11. Let $(X, \leq)$ be a partially ordered set. A mapping f is called weak annihilator of g if $fgx \leq x$ for all $x \in X$.

Example 2.12. Let $X = [0, 1]$ be endowed with a usual ordering and $f, g : X \to X$ be defined by $fx = x^2$ and $gx = x^3$. Thus f is a weak annihilator of g.

Definition 2.13. Let $(X, \leq)$ be a partially ordered set. A mapping f is called domination if $x \leq fx$ for each $x \in X$.

Example 2.14. Let $X = [0, 1]$ be endowed with a usual ordering and $f : X \to X$ be defined by $fx = \sqrt[n]{x}$. Thus f is domination for each $x \in X$.

Definition 2.15. A subset K of a partially ordered set X is called totally ordered when every two elements of K are comparable.

3. Main results

Now, we proof our main results of this section.

Theorem 3.1. *Let $(X, d, \leq)$ be an ordered complete metric linear space. Let $T, S, I, J : X \to W(X)$ be four fuzzy mappings satisfying for every pair $(x, y) \in X \times X$ such that x and y are comparable,*

$$\varphi_1(\delta_1(Sx, Ty)) \leq \psi_1(M(Ix, Sx)) - \psi_2(M(Ix, Sx)) \tag{3.1}$$

where

$$M(Ix, Sx) = \{d(Ix, Jy), D_1(Ix, Sx), D_1(Jy, Ty), \frac{1}{2}[D_1(Ix, Ty) + D_1(Jy, Sx)]\}$$

and ψ_1 and ψ_2 are generalized altering distance functions (in Ψ_4) and $\varphi_1(x) = \psi_1(x, x, x, x)$. Suppose that

(i) *(I, T) and (J, S) be partially weakly increasing,*

(ii) *$T(X) \subseteq I(X)$ and $S(X) \subseteq J(X)$,*

(iii) *S and T are dominating maps,*

(iv) *T is weak annihilator of I and S is weak annihilator of J,*

(v) *if for a nondecreasing sequence* $\{x_n\}$ *with* $x_n \leq y_n$ *for all* n *and* $y_n \to u$ *implies that* $x_n \leq u$.

Assume either

(a) $\{S, I\}$ *are comparable,* S *or* I *is continuous and* $\{T, J\}$ *are weakly comparable or*
(b) $\{T, J\}$ *are comparable,* T *or* J *is continuous and* $\{S, I\}$ *are weakly comparable .*

Then S, T, I and J have a common fixed point. Moreover, the set of common fixed points of S, T, I and J is totally ordered if and only if S, T, I and J have one and only one common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X. Since $T(X) \subseteq I(X)$ and $S(X) \subseteq J(X)$, we can define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$\{y_{2n-1}\} = Jx_{2n-1} \subset Sx_{2n-2}, \quad \{y_{2n}\} = Ix_{2n} \subset Tx_{2n-1}, \tag{3.2}$$

for all $n \in \mathrm{N}$.

By given assumptions

$$x_{2n-2} \leq Sx_{2n-2} = Jx_{2n-1} \leq SJx_{2n-1} \leq x_{2n-1}$$

and

$$x_{2n-1} \leq Tx_{2n-1} = Ix_{2n} \leq TIx_{2n} \leq x_{2n}.$$

Thus for all $n \geq 1$, we have

$$x_n \leq x_{n+1}. \tag{3.3}$$

Without loss of generality, we may assume that

$$d(y_{2n}, y_{2n+1}) > 0 \quad \forall n \in \mathrm{N}. \tag{3.4}$$

If not, then $y_{2n} = y_{2n+1}$, for some n. Putting $x = x_{2n+1}$ and $y = x_{2n}$, form (3.3) and the considered contraction (3.1), we have

