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A Parametric Study of Tangents, Normals & Conjugate Diameters of An Ellipse

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ABSTRACT

This research article delves into the geometric and mathematical significance of the Ellipse; a fundamental conic section defined as the locus of a point such that the ratio of its distance from a fixed point (focus) to a fixed line (directrix) is a constant less than or equal to one. Beyond its classical geometric definition, the ellipse holds substantial importance in astronomy, particularly through Kepler's laws of planetary motion, where celestial bodies orbit stars in elliptical paths.

The study presents a collection of Twenty-nine newly established theorems focusing on the ellipse's tangent & normal of pair conjugate diameters. Each theorem is rigorously formulated, accompanied by detailed mathematical proofs and illustrative diagrams. Furthermore, the author introduces an additional set of 29 original theorems describing precise mathematical relationships between tangent, normal of pair conjugate diameters of an ellipse, and other key elements of the ellipse. Basic fundamental formulae required for derivations have been given in Preamble & Table-1. These contributions offer valuable insights and serve as significant references for scholars pursuing advanced research in geometry and related disciplines.

Keywords: Astronomy, Planetary motion, Ellipse, Conic sections.

INTRODUTION

The ellipse, a key conic section, is defined as the set of points where the sum of distances to two fixed points (foci) remains constant. It has wide-ranging applications in mathematics, physics, astronomy, optics, and engineering due to its rich geometric and physical properties.

A notable feature in ellipse geometry is the concept of conjugate diameters is a pair of diameters where each bisects all chords parallel to the other. These diameters provide insight into the shape and orientation of the ellipse, with one aligned to a set of parallel chords and the other formed by the midpoints of those chords.

The ellipse's reflective property, where rays from one focus reflect to the other, underpins its role in acoustics and optics. Additionally, the behavior of tangents and normals relative to conjugate diameters offers deeper geometric understanding about an ellipse.

While classical properties are well studied, this article introduces and proves several new theorems using parametric equations, focusing on the interplay between tangents, normals, and conjugate diameters. These findings enrich the theoretical framework of ellipse geometry and support further research across scientific disciplines.

PREAMBLE- A

Common description of the ellipse with respect to point P_1 for the entire article is

1. Point O is centre of ellipse,
2. a & b are semi-major axis [1] & semi-minor axis [2] of the ellipse respectively.
3. T_1S_1 is a tangent [3] & P_1N_1 is normal [4] are drawn on ellipse at point 'P' respectively.
4. Points T_1 & S_1 are the intersection of tangent with transverse axis [5] & conjugate axis [6] respectively.
5. Points N_1 & M_1 are intersection of normal with transverse axis & conjugate axis respectively.
6. OT_1 & OS_1 are x-intercepts [7] & y-intercepts [8] of tangent respectively.
7. OT_1 is called as sub-tangent [9] with respect to transverse axis.
8. ON_1 is called as sub-normal [10] with respect to transverse axis.
9. Points Q_1 & R_1 are the projection of point P_1 on transverse axis & conjugate axis respectively.
10. OQ_1 & OR_1 are abscissa [11] & ordinate [12] of the point P_1 respectively.
11. Points F_1 & F_2 are the foci [13] of the ellipse.

Referring fig.1,

Point U_1 is intersection of auxiliary circle [14] and semi-diameter [15] drawn at angle θ°, which is called as eccentric angle [16] for point P_1.

Let, $\angle U_1OQ_1 = \theta°$, hence $\angle Q_1T_1U_1 = (90° - \theta°), \angle S_1OV_1 = (90° - \theta°), \angle OS_1V_1 = \theta, °OU_1 = a,$

$OV_1 = b$, parametric equation of ellipse [17] is $P\ (x, y) = a\cos(\theta°)\ , b\sin(\theta°)$.

$$\therefore OQ_1 = a\cos(\theta°) \quad (A.1)$$

$$\therefore OR_1 = P_1Q_1 = b\sin(\theta°) \quad (A.2)$$

In right $\Delta\ OQ_1U_1, \angle Q_1T_1U_1 = (90° - \theta°)$

$$\sin(\theta°) = \frac{U_1Q_1}{OU_1}$$

$$\therefore U_1Q_1 = OU_1 \times \sin(\theta°)$$

$$\therefore U_1Q_1 = a \times \sin(\theta°) \quad (A.3)$$

In right $\Delta\ T_1Q_1U_1, \angle Q_1T_1U_1 = (90° - \theta°)$

$$\tan(90° - \theta°) = \frac{U_1Q_1}{Q_1T_1}$$

$$\therefore \cot(\theta°) = \frac{U_1Q_1}{Q_1T_1}$$

$$\therefore \tan(\theta°) = \frac{Q_1T_1}{U_1Q_1}$$

$$\therefore Q_1T_1 = U_1Q_1 \times \tan(\theta°)$$

Substituting eqn. (A.3) in above,

$$\therefore Q_1T_1 = a\sin(\theta°) \times \tan(\theta°)$$

$$\therefore Q_1T_1 = \frac{a\sin^2(\theta°)}{\cos(\theta°)} \quad (A.4)$$

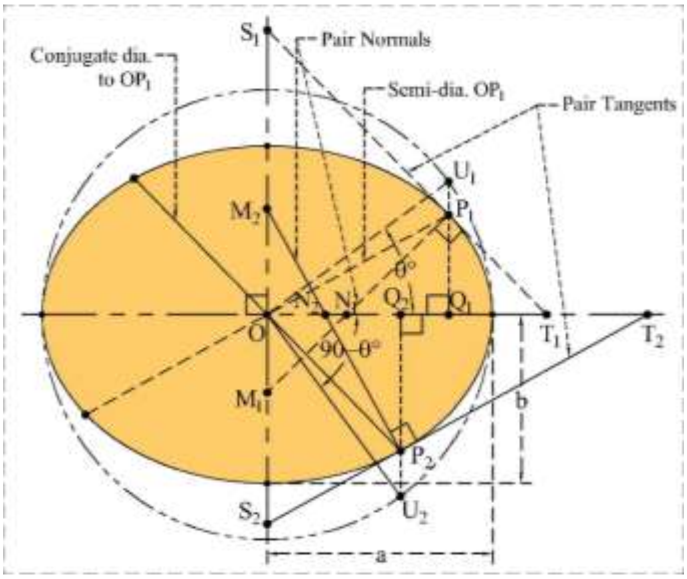

Fig: 1

Referring fig.1, $OT_1 = OQ_1 + Q_1T_1$

Substituting eqn. (A.1) in above,

$$\therefore OT_1 = a\cos(\theta°) + \left(\frac{a\sin^2(\theta°)}{\cos(\theta°)}\right)$$

Simplifying the above eqn.,

$$\therefore OT_1 = \frac{a}{\cos(\theta°)} \qquad (A.5)$$

In right $\Delta\, OR_1V_1, \angle R_1OV_1 = (90° - \theta°)$

$$\tan(90° - \theta°) = \frac{V_1R_1}{OR_1}$$

$$\therefore \cot(\theta°) = \frac{V_1R_1}{OR_1}$$

$$\therefore \tan(\theta°) = \frac{OR_1}{V_1R_1}$$

$$\therefore V_1R_1 = \frac{OR_1}{\tan(\theta°)}$$

Substituting eqn. (A.2) in above,

$$\therefore V_1R_1 = \frac{b\,\cancel{\sin(\theta°)} \times \cos(\theta°)}{\cancel{\sin(\theta°)}}$$

$$\therefore V_1R_1 = b\cos(\theta°) \qquad (A.6)$$

In right $\Delta\, S_1R_1V_1, \angle R_1S_1V_1 = (\theta°)$

$$\tan(\theta°) = \frac{V_1R_1}{R_1S_1}$$

$$\therefore R_1S_1 = \frac{V_1R_1}{\tan(\theta°)}$$

Substituting eqn. (A.6) in above,

$$\therefore R_1S_1 = \frac{b\,cos(\theta°) \times cos(\theta°)}{sinn(\theta°)}$$

$$\therefore R_1S_1 = \frac{b\,cos^2(\theta°)}{sin(\theta°)} \quad (A.7)$$

In fig.1, $OS_1 = OR_1 + R_1S_1$

Substituting eqns. (A.2) and (A.7) in above,

$$\therefore OS_1 = bsin(\theta°) + \frac{b\,cos^2(\theta°)}{sin(\theta°)}$$

Simplifying the above eqn.,

$$\therefore OS_1 = \frac{b}{sin(\theta°)} \quad (A.8)$$

Multiplying eqns. (A.1) and (A.5),

$$OQ_1 \times OT_1 = a\,\cancel{cos(\theta°)} \times \frac{a}{\cancel{cos(\theta°)}}$$

$$\therefore OQ_1 \times OT_1 = a^2 \quad (A.9)$$

Multiplying eqns. (A.2) and (A.5),

$$OR_1 \times OS_1 = b\,\cancel{sin(\theta°)} \times \frac{b}{\cancel{sin(\theta°)}}$$

$$\therefore OR_1 \times OS_1 = b^2 \quad (A.10)$$

According to Pythagoras theorem on right $\Delta\, T_1Q_1P_1$,

$$P_1T_1{}^2 = Q_1T_1{}^2 + P_1Q_1{}^2$$

Substituting eqns. (A.4) and (A.2) in above,

$$\therefore P_1T_1{}^2 = \left(\frac{asin^2(\theta°)}{cos(\theta°)}\right)^2 + [bsin(\theta°)]^2$$

Simplifying the above eqn.,

$$\therefore P_1T_1{}^2 = \frac{sin^2(\theta°)[a^2sin^2(\theta°) + b^2\,cos^2(\theta°)]}{cos^2(\theta°)}$$

$$\therefore P_1T_1 = \frac{sin(\theta°)\sqrt{a^2sin^2(\theta°) + b^2\,cos^2(\theta°)}}{cos(\theta°)} \quad (A.12)$$

According to Pythagoras theorem on right $\Delta\, S_1R_1P_1$,

$$P_1S_1{}^2 = R_1S_1{}^2 + P_1R_1{}^2$$

$$\therefore P_1S_1{}^2 = R_1S_1{}^2 + OQ_1{}^2$$

Substituting eqns. (A.7) and (A.1) in above,

$$\therefore P_1S_1{}^2 = \left(\frac{b\,cos^2(\theta°)}{sin(\theta°)}\right)^2 + [a\,cos(\theta°)]^2$$

Simplifying the above eqn.,

$$\therefore P_1S_1{}^2 = \frac{cos^2(\theta°)[b^2cos^2(\theta°) + a^2sin^2(\theta°)]}{sin^2(\theta°)}$$

$$\therefore P_1S_1 = \frac{cos(\theta°)\sqrt{a^2sin^2(\theta°) + b^2\,cos^2(\theta°)}}{sin(\theta°)} \qquad (A.13)$$

According to Pythagoras theorem on right $\Delta\, T_1OS_1$,

$$T_1S_1{}^2 = OT_1{}^2 + OS_1{}^2$$

Adding eqns. (A.5) & (A.8)

$$T_1S_1{}^2 = \left(\frac{a}{cos(\theta°)}\right)^2 + \left(\frac{b}{sin(\theta°)}\right)^2$$