$$\begin{aligned}
\varphi_1(d(y_{2n+2}, y_{2n+1})) &= \varphi_1(\delta_1(Sx_{2n+1}, Tx_{2n})) \\
&\leq \psi_1\Big(d(Ix_{2n+1}, Jx_{2n}), D_1(Ix_{2n+1}, Sx_{2n+1}), D_1(Jx_{2n}, Tx_{2n}), \\
&\quad \frac{1}{2}[D_1(Ix_{2n+1}, Tx_{2n}) + D_1(Jx_{2n}, Sx_{2n+1})]\Big) \\
&\quad -\psi_2\Big(d(Ix_{2n+1}, Jx_{2n}), D_1(Ix_{2n+1}, Sx_{2n+1}), D_1(Jx_{2n}, Tx_{2n}), \\
&\quad \frac{1}{2}[D_1(Ix_{2n+1}, Tx_{2n}) + D_1(Jx_{2n}, Sx_{2n+1})]\Big) \\
&\leq \psi_1\Big(d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), \\
&\quad \frac{1}{2}[d(y_{2n+1}, y_{2n+1}) + d(y_{2n}, y_{2n+2})]\Big) \\
&\quad -\psi_2\Big(d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), \\
&\quad \frac{1}{2}[d(y_{2n+1}, y_{2n+1}) + d(y_{2n}, y_{2n+2})]\Big) \qquad (3.5) \\
&\leq \psi_1\Big(0, d(y_{2n+1}, y_{2n+2}), 0, \frac{1}{2}d(y_{2n}, y_{2n+2})\Big) \\
&\quad -\psi_2\Big(0, d(y_{2n+1}, y_{2n+2}), 0, \frac{1}{2}d(y_{2n}, y_{2n+2})\Big). \qquad (3.6)
\end{aligned}$$

Using a triangular inequality, we have

$$\frac{1}{2}d(y_{2n}, y_{2n+2}) \leq \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})] \leq \frac{1}{2}d(y_{2n+1}, y_{2n+2}).$$

Using this together with a property of the generalized altering function ψ_1, we get

$$\psi_1\Big(0, d(y_{2n+1}, y_{2n+2}), 0, \frac{1}{2}d(y_{2n}, y_{2n+2})\Big) \leq \varphi_1(d(y_{2n+1}, y_{2n+2})).$$

Hence, we obtain

$$\varphi_1(d(y_{2n+1}, y_{2n+2})) \leq \varphi_1(d(y_{2n+1}, y_{2n+2})) - \psi_2\Big(0, d(y_{2n+1}, y_{2n+2}), 0, \frac{1}{2}d(y_{2n}, y_{2n+2})\Big).$$

This implies that

$$\psi_2\Big(0, d(y_{2n+1}, y_{2n+2}), 0, \frac{1}{2}d(y_{2n}, y_{2n+2})\Big) = 0$$

which yields that

$$d(y_{2n}, y_{2n+1}) = 0.$$

Following the similar arguments, we obtain $y_{2n+2} = y_{2n+3}$ and so on. Thus $\{y_n\}$ becomes a constant sequence and $\{y_{2n}\}$ is the common fixed point of I, J, S and T.

Take for each n, $d(y_{2n}, y_{2n+1}) > 0$. We claim that

$$\lim_{n\to+\infty} d(y_{2n}, y_{2n+1}) = 0. \tag{3.7}$$

By (3.6), we have

$$\begin{aligned}\varphi_1(d(y_{2n+2}, y_{2n+1})) &= \varphi_1(\delta_1(Sx_{2n+1}, Tx_{2n})) \\ &\leq \psi_1(M(y_{2n+1}, y_{2n})) - \psi_2(M(y_{2n+1}, y_{2n}))\end{aligned} \tag{3.8}$$

where

$M(y_{2n+1}, y_{2n}) = \{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), \frac{1}{2} d(y_{2n}, y_{2n+2})\}$.

Suppose for some $n \in \mathbb{N}$, that

$$d(y_{2n+2}, y_{2n+1}) > d(y_{2n}, y_{2n+1}). \tag{3.9}$$

Using (3.9) and a triangular inequality, we have

$$\frac{1}{2} d(y_{2n}, y_{2n+2}) \leq \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})] < d(y_{2n+1}, y_{2n+2}).$$

Using this and (3.9) together with a property of the generalized altering distance function ψ_1, we get

$$\psi_1(M(y_{2n+1}, y_{2n})) \leq \varphi_1(d(y_{2n+1}, y_{2n+2})).$$

where

$M(y_{2n+1}, y_{2n}) = \{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), \frac{1}{2} d(y_{2n}, y_{2n+2})\}$.

Hence, we obtain

$$\varphi_1(d(y_{2n+2}, y_{2n+1})) \leq \varphi_1(d(y_{2n+2}, y_{2n+1})) - \psi_2(M(y_{2n+1}, y_{2n})).$$

where

$M(y_{2n+1}, y_{2n}) = \{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), \frac{1}{2} d(y_{2n}, y_{2n+2})\}$.

This implies that

$$\psi_2\left(d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n}), \frac{1}{2} d(y_{2n}, y_{2n+2})\right) = 0$$

which yields that

$$d(y_{2n+1}, y_{2n}) = 0.$$

Hence, we obtain a contradiction with (3.4). We deduce that

$$d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n}, y_{2n+1}), \quad \forall n \in \mathbb{N}. \tag{3.10}$$

Similarly, putting $x = x_{2n+1}$ and $y = x_{2n+2}$, form (3.3) and the considered contraction (3.1), we have

$$\begin{aligned} \varphi_1(d(y_{2n+2}, y_{2n+3})) &= \varphi_1(\delta_1(Sx_{2n+1}, Tx_{2n+2})) \\ &\leq \psi_1\Big(d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+3}), d(y_{2n+1}, y_{2n+2}), \\ &\quad \frac{1}{2}d(y_{2n+1}, y_{2n+3})\Big) \\ &\quad -\psi_2\Big(d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+3}), d(y_{2n+1}, y_{2n+2}), \\ &\quad \frac{1}{2}d(y_{2n+1}, y_{2n+3})\Big). \end{aligned} \tag{3.11}$$

Suppose, for some $n \in \mathbb{N}$, that

$$d(y_{2n+2}, y_{2n+3}) > d(y_{2n+1}, y_{2n+2}). \tag{3.12}$$

Then, by a triangular inequality, we have

$$\frac{1}{2}d(y_{2n+1}, y_{2n+3}) \leq \frac{1}{2}[d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, y_{2n+3})] < d(y_{2n+2}, y_{2n+3}).$$

Hence, from this, (3.11) and (3.12), we obtain

$$\begin{aligned} \varphi_1(d(y_{2n+1}, y_{2n+3})) &\leq \varphi_1(d(y_{2n+2}, y_{2n+3})) \\ &\quad -\psi_2\Big(d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+3}), d(y_{2n+1}, y_{2n+2}), \\ &\quad \frac{1}{2}d(y_{2n+1}, y_{2n+3})\Big). \end{aligned}$$

This implies that

$$\psi_2\Big(d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+3}), \frac{1}{2}d(y_{2n+1}, y_{2n+3})\Big) = 0$$

which yields that

$$d(y_{2n+1}, y_{2n+2}) = 0.$$

Hence, we obtain a contradiction with (3.4). We deduce that

$$d(y_{2n+1}, y_{2n+2}) \geq d(y_{2n+2}, y_{2n+3}), \quad \forall n \in \mathbb{N}. \tag{3.13}$$

Combining (3.10) and (3.13), we obtain

$$d(y_{2n}, y_{2n+1}) > d(y_{2n+2}, y_{2n+3}), \quad \forall n \in \mathbb{N}. \tag{3.14}$$

Then, $\{d(y_{2n+1}, y_{2n+2})\}$ is a nonincreasing sequence of positive real numbers. This implies that there exists $r \geq 0$ such that

$$\lim_{n\to+\infty} d(y_{2n+1}, y_{2n+2}) = r. \quad (3.15)$$

By (3.8), we have

$$\begin{aligned}\varphi_1(d(y_{2n+2}, y_{2n+1})) &= \varphi_1(\delta_1(Sx_{2n+1}, Tx_{2n})) \\ &\leq \psi_1\, (d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), \\ &\quad \frac{1}{2} d(y_{2n}, y_{2n+2}) \\ &\quad -\psi_2\, (d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), \\ &\quad \frac{1}{2} d(y_{2n}, y_{2n+2}) \\ &\leq \varphi(d(y_{2n+1}, y_{2n})) - \psi_2\, (d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), \\ &\quad d(y_{2n}, y_{2n+1}), 0)\,. \end{aligned} \quad (3.16)$$

Letting $n \to +\infty$ in (3.16) and using the continuities of φ_1 and ψ_2, we obtain

$$\varphi_1(r) \leq \varphi_1(r) - \psi_2(r, r, r, 0), \text{ which}$$

implies that $\psi_2(r, r, r, 0) = 0$ so $r = 0$. Hence

$$\lim_{n\to+\infty} d(y_{2n+1}, y_{2n+2}) = 0.$$

Hence, (3.7) is proved.

Next, we claim that $\{y_n\}$ is a Cauchy sequence.