Simplifying the above eqn.,

$$T_1S_1{}^2 = \frac{a^2sin^2(\theta°) + b^2\,cos^2(\theta°)}{sin^2(\theta°)\,cos^2(\theta°)}$$

$$\therefore T_1S_1 = \frac{\sqrt{a^2sin^2(\theta°) + b^2\,cos^2(\theta°)}}{sin(\theta°) \times cos(\theta°)} \qquad (A.14)$$

$In\ right\ triangle\ P_1Q_1N_1, \qquad tan(\alpha°) = \frac{P_1Q_1}{Q_1T_1}$

$$tan(\alpha°) = \frac{P_1Q_1}{Q_1T_1} = \frac{OR_1}{Q_1T_1}$$

Substituting eqn. (A.2) & (A.4) in above,

$$tan(\alpha°) = bsin(\theta°) \div \left(\frac{asin^2(\theta°)}{cos(\theta°)}\right)$$

$$tan(\alpha°) = bsin(\theta°) \times \left(\frac{cos(\theta°)}{asin^2(\theta°)}\right)$$

$$\therefore tan(\alpha°) = \frac{b\,cos(\theta°)}{asin(\theta°)} \qquad (A.15)$$

$In\ right\ triangle\ P_1Q_1N_1, \angle Q_1P_1N_1 = \alpha°$

$$\therefore tan(\alpha°) = \frac{N_1Q_1}{P_1Q_1}$$

$$\therefore N_1Q_1 = OR_1 \times tan(\alpha°)$$

$$\therefore N_1Q_1 = bsin(\theta°) \times tan(\alpha°)$$

Substituting eqn. (A.15) in above,

$$\therefore N_1Q_1 = bsin(\theta°) \times \frac{b\,cos(\theta°)}{asin(\theta°)}$$

$$\therefore N_1Q_1 = \frac{b^2cos(\theta°)}{a} \qquad (A.16)$$

$$ON_1 = OQ_1 - N_1Q_1$$

Substituting eqns. (A.1) & (A.16) in above,

$\therefore ON_1 = acos(\theta°) - \left(\frac{b^2 cos(\theta°)}{a}\right)$

$Simplifying\ the\ above\ eqn.,$

$$\therefore ON_1 = \frac{(a^2 - b^2)cos(\theta°)}{a} \qquad (A.17)$$

$R_1S_1 = OS_1 - OR_1$

$Substituting\ eqn.(A.8)and\ (A.2)in\ above\ eqn.,$

$\therefore R_1S_1 = \left(\frac{b}{sin(\theta°)}\right) - bsin(\theta°)$

$Simplifying\ the\ above\ eqn.,$

$$\therefore R_1S_1 = \frac{bcos^2(\theta°)}{sin(\theta°)} \qquad (A.18)$$

$In\ right\ triangle\ N_1OM_1, \angle N_1OM_1 = \alpha°.$

$In\ this\ right\ triangle, tan(\alpha°) = \frac{ON_1}{OM_1}$

$\therefore OM_1 = \frac{ON_1}{tan(\alpha°)}$

Substituting eqn. (A.17) in above eqn.,

$\therefore OM_1 = \left(\frac{(a^2 - b^2)cos(\theta°)}{a}\right) \div tan(\alpha°)$

Substituting eqn. (A.15) in above,

$\therefore OM_1 = \left(\frac{(a^2 - b^2)cos(\theta°)}{a}\right) \div \left(\frac{b\ cos(\theta°)}{asin(\theta°)}\right)$

$Simplifying\ the\ above\ eqn.,$

$$\therefore OM_1 = \frac{(a^2 - b^2)sin(\theta°)}{b} \qquad (A.19)$$

Referring fig. A.2, $R_1M_1 = OR_1 + OM_1$

Substituting eqn. (A.19) and $OR = bsin(\theta°)$ in above,

$\therefore R_1M_1 = bsin(\theta°) + \left(\frac{(a^2 - b^2)sin(\theta°)}{b}\right)$

$Simplifying\ the\ above\ eqn.,$

$$\therefore R_1M_1 = \frac{a^2 sin(\theta°)}{b} \qquad (A.20)$$

According to Pythagoras theorem on right $\Delta\ N_1Q_1P_1$,

$P_1{N_1}^2 = N_1Q^2 + P_1{Q_1}^2$

Substituting eqn. (A.16) and (A.2) in above eqn.,

$P_1{N_1}^2 = \left(\frac{b^2 cos(\theta°)}{a}\right)^2 + \left(bsin(\theta°)\right)^2$

Simplifying the above eqn.,

$$\therefore P_1N_1{}^2 = \frac{b^2[a^2sin^2(\theta°) + b^2cos^2(\theta°)]}{a^2}$$

$$\therefore P_1N_1 = \frac{b\sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)}}{a} \qquad (A.21)$$

$$M_1N_1{}^2 = OM_1{}^2 + ON_1{}^2$$

Substituting eqn. (A.19) and eqn. (A.17) in above, we get

$$M_1N_1{}^2 = \left(\frac{(a^2 - b^2)sin(\theta°)}{b}\right)^2 + \left(\frac{(a^2 - b^2)cos(\theta°)}{a}\right)^2$$

Simplifying the above eqn.,

$$\therefore M_1N_1{}^2 = \frac{(a^2 - b^2)^2[a^2sin^2(\theta°) + b^2cos^2(\theta°)]}{a^2b^2}$$

$$\therefore M_1N_1 = \frac{(a^2 - b^2)\sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)}}{ab} \qquad (A.22)$$

Referring fig. 1, $P_1M_1 = P_1N_1 + M_1N_1$

Adding eqns. $(A.21)$ *and* $(A.22)$,

$$\therefore P_1M_1 = P_1N_1 + M_1N_1 = \left(\frac{(a^2 - b^2)\sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)}}{ab}\right) + \left(\frac{b\sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)}}{a}\right)$$

$$\therefore P_1M_1 = \frac{a^{\cancel{2}}\sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)} \cancel{- b^2\sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)} + b^2\sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)}}}{\cancel{a}b}$$

$$\therefore P_1M_1 = \frac{a\sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)}}{b} \qquad (A.23)$$

Referring the fig. 1, $N_1T_1 = OT_1 - ON_1$

Substituting the eqn. $(A.5)$ *and* $(A.17)$ *in above*

$$\therefore N_1T_1 = \left(\frac{a}{cos(\theta°)}\right) - \left(\frac{(a^2 - b^2)cos(\theta°)}{a}\right)$$

Simplifying the above eqn.,

$$\therefore N_1T_1 = \frac{a^2sin^2(\theta°) + b^2cos^2(\theta°)}{acos(\theta°)} \qquad (A.24)$$

In fig. 1, $M_1S_1 = OS_1 + OM_1$

Substituting the eqn. $(A.8)$ *and* $(A.19)$ *in above*

$$\therefore M_1S_1 = \left(\frac{b}{sin(\theta°)}\right) + \left(\frac{(a^2 - b^2)sin(\theta°)}{b}\right)$$

Simplifying the above eqn.,

$\therefore M_1S_1$

$$= \frac{a^2 sin^2(\theta°) + b^2 cos^2(\theta°)}{b sin(\theta°)} \quad (A.25)$$

$Eqn. (A.8) \Longrightarrow OS_1 = \dfrac{b}{sin(\theta°)}$

In right-triangle T_1OS_1, as per Pythagoras theorem

$S_1{F_1}^2 = O{F_1}^2 + O{S_1}^2$

$\therefore S_1{F_1}^2 = \left(\sqrt{a^2 - b^2}\right)^2 + \left(\dfrac{b}{sin(\theta°)}\right)^2$

$Simplifying\ the\ above\ eqn.,$

$\therefore S_1{F_1}^2 = \dfrac{a^2 sin^2(\theta°) + b^2 \cos^2 \theta°}{sin^2(\theta°)}$

$\therefore S_1F_1 = S_1F_2$

$$= \frac{\sqrt{a^2 sin^2(\theta°) + b^2 \cos^2 \theta°}}{sin(\theta°)} \quad (A.26)$$

$Eqn. (A.5) \Longrightarrow OT_1 = \dfrac{a}{cos(\theta°)}$

We already know that $OF_1 = OF_2 = \sqrt{a^2 - b^2}$ which are linear eccentricity of an ellipse

Referring fig. 1, $T_1F_1 = OT_1 - OF_1$

$Substituting\ OT_1 = \dfrac{a}{cos(\theta°)}\ \&\ OF_1 = \sqrt{a^2 - b^2}\ in\ above\ eqn,$

$\therefore T_1F_1 = \left(\dfrac{a}{cos(\theta°)}\right) - \sqrt{a^2 - b^2}$

$\therefore T_1F_1$

$$= \frac{a - \left[\sqrt{a^2 - b^2} \times cos(\theta°)\right]}{cos(\theta°)} \quad (A.27)$$

Referring fig. 1, $T_1F_2 = OT + OF_2$

$\therefore T_1F_2 = \left(\dfrac{a}{cos(\theta°)}\right) + \sqrt{a^2 - b^2}$

$\therefore T_1F_2$

$$= \frac{a + \left[\sqrt{a^2 - b^2} \times cos(\theta°)\right]}{cos(\theta°)} \quad (A.28)$$

$Eqn. (A.5) \Longrightarrow ON_1 = \dfrac{(a^2 - b^2)cos(\theta°)}{a}$

In fig. 1, we know that $N_1F_1 = OF_1 - ON_1$

Substituting eqn. $OF_1 = \sqrt{a^2 - b^2}$ and eqn. (A.5) in above,

$\therefore N_1F_1 = \sqrt{a^2 - b^2} - \left(\dfrac{(a^2 - b^2)cos(\theta°)}{a}\right)$

$\therefore N_1F_1$

$$= \frac{a\sqrt{a^2 - b^2} - (a^2 - b^2)cos(\theta°)}{a} \quad (A.29)$$

In fig. 1, we know that $N_1F_2 = F_1F_2 - N_1F_1$

Substituting eqns. $F_1F_2 = 2\sqrt{a^2 - b^2}$ and eqn. (A.29) in above, we get

$$N_1F_2 = 2\sqrt{a^2 - b^2} - \left(\frac{a\sqrt{a^2 - b^2} - (a^2 - b^2)cos(\theta°)}{a}\right)$$

$$\therefore N_1F_2 = \frac{a\sqrt{a^2 - b^2} + (a^2 - b^2)cos(\theta°)}{a} \qquad (A.30)$$

In fig. 1, we know that, ${M_1Q_1}^2 = {OQ_1}^2 + {OM_1}^2$

Substituting eqns. (A.1) & (A.29) in above, we get

$${M_1Q_1}^2 = [a\,cos(\theta°)]^2 + \left(\frac{(a^2 - b^2)sin(\theta°)}{b}\right)^2$$

Simplifying the above eqn.,

$${M_1Q_1}^2 = \frac{a^2b^2\,cos^2\,\theta° + (a^2 - b^2)^2 sin^2(\theta°)}{b^2} \qquad (A.31)$$

In fig. 1, we know that, ${N_1R_1}^2 = {ON_1}^2 + {OR_1}^2$

Substituting eqns. (A.17) & (A.2) in above, we get

$${N_1R_1}^2 = \left(\frac{(a^2 - b^2)cos(\theta°)}{a}\right)^2 + [b\,sin(\theta°)]^2$$

Simplifying the above eqn.,

$${N_1R_1}^2 = \frac{a^2b^2 sin^2(\theta°) + (a^2 - b^2)^2 cos^2(\theta°)}{a^2} \qquad (A.32)$$

We already known that the parametric equation of ellipse is $P_1\,(x, y) = (a\ cos\,\theta°, b\,sin\,\theta°)$. $\theta°$ is eccentric angle of the ellipse at point P_1.