From (3.7), it will be sufficient to prove that $\{y_{2n}\}$ is a Cauchy sequence. We proceed by negation and suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(i)\}$ and $\{n(i)\}$ such that for all positive integer i,

$$n(i) > m(i) > i, \quad d(y_{m(i)}, y_{n(i)}) \geq \varepsilon, \quad d(y_{m(i)}, y_{n(i)-2}) < \varepsilon. \quad (3.17)$$

From (3.17) and using a triangular inequality, we get

$$\begin{aligned}\varepsilon &\leq d(y_{m(i)}, y_{n(i)}) \\ &\leq d(y_{m(i)}, y_{n(i)-2}) + d(y_{n(i)-2}, y_{n(i)-1}) + d(y_{n(i)-1}, y_{n(i)}) \\ &\leq \varepsilon + d(y_{n(i)-2}, y_{n(i)-1}) + d(y_{n(i)-1}, y_{n(i)}).\end{aligned}$$

Letting $i \to +\infty$ in the above inequality and using (3.7), we obtain

$$\lim_{i\to+\infty} d(y_{m(i)}, y_{n(i)}) = \varepsilon. \quad (3.18)$$

Again, a triangular inequality gives us

$$|d(y_{n(i)}, y_{m(i)-1}) - d(y_{n(i)}, y_{m(i)})| \leq d(y_{m(i)-1}, y_{m(i)}). \tag{3.19}$$

Letting $i \to +\infty$ in the above inequality and using (3.7) and (3.18), we get

$$\lim_{i\to+\infty} d(y_{n(i)}, y_{m(i)-1}) = \varepsilon. \tag{3.20}$$

Similarly, we have

$$\lim_{i\to+\infty} d(y_{n(i)+1}, y_{m(i)-1}) = \varepsilon. \tag{3.21}$$

On the other hand, we have

$$\begin{aligned} d(y_{n(i)}, y_{m(i)}) &\leq d(y_{n(i)}, y_{n(i)+1}) + d(y_{n(i)+1}, y_{m(i)}) \\ &= d(y_{n(i)}, y_{n(i)+1}) + d(Tx_{n(i)}, Sx_{m(i)-1}). \end{aligned}$$

Then, from (3.7), (3.18) and the continuity of φ_1, we get by letting $i \to +\infty$ in the above inequality

$$\varphi_1(\varepsilon) \leq \lim_{i\to+\infty} d(Tx_{n(i)}, Sx_{m(i)-1}). \tag{3.22}$$

Now, using the considered contractive condition (3.1) for $x = x_{2m(i)-1}$ and $y = x_{2n(i)}$, we have

$$\varphi_1(\delta_1(Sx_{2m(i)-1}, Tx_{2n(i)})) \leq \psi_1 \ M(x_{2m(i)-1}, Tx_{2n(i)}) - \psi_2 \ M(x_{2m(i)-1}, Tx_{2n(i)})$$

$$\begin{aligned} M(x_{2m(i)-1}, Tx_{2n(i)}) = &\{d(Ix_{2m(i)-1}, Jx_{2n(i)}), D_1(Ix_{2m(i)-1}, Sx_{2m(i)-1}), \\ &D_1(Jx_{2n(i)}, Tx_{2n(i)}), \\ &\frac{1}{2}[D_1(Ix_{2m(i)-1}, Tx_{2n(i)}) + D_1(Jx_{2n(i)}, Sx_{2m(i)-1})]\} \end{aligned}$$

$$\begin{aligned} \varphi_1(\delta_1(y_{2m(i)-1}, y_{2n(i)})) \leq \ &\psi_1 \ d(y_{2m(i)-1}, y_{2n(i)}), d(y_{2m(i)-1}, y_{2m(i)}), d(y_{2n(i)}, y_{2n(i)+1}), \\ &\frac{1}{2}[d(y_{2m(i)-1}, y_{2n(i)+1}) + d(y_{2n(i)}, y_{2m(i)})] \\ &-\psi_2 \ d(y_{2m(i)-1}, y_{2n(i)}), d(y_{2m(i)-1}, y_{2m(i)}), d(y_{2n(i)}, y_{2n(i)+1}), \\ &\frac{1}{2}[d(y_{2m(i)-1}, y_{2n(i)+1}) + d(y_{2n(i)}, y_{2m(i)})] \ . \end{aligned}$$

Then, from (3.7), (3.20), (3.21) and the continuity of ψ_1 and ψ_2, we get by letting $i \to +\infty$ in the above inequality

$$\begin{aligned}\lim_{i\to+\infty} \varphi_1(\delta_1(Sx_{2m(i)-1}, Tx_{2n(i)})) &\leq \psi_1(\varepsilon, 0, 0, \varepsilon) - \psi_2(\varepsilon, 0, 0, \varepsilon) \\ &\leq \varphi_1(\varepsilon) - \psi_2(\varepsilon, 0, 0, \varepsilon).\end{aligned}$$

Now, combining (3.1) with the above inequality, we get

$$\varphi_1(\varepsilon) \leq \varphi_1(\varepsilon) - \psi_2(\varepsilon, 0, 0, \varepsilon),$$

which implies that $\psi_2(\varepsilon, 0, 0, \varepsilon) = 0$, that is a contradiction since $\varepsilon > 0$. We deduce that $\{y_n\}$ is a Cauchy sequence.

Finally, we prove existence of a common fixed point of the four mappings I, J, S and T.

Since $\{y_n\}$ is a Cauchy sequence in complete metric linear space (X, d), there exists a point $z \in X$, such that $\{y_{2n}\}$ converges to z. Therefore,

$$\{y_{2n+1}\} = Jx_{2n+1} \subset Sx_{2n} \to z, \quad as \quad n \to +\infty \tag{3.23}$$

and

$$\{y_{2n+2}\} = Ix_{2n+2} \subset Tx_{2n+1} \to z, \quad as \quad n \to +\infty. \tag{3.24}$$

Suppose that (a) holds.

Since $\{S, I\}$ are comparable, we have

$$\lim_{n\to+\infty} SIx_{2n+2} = \lim_{n\to+\infty} SIx_{2n+2} = Iz.$$

Also, $x_{2n+1} \leq Tx_{2n+1} = Ix_{2n+2}$. Now

$$\begin{aligned}\varphi_1(\delta_1(SIx_{2n+2}, Tx_{2n+1})) \leq\ & \psi_1\Big(d(IIx_{2n+2}, Jx_{2n+1}), D_1(IIx_{2n+2}, SIx_{2n+2}), D_1(Jx_{2n+1}, Tx_{2n+1}), \\ & \tfrac{1}{2}[D_1(IIx_{2n+2}, Tx_{2n+1}) + D_1(Jx_{2n+1}, SIx_{2n+2})]\Big) \\ & -\psi_2\Big(d(IIx_{2n+2}, Jx_{2n+1}), D_1(IIx_{2n+2}, SIx_{2n+2}), D_1(Jx_{2n+1}, Tx_{2n+1}), \\ & \tfrac{1}{2}[D_1(IIx_{2n+2}, Tx_{2n+1}) + D_1(Jx_{2n+1}, SIx_{2n+2})]\Big)\end{aligned}$$

Assume that I is continuous. On passing limit as $n \to +\infty$, we obtain

$$\begin{aligned}\varphi_1(d(Iz, z)) &\leq \psi_1(d(Iz, z), 0, 0, d(Iz, z)) - \psi_2(d(Iz, z), 0, 0, d(Iz, z)) \\ &\leq \varphi_1(d(Iz, z)) - \psi_2(d(Iz, z), 0, 0, d(Iz, z)),\end{aligned}$$

so $\psi_2(d(Iz, z), 0, 0, d(Iz, z)) = 0$, which implies that

$$Iz = z. \tag{3.25}$$

Now, $x_{2n+1} \preceq Tx_{2n+1}$ and $Tx_{2n+1} \to z$ as $n \to +\infty$, so by assumption we have $x_{2n+1} \preceq z$ and (3.1) becomes

$$\varphi_1(\delta_1(Sz, Tx_{2n+1})) \leq \psi_1(M(z, x_{2n+1})) - \psi_2(M(z, x_{2n+1})).$$

where

$$M(z, x_{2n+1}) = (d(Iz, Jx_{2n+1}), D_1(Iz, Sz), D_1(Jx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[D_1(Iz, Tx_{2n+1}) + D_1(Jx_{2n+1}, Sz)]$$