Referring fig. 1. In right-triangle OU_1Q_1, $\angle\,OU_1Q_1 = 90°$ and $\angle\,U_1OQ_1 = \theta°$

P_1F_1 & P_1F_2 are pair focal distances at a point P

$${P_1F_1}^2 = {F_1Q_1}^2 + {P_1Q_1}^2$$

$$\therefore {P_1F_1}^2 = (OQ_1 - OF_1)^2 + {P_1Q_1}^2$$

$\therefore {P_1F_1}^2 = \left(a\,cos\,\theta° - \sqrt{a^2 - b^2}\right)^2 + (b\,sin\,\theta°)^2$ [where $OF_1 = \sqrt{a^2 - b^2}$ is called linear eccentricity

$$\therefore {P_1F_1}^2 = a^2\,cos^2\,\theta° + a^2 - b^2 - 2a\sqrt{a^2 - b^2}\,cos\,\theta° + b^2\,sin^2\,\theta°$$

$$\therefore {P_1F_1}^2 = a^2\,cos^2\,\theta° + a^2 - 2a\sqrt{a^2 - b^2}\,cos\,\theta° + b^2\,sin^2\,\theta° - b^2$$

$$\therefore {P_1F_1}^2 = a^2\,cos^2\,\theta° + a^2 - 2a\sqrt{a^2 - b^2}\,cos\,\theta° + b^2[sin^2\,\theta° - 1]$$

$$\therefore {P_1F_1}^2 = a^2\,cos^2\,\theta° + a^2 - 2a\sqrt{a^2 - b^2}\,cos\,\theta° - b^2\,cos^2\,\theta°$$

$$\therefore {P_1F_1}^2 = a^2 + a^2\,cos^2\,\theta° - b^2\,cos^2\,\theta° - 2a\sqrt{a^2 - b^2}\,cos\,\theta°$$

$$\therefore {P_1F_1}^2 = a^2 + (a^2 - b^2)\,cos^2\,\theta° - 2a\sqrt{a^2 - b^2}\,cos\,\theta°$$

$$\therefore {P_1F_1}^2 = \left(a - \sqrt{a^2 - b^2}\,cos\,\theta°\right)^2$$

$$\therefore P_1F_1 = a - \left(\sqrt{a^2 - b^2} \times cos\,\theta°\right) \qquad (A.33)$$

We already known that the parametric equation of ellipse is $P_1\,(x, y) = (a\ cos\,\theta°\,, b\,sin\,\theta°)$. $\theta°$ is eccentric angle of the ellipse.
Referring fig. 1, in right-triangle OU_1Q_1, $\angle\,OU_1Q_1 = 90°$ and $\angle\,U_1OQ_1 = \theta°$
P_1F_1 & P_1F_2 are pair focal distances at a point P_1

$P_1F_2{}^2 = F_2Q_1{}^2 + P_1Q_1{}^2$

$\therefore P_1F_2{}^2 = (OQ_1 + OF_2)^2 + P_1Q_1{}^2$

$\therefore P_1F_2{}^2 = \left(a\,cos\,\theta° + \sqrt{a^2 - b^2}\right)^2 + (b\,sin\,\theta°)^2$ [where, $OF_2 = \sqrt{a^2 - b^2}$ is called linear eccentricity

$\therefore P_1F_2{}^2 = a^2\,cos^2\,\theta° + a^2 - b^2 + 2a\sqrt{a^2 - b^2}\,cos\,\theta° + b^2\,sin^2\,\theta°$

$\therefore P_1F_2{}^2 = a^2 + a^2\,cos^2\,\theta° + 2a\sqrt{a^2 - b^2}\,cos\,\theta° + b^2(sin^2\,\theta° - 1)$

$\therefore P_1F_2{}^2 = a^2 + a^2\,cos^2\,\theta° + 2a\sqrt{a^2 - b^2}\,cos\,\theta° - b^2\,cos^2\,\theta°$

$\therefore P_1F_2{}^2 = a^2 + a^2\,cos^2\,\theta° - b^2\,cos^2\,\theta° + 2a\sqrt{a^2 - b^2}\,cos\,\theta°$

$\therefore P_1F_2{}^2 = a^2 + (a^2 - b^2)\,cos^2\,\theta° + 2a\sqrt{a^2 - b^2}\,cos\,\theta°$

$\therefore P_1F_2{}^2 = \left(a + \sqrt{a^2 - b^2}\,cos\,\theta°\right)^2$

$$\therefore P_1F_2 = a + \left(\sqrt{a^2 - b^2} \times cos\,\theta°\right) \qquad (A.34)$$

The above calculated equation for elements of ellipse with respect to both semi-diameter with respect to P_1 are given in L.H.S column in the following table-1.

PREAMBLE- B

If OP_1 & OP_2 are pair conjugate diameters [18] and if $\angle T_1OU_1 = \theta°$, $\angle T_2OU_2 = (90° - \theta°)$, $OU_2 = a, OV_2 = b$, parametric co-ordinates of $P_2 = a\,sin(\theta°)\,, b\,cos(\theta°)$. Point P_2 is tangent point of semi-conjugate diameter.
Referring fig. 1, the elements of ellipse with respect to semi-conjugate diameter (OP_2) is calculated by substituting $(90° - \theta°)$ in place of $\theta°$. The calculated equation for elements of ellipse with respect to both semi-diameter and semi-conjugate diameter (OP_2) are given in R.H.S column in the following table-1.

Table-1

Eqn. No	Eqns. for Semi-diameter OP_1	Eqn. No	Eqns. for Semi-conjugate diameter OP_2
A.1	$OQ_1 = a\,cos(\theta°)$	B.1	$OQ_2 = a\,sin(\theta°)$
A.2	$OR_1 = b\,sin(\theta°)$	B.2	$OR_2 = b\,cos(\theta°)$
A.4	$Q_1T_1 = \dfrac{asin^2(\theta°)}{cos(\theta°)}$	B.4	$Q_2T_2 = \dfrac{acos^2(\theta°)}{sin(\theta°)}$
A.5	$OT_1 = \dfrac{a}{cos(\theta°)}$	B.5	$OT_2 = \dfrac{a}{sin(\theta°)}$
A.8	$OS_1 = \dfrac{b}{sin(\theta°)}$	B.8	$OS_2 = \dfrac{b}{cos(\theta°)}$
A.12	$P_1T_1 = \dfrac{sin(\theta°)\sqrt{a^2sin^2(\theta°) + b^2\,cos^2(\theta°)}}{cos(\theta°)}$	B.12	$P_2T_2 = \dfrac{cos(\theta°)\sqrt{a^2cos^2(\theta°) + b^2\,sin^2(\theta°)}}{sin(\theta°)}$

A.13	$P_1S_1 = \frac{cos(\theta°)\sqrt{a^2sin^2(\theta°)+b^2\,cos^2(\theta°)}}{sin(\theta°)}$	B.13	$P_2S_2 = \frac{sin(\theta°)\sqrt{a^2cos^2(\theta°)+b^2\,sin^2(\theta°)}}{cos(\theta°)}$
A.14	$T_1S_1 = \frac{\sqrt{a^2sin^2(\theta°)+b^2\,cos^2(\theta°)}}{sin(\theta°)\times cos(\theta°)}$	B.14	$T_2S_2 = \frac{\sqrt{a^2cos^2(\theta°)+b^2\,sin^2(\theta°)}}{sin(\theta°)\times cos(\theta°)}$
A.16	$N_1Q_1 = \frac{b^2cos(\theta°)}{a}$	B.16	$N_2Q_2 = \frac{b^2sin(\theta°)}{a}$
A.17	$ON_1 = \frac{(a^2-b^2)cos(\theta°)}{a}$	B.17	$ON_2 = \frac{(a^2-b^2)\,sin(\theta°)}{a}$
A.18	$R_1S_1 = \frac{bcos^2(\theta°)}{sin(\theta°)}$	B.18	$R_2S_2 = \frac{bsin^2(\theta°)}{cos(\theta°)}$
A.19	$OM_1 = \frac{(a^2-b^2)sin(\theta°)}{b}$	B.19	$OM_2 = \frac{(a^2-b^2)cos(\theta°)}{b}$
A.20	$R_1M_1 = \frac{a^2sin(\theta°)}{b}$	B.20	$R_2M_2 = \frac{a^2cos(\theta°)}{b}$
A.21	$P_1N_1 = \frac{b\sqrt{a^2sin^2(\theta°)+b^2cos^2(\theta°)}}{a}$	B.21	$P_2N_2 = \frac{b\sqrt{a^2cos^2(\theta°)+b^2sin^2(\theta°)}}{a}$
A.22	$M_1N_1 = \frac{(a^2-b^2)\sqrt{a^2sin^2(\theta°)+b^2cos^2(\theta°)}}{ab}$	B.22	$M_2N_2 = \frac{(a^2-b^2)\sqrt{a^2cos^2(\theta°)+b^2\,sin^2(\theta°)}}{ab}$
A.23	$P_1M_1 = \frac{a\sqrt{a^2sin^2(\theta°)+b^2cos^2(\theta°)}}{b}$	B.23	$P_2M_2 = \frac{a\sqrt{a^2cos^2(\theta°)+b^2sin^2(\theta°)}}{b}$
A.24	$N_1T_1 = \frac{a^2sin^2(\theta°)+b^2cos^2(\theta°)}{acos(\theta°)}$	B.24	$N_2T_2 = \frac{a^2cos^2(\theta°)+b^2\,sin^2(\theta°)}{a\,sin(\theta°)}$
A.25	$M_1S_1 = \frac{a^2sin^2(\theta°)+b^2cos^2(\theta°)}{bsin(\theta°)}$	B.25	$M_2S_2 = \frac{a^2cos^2(\theta°)+b^2sin^2(\theta°)}{bcos(\theta°)}$
A.26	$S_1F_1 = S_1F_2 = \frac{\sqrt{a^2sin^2(\theta°)+b^2\,cos^2\,\theta°}}{sin(\theta°)}$	B.26	$S_2F_1 = S_2F_2 = \frac{\sqrt{a^2cos^2(\theta°)+b^2\,sin^2\,\theta°}}{cos(\theta°)}$
A.27	$T_1F_1 = \frac{a-\left[\sqrt{a^2-b^2}\times cos(\theta°)\right]}{cos(\theta°)}$	B.27	$T_2F_1 = \frac{a-\left[\sqrt{a^2-b^2}\times sin(\theta°)\right]}{sin(\theta°)}$
A.28	$T_1F_2 = \frac{a+\left[\sqrt{a^2-b^2}\times cos(\theta°)\right]}{cos(\theta°)}$	B.28	$T_2F_2 = \frac{a+\left[\sqrt{a^2-b^2}\times sin(\theta°)\right]}{sin(\theta°)}$
A.29	$N_1F_1 = \frac{a\sqrt{a^2-b^2}-(a^2-b^2)cos(\theta°)}{a}$	B.29	$N_2F_1 = \frac{a\sqrt{a^2-b^2}-(a^2-b^2)sin(\theta°)}{a}$
A.30	$N_1F_2 = \frac{a\sqrt{a^2-b^2}+(a^2-b^2)cos(\theta°)}{a}$	B.30	$N_2F_2 = \frac{a\sqrt{a^2-b^2}+(a^2-b^2)sin(\theta°)}{a}$
A.31	${M_1Q_1}^2 = \frac{a^2b^2\,cos^2\,\theta°+(a^2-b^2)^2sin^2(\theta°)}{b^2}$	B.31	${M_2Q_2}^2 = \frac{a^2b^2\,sin^2\,\theta°+(a^2-b^2)^2cos^2(\theta°)}{b^2}$

A.32	${N_1R_1}^2 = \frac{a^2b^2sin^2(\theta°) + (a^2 - b^2)^2cos^2(\theta°)}{a^2}$	B.32	${N_2R_2}^2 = \frac{a^2b^2cos^2(\theta°) + (a^2 - b^2)^2sin^2(\theta°)}{a^2}$
A.33	$P_1F_1 = a - \left(\sqrt{a^2 - b^2} \times cos\,\theta°\right)$	B.33	$P_2F_1 = a - \left(\sqrt{a^2 - b^2} \times sin\,\theta°\right)$
A.34	$P_1F_2 = a + \left(\sqrt{a^2 - b^2} \times cos\,\theta°\right)$	B.34	$P_2F_2 = a + \left(\sqrt{a^2 - b^2} \times sin\,\theta°\right)$
A.35	$OP_1 = \sqrt{a^2cos^2(\theta°) + b^2sin^2(\theta°)}$	B.35	$OP_2 = \sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)}$

ANALYSIS & DERIVATIONS OF THE THEOREMS

THEOREM- 1:

$\boldsymbol{{OQ_1}^2 + {OQ_2}^2 = a^2}$

Derivation of equations for proof of the theorem

$Eqn.(A.1) \Longrightarrow OQ_1 = acos(\theta°)$

$\therefore {OQ_1}^2 = a^2cos^2(\theta°)$ (1.1)

$Eqn.(B.23) \Longrightarrow OQ_2 = asin(\theta°)$

$\therefore {OQ_2}^2 = a^2sin^2(\theta°)$ (1.2)

Adding eqns. (1.1) & (1.2),

${OQ_1}^2 + {OQ_2}^2 = a^2cos^2(\theta°) + a^2sin^2(\theta°)$

$\therefore {OQ_1}^2 + {OQ_2}^2 = a^2[cos^2(\theta°) + sin^2(\theta°)]$

$\therefore {OQ_1}^2 + {OQ_2}^2 = a^2$ (1.3)

Eqn. (1.3) is mathematical expression of the theorem.

THEOREM- 2:

$\boldsymbol{{OR_1}^2 + {OR_2}^2 = b^2}$

Derivation of equations for proof of the theorem

$Eqn.(A.1) \Longrightarrow OQ_1 = bsin(\theta°)$

$\therefore {OR_1}^2 = b^2sin^2(\theta°)$ (2.1)

$Eqn.(B.23) \Longrightarrow OR_2 = bcos(\theta°)$

$\therefore {OR_2}^2 = b^2cos^2(\theta°)$ (2.2)

Adding eqns. (2.1) & (2.2),

${OR_1}^2 + {OR_2}^2 = b^2sin^2(\theta°) + b^2cos^2(\theta°)$

$\therefore {OR_1}^2 + {OR_2}^2 = b^2[sin^2(\theta°) + cos^2(\theta°)]$

$\therefore {OR_1}^2 + {OR_2}^2 = b^2$ (2.3)

Eqn. (2.3) is mathematical expression of the theorem.

THEOREM- 3:

$$(\boldsymbol{OQ_1} \times \boldsymbol{OR_2}) + (\boldsymbol{OR_1} \times \boldsymbol{OQ_2}) = \boldsymbol{ab}$$

Derivation of equations for proof of the theorem

$Eqn.(A.1) \Rightarrow OQ_1$

$= acos(\theta°)$ (3.1)

$Eqn.(A.2) \Rightarrow OR_1$

$= bsin(\theta°)$ (3.2)

$Eqn.(B.1) \Rightarrow OQ_2$

$= asin(\theta°)$ (3.3)

$Eqn.(B.2) \Rightarrow OR_2$

$= bcos(\theta°)$ (3.4)

Multiplying eqns. (3.1) & (3.4),

$OQ_1 \times OR_2 = acos(\theta°) \times bcos(\theta°)$

$\therefore OQ_1 \times OR_2$

$= abcos^2(\theta°)$ (3.5)

Multiplying eqns. (3.2) & (3.3),

$OQ_2 \times OR_1 = asin(\theta°) \times bsin(\theta°)$

$\therefore OQ_2 \times OR_1$

$= absin^2(\theta°)$ (3.6)

Adding eqns. (3.5) & (3.6),

$(OQ_1 \times OR_2) + (OQ_2 \times OR_1) = abcos^2(\theta°) + absin^2(\theta°)$

$\therefore (OQ_1 \times OR_2) + (OQ_2 \times OR_1) = ab[cos^2(\theta°) + sin^2(\theta°)]$

$\therefore (OQ_1 \times OR_2) + (OQ_2 \times OR_1)$

$= ab$ (3.7)

Eqn. (3.7) is mathematical expression of the theorem.

THEOREM- 4:

$$\left(\frac{\mathbf{1}}{\boldsymbol{OT_1} \times \boldsymbol{OS_2}}\right) + \left(\frac{\mathbf{1}}{\boldsymbol{OT_2} \times \boldsymbol{OS_1}}\right) = \frac{\mathbf{1}}{\boldsymbol{ab}}$$

Derivation of equations for proof of the theorem

$Eqn.(A.5) \Rightarrow OT_1$

$= \frac{a}{cos(\theta°)}$ (4.1)

$Eqn.(A.8) \Rightarrow OS_1$

$= \frac{b}{sin(\theta°)}$ (4.2)

$Eqn.(B.5) \Rightarrow OT_2$

$= \frac{a}{sin(\theta°)}$ (4.3)

$Eqn.(B.8) \Rightarrow OS_2$

$= \frac{b}{cos(\theta°)}$ (4.4)

Multiplying eqns. (4.1) & (4.4),

$$OT_1 \times OS_2 = \left(\frac{a}{cos(\theta°)}\right) \times \left(\frac{b}{cos(\theta°)}\right)$$

$$\therefore OT_1 \times OS_2 = \frac{ab}{cos^2(\theta°)} \qquad (4.5)$$

Multiplying eqns. (4.2) & (4.3),

$$OT_2 \times OS_1 = \left(\frac{a}{sin(\theta°)}\right) \times \left(\frac{b}{sin(\theta°)}\right)$$

$$\therefore OT_2 \times OS_1 = \frac{ab}{sin^2(\theta°)} \qquad (4.6)$$

Adding reciprocal of eqns. (4.5) & (4.6),

$$\left(\frac{1}{OT_1 \times OS_2}\right) + \left(\frac{1}{OT_2 \times OS_1}\right) = \frac{cos^2(\theta°)}{ab} + \frac{sin^2(\theta°)}{ab}$$

$$\therefore \left(\frac{1}{OT_1 \times OS_2}\right) + \left(\frac{1}{OT_2 \times OS_1}\right) = \frac{sin^2(\theta°) + cos^2(\theta°)}{ab}$$

$$\therefore \left(\frac{1}{OT_1 \times OS_2}\right) + \left(\frac{1}{OT_2 \times OS_1}\right) = \frac{1}{ab} \qquad (4.7)$$

Eqn. (4.7) is mathematical expression of the theorem.

THEOREM- 5:

$$\boldsymbol{T_1S_1 \times P_2M_2 = T_2S_2 \times P_1M_1}$$

Derivation of equations for proof of the theorem

$$Eqn. (A.14) \Longrightarrow T_1S_1 = \frac{\sqrt{a^2 sin^2(\theta°) + b^2\, cos^2(\theta°)}}{sin(\theta°) \times cos(\theta°)} \qquad (5.1)$$

$$Eqn. (B.23) \Longrightarrow P_2M_2 = \frac{a\sqrt{a^2 cos^2(\theta°) + b^2 sin^2(\theta°)}}{b} \qquad (5.2)$$

$$Eqn. (B.14) \Longrightarrow T_2S_2 = \frac{\sqrt{a^2 cos^2(\theta°) + b^2\, sin^2(\theta°)}}{sin(\theta°) \times cos(\theta°)} \qquad (5.3)$$

$$Eqn. (A.23) \Longrightarrow P_1M_1 = \frac{a\sqrt{a^2 sin^2(\theta°) + b^2 cos^2(\theta°)}}{b} \qquad (5.4)$$

Multiplying eqns. (5.1) & (5.2),

$$T_1S_1 \times P_2M_2 = \left(\frac{\sqrt{a^2 sin^2(\theta°) + b^2\, cos^2(\theta°)}}{sin(\theta°) \times cos(\theta°)}\right) \times \left(\frac{a\sqrt{a^2 cos^2(\theta°) + b^2 sin^2(\theta°)}}{b}\right)$$

$\therefore T_1S_1 \times P_2M_2$

$$= \frac{a \times \sqrt{a^2 sin^2(\theta°) + b^2\, cos^2(\theta°)} \times \sqrt{a^2 cos^2(\theta°) + b^2 sin^2(\theta°)}}{b \times sin(\theta°)cos(\theta°)} \quad (5.5)$$

Multiplying eqns. (5.3) & (5.4),

$$T_2S_2 \times P_1M_1 = \left(\frac{\sqrt{a^2 cos^2(\theta°) + b^2\, sin^2(\theta°)}}{sin(\theta°) \times cos(\theta°)}\right) \times \left(\frac{a\sqrt{a^2 sin^2(\theta°) + b^2 cos^2(\theta°)}}{b}\right)$$

$\therefore T_2S_2 \times P_1M_1$

$$= \frac{a \times \sqrt{a^2 cos^2(\theta°) + b^2 sin^2(\theta°)} \times \sqrt{a^2 sin^2(\theta°) + b^2\, cos^2(\theta°)}}{b \times sin(\theta°)cos(\theta°)} \quad (5.6)$$

Equating eqns. (5.5) & (5.6),

$T_1S_1 \times P_2M_2$

$$= T_2S_2 \times P_1M_1 \quad (5.7)$$

Eqn. (5.7) is mathematical expression of the theorem.