Passing to the limit $n \to +\infty$ in the above inequality and using (3.25),

$$\varphi_1(\delta_1(Sz, z)) \leq \psi_1\left(0, D_1(z, Sz), 0, \frac{1}{2}D_1(z, Sz)\right) - \psi_2\left(0, D_1(z, Sz), 0, \frac{1}{2}D_1(z, Sz)\right)$$

which holds unless $\psi_2\left(0, d(z, Sz), 0, \frac{1}{2}D_1(z, Sz)\right) = 0$, so

$$Sz = z. \tag{3.26}$$

Since $S(X) \subseteq J(X)$, there exists a point $w \in X$ such that $Sz = Jw$. Suppose that $Tw \neq Jw$. Since $z \preceq Sz = Jw \preceq SJw \preceq w$ implies $z \preceq w$. From (3.1), we obtain

$$\begin{aligned}\varphi_1(\delta_1(Sz, Tw)) &\leq \psi_1\Big(d(Iz, Jw), D_1(Iz, Sz), D_1(Jw, Tw), \frac{1}{2}[D_1(Iz, Tw) + D_1(Jw, Sz)]\Big) \\ &\quad - \psi_2\Big(d(Iz, Jw), D_1(Iz, Sz), D_1(Jw, Tw), \frac{1}{2}[D_1(Iz, Tw) + D_1(Jw, Sz)]\Big) \\ &\leq \psi_1\left(0, 0, D_1(Jw, Tw), \frac{1}{2}D_1(Jw, Tw)\right) \\ &\quad - \psi_2\left(0, 0, D_1(Jw, Tw), \frac{1}{2}D_1(Jw, Tw)\right)\end{aligned}$$

Hence

$$Jw = Tw. \tag{3.27}$$

Since T and J are weakly compatible, $Tz = TSz = TJw = JTw = JSz = Jz$. Thus z is a coincidence point of T and J.

Now, since $x_{2n} \leq Sx_{2n}$ and $Sx_{2n} \to z$ as $n \to +\infty$, implies that $x_{2n} \leq z$, from (3.1)

$$\varphi_1(\delta_1(Sx_{2n}, Tz)) \leq \psi_1(M((x_{2n}, z))) - \psi_2(M((x_{2n}, z)))$$

where

$$M((x_{2n}, z) = (d(Ix_{2n}, Jz), D_1(Ix_{2n}, Sx_{2n}), D_1(Jz, Tz), \frac{1}{2}[D_1(Ix_{2n}, Tz) + D_1(Jz, Sx_{2n})])$$

Passing to the limit $n \to +\infty$ in the above inequality, we have

$$\varphi_1(\delta_1(z, Tz)) \leq \psi_1(d(z, Tz), 0, 0, d(z, Tz)) - \psi_2(d(z, Tz), 0, 0, d(z, Tz))$$

which gives that

$$z = Tz. \tag{3.28}$$

Therefore, $Sz = Tz = Iz = Jz = z$, so z is a common fixed point of I, J, S and T. The proof is similar when S is continuous.

Similarly, the result follows when (b) holds.

Now, suppose that set of common fixed points of I, J, S and T is totally ordered. We claim that there is a unique common fixed point of I, J, S and T. Assume on contrary that, $Su = Tu = Iu = Ju = u$ and $Sv = Tv = Iv = Jv = v$ but $u = v$.

By supposition, we can replace $x = u$ and $y = v$ in (3.1) to obtain

$$\begin{aligned}\varphi_1(d(u, v)) &\leq \varphi_1(\delta_1(Su, Ty)) \\ &\leq \psi_1(d(Iu, Jv), D_1(Iu, Su), D_1(Jv, Tv), \frac{1}{2}[D_1(Iu, Tv) + D_1(Jv, Su)] \\ &\quad -\psi_2(d(Iu, Jv), D_1(Iu, Su), D_1(Jv, Tv), \frac{1}{2}[D_1(Iu, Tv) + D_1(Jv, Su)] \\ &\leq \psi_1(d(u, v), 0, 0, d(u, v)) - \psi_2(d(u, v), 0, 0, d(u, v)) \\ &\leq \varphi_1(d(u, v))\end{aligned}$$ a

contraction, so $u = v$.

Conversely, if I, J, S and T have only one common fixed point, then the set of common fixed point of I, J, S and T being singleton is totally ordered.