THEOREM- 6:

$\boldsymbol{OP_1 \times P_1M_1 = OP_2 \times P_2M_2}$

Derivation of equations for proof of the theorem

$Eqn. (A.35) \Longrightarrow OP_1$

$$= \sqrt{a^2 cos^2(\theta°) + b^2 sin^2(\theta°)} \quad (6.1)$$

$Eqn. (A.23) \Longrightarrow P_1M_1$

$$= \frac{a\sqrt{a^2 sin^2(\theta°) + b^2 cos^2(\theta°)}}{b} \quad (6.2)$$

$Eqn. (B.35) \Longrightarrow OP_2$

$$= \sqrt{a^2 sin^2(\theta°) + b^2 cos^2(\theta°)} \quad (6.3)$$

$Eqn. (B.23) \Longrightarrow P_2M_2$

$$= \frac{a\sqrt{a^2 cos^2(\theta°) + b^2 sin^2(\theta°)}}{b} \quad (6.4)$$

Multiplying eqns. (6.1) & (6.2),

$$OP_1 \times P_1M_1 = \sqrt{a^2 cos^2(\theta°) + b^2 sin^2(\theta°)} \times \left(\frac{a\sqrt{a^2 sin^2(\theta°) + b^2 cos^2(\theta°)}}{b}\right)$$

$\therefore OP_1 \times P_1M_1$

$$= \frac{a\sqrt{a^2 sin^2(\theta°) + b^2 cos^2(\theta°)} \times \sqrt{a^2 cos^2(\theta°) + b^2 sin^2(\theta°)}}{b} \quad (6.5)$$

Multiplying eqns. (6.3) & (6.4),

$$OP_2 \times P_2M_2 = \sqrt{a^2 sin^2(\theta°) + b^2 cos^2(\theta°)} \times \left(\frac{a\sqrt{a^2 cos^2(\theta°) + b^2 sin^2(\theta°)}}{b}\right)$$

$\therefore OP_2 \times P_2M_2$

$$= \frac{a \times \sqrt{a^2 sin^2(\theta°) + b^2 cos^2(\theta°)} \times \sqrt{a^2 cos^2(\theta°) + b^2 sin^2(\theta°)}}{b} \quad (6.6)$$

Equating eqns. (3.5) & (3.6),

$$OP_1 \times P_1M_1 = OP_2 \times P_2M_2 \tag{6.7}$$

Eqn. (6.7) is mathematical expression of the theorem.

THEOREM- 7:

$$\boldsymbol{P_1N_1 \times P_2M_2 = P_2N_2 \times P_1M_1}$$

Derivation of equations for proof of the theorem

$$Eqn.(A.21) \Longrightarrow P_1N_1 = \frac{b\sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)}}{a} \tag{7.1}$$

$$Eqn.(B.23) \Longrightarrow P_2M_2 = \frac{a\sqrt{a^2cos^2(\theta°) + b^2sin^2(\theta°)}}{b} \tag{7.2}$$

$$Eqn.(B.21) \Longrightarrow P_2N_2 = \frac{b\sqrt{a^2cos^2(\theta°) + b^2sin^2(\theta°)}}{a} \tag{7.3}$$

$$Eqn.(A.23) \Longrightarrow P_1M_1 = \frac{a\sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)}}{b} \tag{7.4}$$

Multiplying eqns. (7.1) & (7.2),

$$P_1N_1 \times P_2M_2 = \left(\frac{\cancel{b}\sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)}}{\cancel{a}}\right) \times \frac{\cancel{a}\sqrt{a^2cos^2(\theta°) + b^2sin^2(\theta°)}}{\cancel{b}}$$

$$\therefore P_1N_1 \times P_2M_2 = \sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)} \times \sqrt{a^2cos^2(\theta°) + b^2sin^2(\theta°)} \tag{7.5}$$

Multiplying eqns. (7.3) & (7.4),

$$P_2N_2 \times P_1M_1 = \left(\frac{\cancel{b}\sqrt{a^2cos^2(\theta°) + b^2sin^2(\theta°)}}{\cancel{a}}\right) \times \left(\frac{\cancel{a}\sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)}}{\cancel{b}}\right)$$

$$\therefore P_2N_2 \times P_1M_1 = \sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)} \times \sqrt{a^2cos^2(\theta°) + b^2sin^2(\theta°)} \tag{7.6}$$

Equating eqns. (7.5) & (7.6),

$$P_1N_1 \times P_2M_2 = P_2N_2 \times P_1M_1 \tag{7.7}$$

Eqn. (7.7) is mathematical expression of the theorem.

THEOREM- 8:

$$\boldsymbol{OT_1 \times OS_1 = OT_2 \times OS_2}$$

Derivation of equations for proof of the theorem

$$Eqn.(A.5) \Longrightarrow OT_1 = \frac{a}{cos(\theta°)} \quad (8.1)$$

$$Eqn.(A.8) \Longrightarrow OS_1 = \frac{b}{sin(\theta°)} \quad (8.2)$$

$$Eqn.(B.5) \Longrightarrow OT_2 = \frac{a}{sin(\theta°)} \quad (8.3)$$

$$Eqn.(B.8) \Longrightarrow OS_2 = \frac{b}{cos(\theta°)} \quad (8.4)$$

Multiplying eqns. (8.1) & (8.2),

$$OT_1 \times OS_1 = \left(\frac{a}{cos(\theta°)}\right) \times \left(\frac{b}{sin(\theta°)}\right)$$

$$\therefore OT_1 \times OS_1 = \frac{ab}{sin(\theta°)\, cos(\theta°)} \quad (8.5)$$

Multiplying eqns. (8.3) & (8.4),

$$OT_2 \times OS_2 = \left(\frac{a}{sin(\theta°)}\right) \times \left(\frac{b}{cos(\theta°)}\right)$$

$$\therefore OT_2 \times OS_2 = \frac{ab}{sin(\theta°)\, cos(\theta°)} \quad (8.6)$$

Equating eqns. (8.5) & (8.6),

$$OT_1 \times OS_1 = OT_2 \times OS_2 \quad (8.7)$$

Eqn. (8.7) is mathematical expression of the theorem.

THEOREM- 9:

$$\boldsymbol{ON_1 \times OM_1 = ON_2 \times OM_2}$$

Derivation of equations for proof of the theorem

$$Eqn.(A.17) \Longrightarrow ON_1 = \frac{(a^2 - b^2)cos(\theta°)}{a} \quad (9.1)$$

$$Eqn.(A.19) \Longrightarrow OM_1 = \frac{(a^2 - b^2)sin(\theta°)}{b} \quad (9.2)$$

$$Eqn.(B.17) \Longrightarrow ON_2 = \frac{(a^2 - b^2)\, sin(\theta°)}{a} \quad (9.3)$$

$$Eqn.(B.19) \Longrightarrow OM_2 = \frac{(a^2 - b^2)cos(\theta°)}{b} \quad (9.4)$$

Multiplying eqns. (9.1) & (9.2),

$$ON_1 \times OM_1 = \left(\frac{(a^2 - b^2)cos(\theta°)}{a}\right) \times \left(\frac{(a^2 - b^2)sin(\theta°)}{b}\right)$$

$$\therefore ON_1 \times OM_1$$
$$= \frac{(a^2 - b^2)^2 sin(\theta°)cos(\theta°)}{ab} \quad (9.5)$$

Multiplying eqns. (9.3) & (9.4),

$$ON_2 \times OM_2 = \left(\frac{(a^2 - b^2)\, sin(\theta°)}{a}\right) \times \left(\frac{(a^2 - b^2)cos(\theta°)}{b}\right)$$

$$\therefore ON_2 \times OM_2$$
$$= \frac{(a^2 - b^2)^2\, sin(\theta°)\, cos(\theta°)}{ab} \quad (9.6)$$

Equating eqns. (9.5) & (9.6),

$$ON_1 \times OM_1 = ON_2 \times OM_2$$

Eqn. (0.0) is mathematical expression of the theorem.

THEOREM- 10:
$$\boldsymbol{OQ_1 \times OR_1 = OQ_2 \times OR_2 = Q_1T_1 \times R_1S_1 = Q_2T_2 \times R_2S_2}$$

Derivation of equations for proof of the theorem

$$Eqn.(A.1) \Longrightarrow OQ_1$$
$$= a\, cos(\theta°) \quad (10.1)$$
$$Eqn.(A.2) \Longrightarrow OR_1$$
$$= b\, sin(\theta°) \quad (10.2)$$
$$Eqn.(B.1) \Longrightarrow OQ_2$$
$$= a\, sin(\theta°) \quad (10.3)$$
$$Eqn.(B.2) \Longrightarrow OR_2$$
$$= b\, cos(\theta°) \quad (10.4)$$
$$Eqn.(A.4) \Longrightarrow Q_1T_1$$
$$= \frac{asin^2(\theta°)}{cos(\theta°)} \quad (10.5)$$
$$Eqn.(A.18) \Longrightarrow R_1S_1$$
$$= \frac{bcos^2(\theta°)}{sin(\theta°)} \quad (10.6)$$
$$Eqn.(B.4) \Longrightarrow Q_2T_2$$
$$= \frac{acos^2(\theta°)}{sin(\theta°)} \quad (10.7)$$
$$Eqn.(B.18) \Longrightarrow R_2S_2$$
$$= \frac{bsin^2(\theta°)}{cos(\theta°)} \quad (10.8)$$

Multiplying eqns. (10.1) & (10.2),

$$OQ_1 \times OR_1 = a\, cos(\theta°) \times b\, sin(\theta°)$$
$$\therefore OQ_1 \times OR_1$$
$$= ab\, sin(\theta°)\, cos(\theta°) \quad (10.9)$$

Multiplying eqns. (10.3) & (10.4),

$OQ_2 \times OR_2 = a\,sin(\theta°) \times b\,cos(\theta°)$

$\therefore OQ_2 \times OR_2$

$= ab\,sin(\theta°)\,cos(\theta°)$ (10.10)

Multiplying eqns. (10.5) & (10.6),

$$Q_1T_1 \times R_1S_1 = \frac{asin^{\not 2}(\theta°)}{\not{cos(\theta°)}} \times \frac{bcos^{\not 2}(\theta°)}{\not{sin(\theta°)}}$$

$\therefore Q_1T_1 \times R_1S_1$

$= ab\,sin(\theta°)\,cos(\theta°)$ (10.11)

Multiplying eqns. (10.7) & (10.8),

$$Q_2T_2 \times R_2S_2 = \frac{acos^{\not 2}(\theta°)}{\not{sin(\theta°)}} \times \frac{bsin^{\not 2}(\theta°)}{\not{cos(\theta°)}}$$

$\therefore Q_2T_2 \times R_2S_2$

$= ab\,sin(\theta°)\,cos(\theta°)$ (10.12)

Equating eqns. (10.9) & (10.10), (10.11) & (10.12),

$OQ_1 \times OR_1 = OQ_2 \times OR_2 = Q_1T_1 \times R_1S_1$

$= Q_2T_2 \times R_2S_2$ (10.13)

Eqn. (10.13) is mathematical expression of the theorem.

THEOREM- 11:

$\boldsymbol{P_1F_1{}^2 + P_1F_2{}^2 + P_2F_1{}^2 + P_2F_2{}^2 = 2(3a^2 - b^2)}$

Derivation of equations for proof of the theorem

$Eqn.(A.33) \Rightarrow P_1F_1 = a - \left(\sqrt{a^2 - b^2} \times cos\,\theta°\right)$

$\therefore P_1F_1{}^2 = a^2 + (a^2 - b^2)\,cos^2\,\theta°$

$- 2a\sqrt{a^2 - b^2} \times cos\,\theta°$ (11.1)

$Eqn.(A.34) \Rightarrow P_1F_2 = a + \left(\sqrt{a^2 - b^2} \times cos\,\theta°\right)$

$\therefore P_1F_2{}^2 = a^2 + (a^2 - b^2)\,cos^2\,\theta°$

$+ 2a\sqrt{a^2 - b^2} \times cos\,\theta°$ (11.2)

$Eqn.(B.33) \Rightarrow P_2F_1 = a - \left(\sqrt{a^2 - b^2} \times sin\,\theta°\right)$

$\therefore P_2F_1{}^2 = a^2 + (a^2 - b^2)\,sin^2\,\theta°$

$- 2a\sqrt{a^2 - b^2} \times sin\,\theta°$ (11.3)

$Eqn.(B.34) \Rightarrow P_2F_2 = a + \left(\sqrt{a^2 - b^2} \times sin\,\theta°\right)$

$\therefore P_2F_2{}^2 = a^2 + (a^2 - b^2)\,sin^2\,\theta°$

$+ 2a\sqrt{a^2 - b^2} \times sin\,\theta°$ (11.4)

Multiplying eqns. (11.1), (11.2), (11.3) & (11.4),

$\therefore P_1F_1{}^2 + P_1F_2{}^2 + P_2F_1{}^2 + P_2F_2{}^2$

$$= \Big[a^2 + (a^2 - b^2)\,cos^2\,\theta° \cancel{- 2a\sqrt{a^2 - b^2}\,cos\,\theta°} + a^2 + (a^2 - b^2)\,cos^2\,\theta° \cancel{+ 2a\sqrt{a^2 - b^2}\,cos\,\theta°} + a^2 + (a^2 - b^2)\,sin^2\,\theta° \cancel{- 2a\sqrt{a^2 - b^2}\,sin\,\theta°} + a^2 + (a^2 - b^2)\,sin^2\,\theta° \cancel{+ 2a\sqrt{a^2 - b^2}\,sin\,\theta°}\Big]$$

$\therefore P_1F_1{}^2 + P_1F_2{}^2 + P_2F_1{}^2 + P_2F_2{}^2 = 4a^2 + 2(a^2 - b^2)\cos^2\theta° + 2(a^2 - b^2)\sin^2\theta°$

$\therefore P_1F_1{}^2 + P_1F_2{}^2 + P_2F_1{}^2 + P_2F_2{}^2 = 4a^2 + 2(a^2 - b^2)(\cos^2\theta° + \sin^2\theta°)$

Simplifying the above eqn.,

$$\therefore P_1F_1{}^2 + P_1F_2{}^2 + P_2F_1{}^2 + P_2F_2{}^2 = 2(3a^2 - b^2) \tag{11.5}$$

Eqn. (11.5) is mathematical expression of the theorem.

THEOREM- 12:

$$\boldsymbol{M_1Q_1{}^2 + M_2Q_2{}^2 = \frac{a^2b^2 + (a^2 - b^2)^2}{b^2}}$$

Derivation of equations for proof of the theorem

$$Eqn.(A.31) \Longrightarrow M_1Q_1{}^2 = \frac{a^2b^2\cos^2\theta° + (a^2 - b^2)^2\sin^2(\theta°)}{b^2} \tag{12.1}$$

$$Eqn.(B.31) \Longrightarrow M_2Q_2{}^2 = \frac{a^2b^2\sin^2\theta° + (a^2 - b^2)^2\cos^2(\theta°)}{b^2} \tag{12.2}$$

Adding eqns. (12.1) & (12.2) in above,

$$M_1Q_1{}^2 + M_2Q_2{}^2 = \left(\frac{a^2b^2\cos^2\theta° + (a^2 - b^2)^2\sin^2(\theta°)}{b^2}\right) + \left(\frac{a^2b^2\sin^2\theta° + (a^2 - b^2)^2\cos^2(\theta°)}{b^2}\right)$$

$$\therefore M_1Q_1{}^2 + M_2Q_2{}^2 = \frac{a^2b^2\cos^2\theta° + a^2b^2\sin^2\theta° + (a^2 - b^2)^2\sin^2(\theta°) + (a^2 - b^2)^2\cos^2(\theta°)}{b^2}$$

$$\therefore M_1Q_1{}^2 + M_2Q_2{}^2 = \frac{a^2b^2[\cos^2\theta° + \sin^2\theta°] + (a^2 - b^2)^2[\sin^2(\theta°) + \cos^2(\theta°)]}{b^2}$$

$$\therefore M_1Q_1{}^2 + M_2Q_2{}^2 = \frac{a^2b^2 + (a^2 - b^2)^2}{b^2} \tag{12.3}$$

Eqn. (12.3) is mathematical expression of the theorem.

THEOREM- 13:

$$\boldsymbol{N_1R_1 + N_2R_2 = \frac{a^2b^2 + (a^2 - b^2)^2}{a^2}}$$

Derivation of equations for proof of the theorem

$$Eqn.(A.31) \Longrightarrow N_1R_1{}^2 = \frac{a^2b^2\sin^2(\theta°) + (a^2 - b^2)^2\cos^2(\theta°)}{a^2} \tag{13.1}$$

$$Eqn.(B.31) \Longrightarrow N_2R_2{}^2 = \frac{a^2b^2\cos^2(\theta°) + (a^2 - b^2)^2\sin^2(\theta°)}{a^2} \tag{13.2}$$

Adding eqns. (13.1) & (13.2)

$$N_1R_1{}^2 + N_2R_2{}^2 = \left(\frac{a^2b^2\sin^2(\theta°) + (a^2 - b^2)^2\cos^2(\theta°)}{a^2}\right) + \left(\frac{a^2b^2\cos^2(\theta°) + (a^2 - b^2)^2\sin^2(\theta°)}{a^2}\right)$$

$$\therefore N_1R_1{}^2 + N_2R_2{}^2 = \frac{a^2b^2\sin^2(\theta°) + a^2b^2\cos^2(\theta°) + (a^2 - b^2)^2\cos^2(\theta°) + (a^2 - b^2)^2\sin^2(\theta°)}{a^2}$$

$$\therefore N_1R_1{}^2 + N_2R_2{}^2 = \frac{a^2b^2[sin^2(\theta°) + cos^2(\theta°)] + (a^2 - b^2)^2[cos^2(\theta°) + sin^2(\theta°)]}{a^2}$$

$$\therefore N_1R_1{}^2 + N_2R_2{}^2 = \frac{a^2b^2 + (a^2 - b^2)^2}{a^2} \qquad (13.3)$$

Eqn. (13.3) is mathematical expression of the theorem.

THEOREM- 14:

$$\boldsymbol{ON_1{}^2 + ON_2{}^2 = \left(\frac{a^2 - b^2}{a}\right)^2}$$

Derivation of equations for proof of the theorem

$$Eqn. (A.17) \Longrightarrow ON_1 = \frac{(a^2 - b^2)cos(\theta°)}{a}$$

$$Eqn. (A.17) \Longrightarrow ON_1{}^2 = \frac{(a^2 - b^2)^2cos^2(\theta°)}{a^2} \qquad (14.1)$$

$$Eqn. (B.17) \Longrightarrow ON_2 = \frac{(a^2 - b^2)\,sin(\theta°)}{a}$$

$$Eqn. (B.17) \Longrightarrow ON_2{}^2 = \frac{(a^2 - b^2)^2\,sin^2(\theta°)}{a^2} \qquad (14.2)$$

Adding eqns. (14.1) & (14.2)

$$ON_1{}^2 + ON_2{}^2 = \left(\frac{(a^2 - b^2)^2cos^2(\theta°)}{a^2}\right) + \left(\frac{(a^2 - b^2)^2\,sin^2(\theta°)}{a^2}\right)$$

$$\therefore ON_1{}^2 + ON_2{}^2 = \frac{(a^2 - b^2)^2[cos^2(\theta°) + sin^2(\theta°)]}{a^2}$$

$$\therefore ON_1{}^2 + ON_2{}^2 = \left(\frac{a^2 - b^2}{a}\right)^2 \qquad (14.3)$$

Eqn. (14.3) is mathematical expression of the theorem.

THEOREM- 15:

$$\boldsymbol{OM_1{}^2 + OM_2{}^2 = \left(\frac{a^2 - b^2}{b}\right)^2}$$

Derivation of equations for proof of the theorem

$$Eqn. (A.19) \Longrightarrow OM_1 = \frac{(a^2 - b^2)sin(\theta°)}{b}$$

$$Eqn. (A.19) \Longrightarrow OM_1{}^2 = \frac{(a^2 - b^2)^2sin^2(\theta°)}{b^2} \qquad (15.1)$$

$$Eqn. (B.19) \Longrightarrow OM_2 = \frac{(a^2 - b^2)\,cos(\theta°)}{b}$$

$Eqn. (B.19) \Longrightarrow OM_2{}^2$

$$= \frac{(a^2 - b^2)^2 \cos^2(\theta°)}{b^2} \tag{15.2}$$

Adding eqns. (15.1) & (15.2)

$$OM_1{}^2 + OM_2{}^2 = \left(\frac{(a^2 - b^2)^2 \sin^2(\theta°)}{b^2}\right) + \left(\frac{(a^2 - b^2)^2 \cos^2(\theta°)}{b^2}\right)$$

$$\therefore OM_1{}^2 + OM_2{}^2 = \frac{(a^2 - b^2)^2[\sin^2(\theta°) + \cos^2(\theta°)]}{b^2}$$

$\therefore OM_1{}^2 + OM_2{}^2$

$$= \left(\frac{a^2 - b^2}{b}\right)^2 \tag{15.3}$$

Eqn. (15.3) is mathematical expression of the theorem.

THEOREM- 16:

$$\boldsymbol{M_1Q_1{}^2 - OM_1{}^2 = OQ_1{}^2}$$

Derivation of equations for proof of the theorem

$Eqn. (A.31) \Longrightarrow M_1Q_1{}^2$

$$= \frac{a^2b^2 \cos^2 \theta° + (a^2 - b^2)^2 \sin^2(\theta°)}{b^2} \tag{16.1}$$

$$Eqn. (A.19) \Longrightarrow OM_1 = \frac{(a^2 - b^2)\sin(\theta°)}{b}$$

$\therefore OM_1{}^2$

$$= \frac{(a^2 - b^2)^2 \sin^2(\theta°)}{b^2} \tag{16.2}$$

Deducting eqn. (16.2) from (16.1)

$$M_1Q_1{}^2 - OM_1{}^2 = \left(\frac{a^2b^2 \cos^2 \theta° + (a^2 - b^2)^2 \sin^2(\theta°)}{b^2}\right) - \left(\frac{(a^2 - b^2)^2 \sin^2(\theta°)}{b^2}\right)$$

$$\therefore M_1Q_1{}^2 - OM_1{}^2 = \left(\frac{a^2b^2 \cos^2 \theta° + \cancel{(a^2 - b^2)^2 \sin^2(\theta°)} - \cancel{(a^2 - b^2)^2 \sin^2(\theta°)}}{b^2}\right)$$

$$\therefore M_1Q_1{}^2 - OM_1{}^2 = \left(\frac{a^2\cancel{b^2} \cos^2 \theta°}{\cancel{b^2}}\right)$$

$$\therefore M_1Q_1{}^2 - OM_1{}^2 = a^2 \cos^2 \theta°$$

Substituting eqn. (A.1) in above,

$\therefore M_1Q_1{}^2 - OM_1{}^2$

$$= OQ_1{}^2 \tag{16.3}$$

Eqn. (16.3) is mathematical expression of the theorem.