Corollary 3.2. *Let $(X, d, \leq)$ be an ordered complete metric linear space. Let $T, S, I, J : X \to W(X)$ be four fuzzy mappings satisfying for every pair $(x, y) \in$*

$X \times X$ such that x and y are comparable and there exists a positive Lebesgue integrable function u on $\mathbb{R}^+$ such that $\int_0^\epsilon u(t)dt > 0$ for each $\epsilon > 0$ and that ,

$$\int_0^{\varphi_1(\delta_1(Sx,Ty))} u(t)dt \leq \int_0^{\psi_1(M(x,y))} u(t)dt - \int_0^{\psi_2(M(x,y))} u(t)dt \quad (3.29)$$

where ψ_1 and ψ_2 are generalized altering distance functions (in Ψ_4) and $\varphi_1(x) = \psi_1(x, x, x, x)$ also

$$M(x, y) = \{d(Ix, Jy), D_1(Ix, Sx), D_1(Jy, Ty), \frac{1}{2}[D_1(Ix, Ty) + D_1(Jy, Sx)]\}$$

Suppose that

(i) *(I, T) and (J, S) be partially weakly increasing,*
(ii) *$T(X) \subseteq I(X)$ and $S(X) \subseteq J(X)$,*
(iii) *S and T are dominating maps,*
(iv) *T is weak annihilator of I and S is weak annihilator of J,*
(v) *if for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies that $x_n \leq u$.*

Assume either

(a) *$\{S, I\}$ are comparable, S or I is continuous and $\{T, J\}$ are weakly comparable or*
(b) *$\{T, J\}$ are comparable, T or J is continuous and $\{S, I\}$ are weakly comparable .*

Then S, T, I and J have a common fixed point. Moreover, the set of common fixed points of S, T, I and J is totally ordered if and only if S, T, I and J have one and only one common fixed point.

Remark 3.3. If we take $\psi_1(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}$ and $\psi_2(t_1, t_2, t_3, t_4) = (1 - k)\max\{t_1, t_2, t_3, t_4\}$, for $k \in (0, 1)$ then $\varphi_1(t) = t$ for all $t_1, t_2, t_3, t_4 \geq 0$ then the we get following result.

Corollary 3.4. *Let $(X, d, \leq)$ be an ordered complete metric linear space. Let $T, S, I, J : X \to W(X)$ be four fuzzy mappings satisfying for every pair $(x, y) \in X \times X$ such that x and y are comparable,*

$$\delta_1(Sx, Ty) \leq k \max\{d(Ix, Jy), D_1(Ix, Sx), D_1(Jy, Ty), \frac{1}{2}[D_1(Ix, Ty) + D_1(Jy, Sx)]\} \quad (3.30)$$

where $k \in (0, 1)$ Suppose that

(i) *(I, T) and (J, S) be partially weakly increasing,*
(ii) *$T(X) \subseteq I(X)$ and $S(X) \subseteq J(X)$,*
(iii) *S and T are dominating maps,*
(iv) *T is weak annihilator of I and S is weak annihilator of J,*
(v) *if for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies that $x_n \leq u$.*

Assume either

(a) *$\{S, I\}$ are comparable, S or I is continuous and $\{T, J\}$ are weakly comparable or*

(b) *$\{T, J\}$ are comparable, T or J is continuous and $\{S, I\}$ are weakly comparable .*

Then S, T, I and J have a common fixed point. Moreover, the set of common fixed points of S, T, I and J is totally ordered if and only if S, T, I and J have one and only one common fixed point.

Remark 3.5. Other results could be derived for other choices of ψ_1 and ψ_2.

Corollary 3.6. *Let $(X, d, \leq)$ be an ordered complete metric linear space. Let $T, S, I : X \to W(X)$ be three fuzzy mappings satisfying for every pair $(x, y) \in X \times X$ such that x and y are comparable,*

$$\begin{aligned}\varphi_1(\delta_1(Sx, Ty)) \leq\ & \psi_1 \Big(d(Ix, Iy), D_1(Ix, Sx), D_1(Iy, Ty), \\ & \frac{1}{2}[D_1(Ix, Ty) + D_1(Iy, Sx)]\Big) \\ & -\psi_2 \Big(d(Ix, Jy), D_1(Ix, Sx), D_1(Iy, Ty), \\ & \frac{1}{2}[D_1(Ix, Ty) + D_1(Iy, Sx)]\Big) \qquad (3.31)\end{aligned}$$

where ψ_1 and ψ_2 are generalized altering distance functions (in Ψ_4) and $\varphi_1(x) = \psi_1(x, x, x, x)$. Suppose that

(i) *(I, T) and (I, S) be partially weakly increasing,*

(ii) *$T(X) \subseteq I(X)$ and $S(X) \subseteq I(X)$,*

(iii) *S and T are dominating maps,*

(iv) *T is weak annihilator of I and S is weak annihilator of I,*

(v) *if for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies that $x_n \leq u$.*

Assume either

(a) *$\{S, I\}$ are comparable, S or I is continuous $\{$and$\}$ T, I are weakly compa- rable or*

(b) *$\{T, I\}$ are comparable, T or I is continuous and $\{S, I\}$ are weakly comparable .*

Then S, T, I have a common fixed point. Moreover, the set of common fixed points of S, T, I is totally ordered if and only if S, T, I have one and only one common fixed point.

Proof. It follows by taking $J = I$ in Theorem 3.1.

Corollary 3.7. *Let $(X, d, \leq)$ be an ordered complete metric linear space. Let $T, I, J : X \to W(X)$ be three fuzzy mappings satisfying for every pair $(x, y) \in X \times X$ such that x and y are comparable,*

$$\begin{aligned}\varphi_1(\delta_1(Tx, Ty)) \leq\ & \psi_1\,(d(Ix, Jy), D_1(Ix, Tx), D_1(Jy, Ty), \\ & \tfrac{1}{2}[D_1(Ix, Ty) + D_1(Jy, Tx)] \\ & -\psi_2\,(d(Ix, Jy), D_1(Ix, Tx), D_1(Jy, Ty), \\ & \tfrac{1}{2}[D_1(Ix, Ty) + D_1(Jy, Tx)] \qquad (3.32)\end{aligned}$$

where ψ_1 and ψ_2 are generalized altering distance functions (in Ψ_4) and $\varphi_1(x) = \psi_1(x, x, x, x)$. Suppose that

(i) *(I, T) and (J, T) be partially weakly increasing,*
(ii) *$T(X) \subseteq I(X)$ and $T(X) \subseteq J(X)$,*
(iii) *T is dominating maps,*
(iv) *T is weak annihilator of I and J,*
(v) *if for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies that $x_n \leq u$.*

Assume either

(a) *$\{T, I\}$ are comparable, T or I is continuous and $\{T, J\}$ are weakly comparable or*
(b) *$\{T, J\}$ are comparable, T or J is continuous and $\{T, I\}$ are weakly comparable .*

Then T, I and J have a common fixed point. Moreover, the set of common fixed points of T, I and J is totally ordered if and only if T, I and J have one and only one common fixed point.

Proof. It follows by taking $S = T$ in Theorem 3.1.

Corollary 3.8. *Let $(X, d, \leq)$ be an ordered complete metric linear space. Let $T, I : X \to W(X)$ be three fuzzy mappings satisfying for every pair $(x, y) \in X \times X$ such that x and y are comparable,*

$$\begin{aligned}\varphi_1(\delta_1(Tx, Ty)) \leq\ & \psi_1\,(d(Ix, Iy), D_1(Ix, Tx), D_1(Iy, Ty), \\ & \tfrac{1}{2}[D_1(Ix, Ty) + D_1(Iy, Tx)] \\ & -\psi_2\,(d(Ix, Iy), D_1(Ix, Tx), D_1(Iy, Ty), \\ & \tfrac{1}{2}[D_1(Ix, Ty) + D_1(Iy, Tx)] \qquad (3.33)\end{aligned}$$

where ψ_1 and ψ_2 are generalized altering distance functions (in Ψ_4) and $\varphi_1(x) = \psi_1(x, x, x, x)$. Suppose that

(i) *(I, T) be partially weakly increasing,*
(ii) *$T(X) \subseteq I(X)$,*
(iii) *T is dominating maps,*
(iv) *T is weak annihilator of I,*

(v) *if for a nondecreasing sequence* $\{x_n\}$ *with* $x_n \leq y_n$ *for all* n *and* $y_n \to u$ *implies that* $x_n \leq u$,

(vi) $\{T, I\}$ *are comparable,* T *or* I *is continuous.*

Then T, I *have a common fixed point. Moreover, the set of common fixed points of* T, I *is totally ordered if and only if* T, I *have one and only one common fixed point.*

Proof. It follows by taking $S = T$ and $J = I$ in Theorem 3.1.

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