THEOREM- 17:

$$\boldsymbol{N_1R_1{}^2 - ON_1{}^2 = OR_1{}^2}$$

Derivation of equations for proof of the theorem

$Eqn. (A.32) \Longrightarrow N_1R_1{}^2$

$$= \frac{a^2b^2 \sin^2(\theta°) + (a^2 - b^2)^2 \cos^2(\theta°)}{a^2} \tag{17.1}$$

$$Eqn. (B.17) \Longrightarrow ON_1 = \frac{(a^2 - b^2)\, cos(\theta°)}{a}$$

$$\therefore {ON_1}^2 = \frac{(a^2 - b^2)^2\, cos^2(\theta°)}{a^2} \qquad (17.2)$$

Deducting eqn. (17.2) from (17.1)

$${N_1R_1}^2 - {ON_1}^2 = \left(\frac{a^2b^2sin^2(\theta°) + (a^2 - b^2)^2cos^2(\theta°)}{a^2}\right) - \left(\frac{(a^2 - b^2)^2\, cos^2(\theta°)}{a^2}\right)$$

$$\therefore {N_1R_1}^2 - {ON_1}^2 = \frac{a^2b^2sin^2(\theta°) + \cancel{(a^2 - b^2)^2cos^2(\theta°)} - \cancel{(a^2 - b^2)^2\, cos^2(\theta°)}}{a^2}$$

$$\therefore {N_1R_1}^2 - {ON_1}^2 = \frac{\cancel{a^2}b^2sin^2(\theta°)}{\cancel{a^2}}$$

$$\therefore {N_1R_1}^2 - {ON_1}^2 = b^2sin^2(\theta°)$$

Substituting eqn. (A.2)

$$\therefore {N_1R_1}^2 - {ON_1}^2 = {OR_1}^2 \qquad (17.3)$$

Eqn. (17.3) is mathematical expression of the theorem.

THEOREM- 18:

$$\boldsymbol{\frac{1}{{OT_1}^2} + \frac{1}{{OT_2}^2} = \frac{1}{a^2}}$$

Derivation of equations for proof of the theorem

$$Eqn. (A.5) \Longrightarrow OT_1 = \frac{a}{cos(\theta°)}$$

$$\therefore {OT_1}^2 = \frac{a^2}{cos^2(\theta°)} \qquad (18.1)$$

$$Eqn. (B.5) \Longrightarrow OT_2 = \frac{a}{sin(\theta°)}$$

$$\therefore {OT_2}^2 = \frac{a^2}{sin^2(\theta°)} \qquad (18.2)$$

Adding reciprocal of eqns. (18.1) & (18.2)

$$\frac{1}{{OT_1}^2} + \frac{1}{{OT_2}^2} = \left(\frac{cos^2(\theta°)}{a^2}\right) + \left(\frac{sin^2(\theta°)}{a^2}\right)$$

$$\therefore \frac{1}{{OT_1}^2} + \frac{1}{{OT_2}^2} = \frac{cos^2(\theta°) + sin^2(\theta°)}{a^2}$$

$$\therefore \frac{1}{{OT_1}^2} + \frac{1}{{OT_2}^2} = \frac{1}{a^2} \qquad (18.3)$$

Eqn. (18.3) is mathematical expression of the theorem.

THEOREM- 19:

$$\frac{1}{OS_1{}^2} + \frac{1}{OS_2{}^2} = \frac{1}{b^2}$$

Derivation of equations for proof of the theorem

$$Eqn. (A.8) \Longrightarrow OS_1 = \frac{b}{sin(\theta°)}$$

$$\therefore OS_1{}^2 = \frac{b^2}{sin^2(\theta°)} \qquad (19.1)$$

$$Eqn. (B.8) \Longrightarrow OS_2 = \frac{b}{cos(\theta°)}$$

$$\therefore OS_2{}^2 = \frac{b^2}{cos^2(\theta°)} \qquad (19.2)$$

Adding reciprocal of eqns. (19.1) & (19.2)

$$\frac{1}{OS_1{}^2} + \frac{1}{OS_2{}^2} = \left(\frac{sin^2(\theta°)}{b^2}\right) + \left(\frac{cos^2(\theta°)}{b^2}\right)$$

$$\therefore \frac{1}{OS_1{}^2} + \frac{1}{OS_2{}^2} = \frac{sin^2(\theta°) + cos^2(\theta°)}{b^2}$$

$$\therefore \frac{1}{OS_1{}^2} + \frac{1}{OS_2{}^2} = \frac{1}{b^2} \qquad (19.3)$$

Eqn. (19.3) is mathematical expression of the theorem.

THEOREM- 20:

$$R_1M_1{}^2 + R_2M_2{}^2 = \left(\frac{a^2}{b}\right)^2$$

Derivation of equations for proof of the theorem

$$Eqn. (A.20) \Longrightarrow R_1M_1 = \frac{a^2 sin(\theta°)}{b}$$

$$\therefore R_1M_1{}^2 = \frac{a^4 sin^2(\theta°)}{b^2} \qquad (20.1)$$

$$Eqn. (B.20) \Longrightarrow R_2M_2 = \frac{a^2 cos(\theta°)}{b}$$

$$\therefore R_2M_2{}^2 = \frac{a^4 cos^2(\theta°)}{b^2} \qquad (20.2)$$

Adding eqns. (20.1) & (18.2) in above,

$$R_1M_1{}^2 + R_2M_2{}^2 = \frac{a^4 sin^2(\theta°)}{b^2} + \frac{a^4 cos^2(\theta°)}{b^2}$$

$$\therefore R_1M_1{}^2 + R_2M_2{}^2 = \frac{a^4[sin^2(\theta°) + cos^2(\theta°)]}{b^2}$$

$\therefore R_1M_1{}^2 + R_2M_2{}^2$

$$= \left(\frac{a^2}{b}\right)^2 \qquad (20.3)$$

Eqn. (20.3) is mathematical expression of the theorem.

THEOREM- 21:

$$\boldsymbol{N_1Q_1{}^2 + N_2Q_2{}^2 = \left(\frac{b^2}{a}\right)^2}$$

Derivation of equations for proof of the theorem

$$Eqn.(A.16) \Longrightarrow N_1Q_1 = \frac{b^2cos(\theta°)}{a}$$

$\therefore N_1Q_1$

$$= \frac{b^4cos^2(\theta°)}{a^2} \qquad (21.1)$$

$$Eqn.(B.16) \Longrightarrow N_2Q_2 = \frac{b^2sin(\theta°)}{a}$$

$\therefore N_2Q_2{}^2$

$$= \frac{b^4sin^2(\theta°)}{a^2} \qquad (21.2)$$

Adding eqns. (21.1) & (21.2) in above,

$$N_1Q_1{}^2 + N_2Q_2{}^2 = \left(\frac{b^4cos^2(\theta°)}{a^2}\right) + \left(\frac{b^4sin^2(\theta°)}{a^2}\right)$$

$$\therefore N_1Q_1{}^2 + N_2Q_2{}^2 = \frac{b^4[sin^2(\theta°) + cos^2(\theta°)]}{a^2}$$

$\therefore N_1Q_1{}^2 + N_2Q_2{}^2$

$$= \left(\frac{b^2}{a}\right)^2 \qquad (21.3)$$

Eqn. (21.3) is mathematical expression of the theorem.

THEOREM- 22:

$$\boldsymbol{P_1N_1{}^2 + P_2N_2{}^2 = \frac{b^2(a^2 + b^2)}{a^2}}$$

Derivation of equations for proof of the theorem

$$Eqn.(A.21) \Longrightarrow P_1N_1 = \frac{b\sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)}}{a}$$

$\therefore P_1N_1{}^2$

$$= \frac{b^2[a^2sin^2(\theta°) + b^2cos^2(\theta°)]}{a^2} \qquad (22.1)$$

$$Eqn.(B.21) \Longrightarrow P_2N_2 = \frac{b\sqrt{a^2cos^2(\theta°) + b^2sin^2(\theta°)}}{a}$$

$\therefore P_2N_2{}^2$

$$= \frac{b^2[a^2cos^2(\theta°) + b^2sin^2(\theta°)]}{a^2} \qquad (22.2)$$

Adding eqns. (22.1) & (22.2),

$$P_1N_1{}^2 + P_2N_2{}^2 = \left(\frac{b^2[a^2sin^2(\theta°) + b^2cos^2(\theta°)]}{a^2}\right) + \left(\frac{b^2[a^2cos^2(\theta°) + b^2sin^2(\theta°)]}{a^2}\right)$$

$$\therefore P_1N_1{}^2 + P_2N_2{}^2 = \frac{b^2[a^2sin^2(\theta°) + b^2cos^2(\theta°) + a^2cos^2(\theta°) + b^2sin^2(\theta°)]}{a^2}$$

$$\therefore P_1N_1{}^2 + P_2N_2{}^2 = \frac{b^2\{a^2[sin^2(\theta°) + a^2cos^2(\theta°)] + b^2[sin^2(\theta°) + +b^2cos^2(\theta°)]\}}{a^2}$$

$$\therefore P_1N_1{}^2 + P_2N_2{}^2 = \frac{b^2(a^2 + b^2)}{a^2} \qquad (22.3)$$

Eqn. (22.3) is mathematical expression of the theorem.

THEOREM- 23:

$$\boldsymbol{P_1M_1{}^2 + P_2M_2{}^2 = \frac{a^2(a^2 + b^2)}{b^2}}$$

Derivation of equations for proof of the theorem

$$Eqn.(A.23) \Longrightarrow P_1M_1 = \frac{a\sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)}}{b}$$

$$\therefore P_1M_1{}^2 = \frac{a^2[a^2sin^2(\theta°) + b^2cos^2(\theta°)]}{b^2} \qquad (23.1)$$

$$Eqn.(B,23) \Longrightarrow P_2M_2 = \frac{a\sqrt{a^2cos^2(\theta°) + b^2sin^2(\theta°)}}{b}$$

$$\therefore P_2M_2{}^2 = \frac{a^2[a^2cos^2(\theta°) + b^2sin^2(\theta°)]}{b^2} \qquad (23.2)$$

Adding eqns. (23.1) & (23.2),

$$P_1M_1{}^2 + P_2M_2{}^2 = \left(\frac{a^2[a^2sin^2(\theta°) + b^2cos^2(\theta°)]}{b^2}\right) + \left(\frac{a^2[a^2cos^2(\theta°) + b^2sin^2(\theta°)]}{b^2}\right)$$

$$\therefore P_1M_1{}^2 + P_2M_2{}^2 = \frac{a^2[a^2sin^2(\theta°) + b^2cos^2(\theta°) + a^2cos^2(\theta°) + b^2sin^2(\theta°)]}{b^2}$$

$$\therefore P_1M_1{}^2 + P_2M_2{}^2 = \frac{a^2\{a^2[sin^2(\theta°) + cos^2(\theta°)] + b^2[cos^2(\theta°) + sin^2(\theta°)]\}}{b^2}$$

$$\therefore P_1M_1{}^2 + P_2M_2{}^2 = \frac{a^2(a^2 + b^2)}{b^2} \qquad (23.3)$$

Eqn. (23.3) is mathematical expression of the theorem.

THEOREM- 24:

$$\boldsymbol{\therefore M_1N_1{}^2 + M_2N_2{}^2 = \frac{(a^2 - b^2)^2(a^2 + b^2)}{a^2b^2}}$$

Derivation of equations for proof of the theorem

$$Eqn.(A.22) \Longrightarrow M_1N_1 = \frac{(a^2 - b^2)\sqrt{a^2sin^2(\theta°) + b^2cos^2(\theta°)}}{ab}$$

$$\therefore M_1N_1{}^2 = \frac{(a^2-b^2)^2[a^2sin^2(\theta°)+b^2cos^2(\theta°)]}{a^2b^2} \quad (24.1)$$

$$Eqn.(B.22) \Longrightarrow M_2N_2 = \frac{(a^2-b^2)\sqrt{a^2cos^2(\theta°)+b^2\,sin^2(\theta°)}}{ab}$$

$$\therefore M_2N_2{}^2 = \frac{(a^2-b^2)^2[a^2cos^2(\theta°)+b^2\,sin^2(\theta°)]}{a^2b^2} \quad (24.2)$$

Adding eqns. (23.1) & (23.2),

$$\therefore M_1N_1{}^2 + M_2N_2{}^2 = \left(\frac{(a^2-b^2)^2[a^2sin^2(\theta°)+b^2cos^2(\theta°)]}{a^2b^2}\right) + \left(\frac{(a^2-b^2)^2[a^2cos^2(\theta°)+b^2\,sin^2(\theta°)]}{a^2b^2}\right)$$

$$\therefore M_1N_1{}^2 + M_2N_2{}^2 = \frac{(a^2-b^2)^2[a^2sin^2(\theta°)+b^2cos^2(\theta°)+a^2cos^2(\theta°)+b^2\,sin^2(\theta°)]}{a^2b^2}$$

$$\therefore M_1N_1{}^2 + M_2N_2{}^2 = \frac{(a^2-b^2)^2\{a^2[sin^2(\theta°)+cos^2(\theta°)]+b^2[sin^2(\theta°)+cos^2(\theta°)]\}}{a^2b^2}$$

$$\therefore M_1N_1{}^2 + M_2N_2{}^2 = \frac{(a^2-b^2)^2(a^2+b^2)}{a^2b^2} \quad (24.3)$$

Eqn. (24.3) is mathematical expression of the theorem.

THEOREM- 25:

$$\boldsymbol{N_1F_1 + N_1F_2 = N_2F_1 + N_2F_2}$$

Derivation of equations for proof of the theorem

$$Eqn.(A.29) \Longrightarrow N_1F_1 = \frac{a\sqrt{a^2-b^2}-(a^2-b^2)cos(\theta°)}{a} \quad (25.1)$$

$$Eqn.(A.30) \Longrightarrow N_1F_2 = \frac{a\sqrt{a^2-b^2}+(a^2-b^2)cos(\theta°)}{a} \quad (25.2)$$

$$Eqn.(B.29) \Longrightarrow N_2F_1 = \frac{a\sqrt{a^2-b^2}-(a^2-b^2)sin(\theta°)}{a} \quad (25.3)$$

$$Eqn.(B.30) \Longrightarrow N_2F_2 = \frac{a\sqrt{a^2-b^2}+(a^2-b^2)sin(\theta°)}{a} \quad (25.4)$$

Adding eqns. (25.1) & (25.2),

$$N_1F_1 + N_1F_2 = \left(\frac{a\sqrt{a^2-b^2}-(a^2-b^2)cos(\theta°)}{a}\right) + \left(\frac{a\sqrt{a^2-b^2}+(a^2-b^2)cos(\theta°)}{a}\right)$$

$$N_1F_1 + N_1F_2 = \frac{a\sqrt{a^2-b^2}+a\sqrt{a^2-b^2}\cancel{+(a^2-b^2)cos(\theta°)-(a^2-b^2)cos(\theta°)}}{a}$$

$$\therefore N_1F_1 + N_1F_2 = \frac{2a\sqrt{a^2 - b^2}}{a} \qquad (25.5)$$

Adding eqns. (25.3) & (25.4),

$$N_2F_1 + N_2F_2 = \left(\frac{a\sqrt{a^2 - b^2} - (a^2 - b^2)sin(\theta°)}{a}\right) + \left(\frac{a\sqrt{a^2 - b^2} + (a^2 - b^2)sin(\theta°)}{a}\right)$$

$$\therefore N_2F_1 + N_2F_2 = \left(\frac{a\sqrt{a^2 - b^2} + a\sqrt{a^2 - b^2} \cancel{+ (a^2 - b^2)sin(\theta°) - (a^2 - b^2)sin(\theta°)}}{a}\right)$$

$$\therefore N_2F_1 + N_2F_2 = \frac{2a\sqrt{a^2 - b^2}}{a} \qquad (25.6)$$

Equating eqns. (25.5) & (25.6),

$$N_1F_1 + N_1F_2 = N_2F_1 + N_2F_2 \qquad (25.7)$$

Eqn. (25.7) is mathematical expression of the theorem.

THEOREM- 26:

$$\left(\boldsymbol{P_1T_1}^2 + \boldsymbol{P_2S_2}^2\right) \times \left(\boldsymbol{P_2T_2}^2 + \boldsymbol{P_1S_1}^2\right) = (\boldsymbol{a}^2 + \boldsymbol{b}^2)^2$$

Derivation of equations for proof of the theorem

$$Eqn.(A.12) \Longrightarrow P_1T_1 = \frac{sin(\theta°)\sqrt{a^2sin^2(\theta°) + b^2\,cos^2(\theta°)}}{cos(\theta°)}$$

$$\therefore P_1T_1{}^2 = \frac{sin^2(\theta°)\,[a^2sin^2(\theta°) + b^2\,cos^2(\theta°)]}{cos^2(\theta°)} \qquad (26.1)$$

$$Eqn.(B.13) \Longrightarrow P_2S_2 = \frac{sin(\theta°)\sqrt{a^2cos^2(\theta°) + b^2\,sin^2(\theta°)}}{cos(\theta°)}$$

$$\therefore P_2S_2{}^2 = \frac{sin^2(\theta°)\,[a^2cos^2(\theta°) + b^2\,sin^2(\theta°)]}{cos^2(\theta°)} \qquad (26.2)$$

$$Eqn.(B.12) \Longrightarrow P_2T_2 = \frac{cos(\theta°)\sqrt{a^2cos^2(\theta°) + b^2\,sin^2(\theta°)}}{sin(\theta°)}$$

$$\therefore P_2T_2{}^2 = \frac{cos^2(\theta°)[a^2cos^2(\theta°) + b^2\,sin^2(\theta°)]}{sin^2(\theta°)} \qquad (26.3)$$

$$Eqn.(A.13) \Longrightarrow P_1S_1 = \frac{cos(\theta°)\sqrt{a^2sin^2(\theta°) + b^2\,cos^2(\theta°)}}{sin(\theta°)}$$

$$\therefore P_1S_1{}^2 = \frac{cos^2(\theta°)\,[a^2sin^2(\theta°) + b^2\,cos^2(\theta°)]}{sin^2(\theta°)} \qquad (26.4)$$

Adding eqns. (26.1) & (26.2),

$$P_1T_1{}^2 + P_2S_2{}^2 = \left(\frac{sin^2(\theta°)\,[a^2 sin^2(\theta°) + b^2\,cos^2(\theta°)]}{cos^2(\theta°)}\right) + \left(\frac{sin^2(\theta°)\,[a^2 cos^2(\theta°) + b^2\,sin^2(\theta°)]}{cos^2(\theta°)}\right)$$

$$\therefore P_1T_1{}^2 + P_2S_2{}^2 = \frac{sin^2(\theta°)\,[a^2 sin^2(\theta°) + a^2 cos^2(\theta°) + b^2\,sin^2(\theta°) + b^2\,cos^2(\theta°)]}{cos^2(\theta°)}$$

$$\therefore P_1T_1{}^2 + P_2S_2{}^2 = \frac{sin^2(\theta°) \times \{a^2[sin^2(\theta°) + cos^2(\theta°)] + b^2[sin^2(\theta°) + cos^2(\theta°)]\}}{cos^2(\theta°)}$$

$$\therefore P_1T_1{}^2 + P_2S_2{}^2 = \frac{sin^2(\theta°) \times \{a^2[sin^2(\theta°) + cos^2(\theta°)] + b^2[sin^2(\theta°) + cos^2(\theta°)]\}}{cos^2(\theta°)}$$

$$\therefore P_1T_1{}^2 + P_2S_2{}^2 = \frac{sin^2(\theta°) \times (a^2 + b^2)}{cos^2(\theta°)} \qquad (26.5)$$

Adding eqns. (26.3) & (26.4),

$$P_2T_2{}^2 + P_1S_1{}^2 = \left(\frac{cos^2(\theta°)[a^2 cos^2(\theta°) + b^2\,sin^2(\theta°)]}{sin^2(\theta°)}\right) + \left(\frac{cos^2(\theta°)\,[a^2 sin^2(\theta°) + b^2\,cos^2(\theta°)]}{sin^2(\theta°)}\right)$$

$$\therefore P_2T_2{}^2 + P_1S_1{}^2 = \frac{cos^2(\theta°)[a^2 cos^2(\theta°) + a^2 sin^2(\theta°) + b^2\,sin^2(\theta°) + b^2\,cos^2(\theta°)]}{sin^2(\theta°)}$$

$$\therefore P_2T_2{}^2 + P_1S_1{}^2 = \frac{cos^2(\theta°)[a^2\{cos^2(\theta°) + sin^2(\theta°)\} + b^2\{sin^2(\theta°) + cos^2(\theta°)\}]}{sin^2(\theta°)}$$

$$\therefore P_2T_2{}^2 + P_1S_1{}^2 = \frac{cos^2(\theta°) \times (a^2 + b^2)}{sin^2(\theta°)} \qquad (26.6)$$

Multiplying eqns. (26.5) & (26.6),

$$\left(P_1T_1{}^2 + P_2S_2{}^2\right) \times \left(P_2T_2{}^2 + P_1S_1{}^2\right) = \left(\frac{\cancel{sin^2(\theta°)} \times (a^2 + b^2)}{\cancel{cos^2(\theta°)}}\right) \times \left(\frac{\cancel{cos^2(\theta°)} \times (a^2 + b^2)}{\cancel{sin^2(\theta°)}}\right)$$

$$\left(P_1T_1{}^2 + P_2S_2{}^2\right) \times \left(P_2T_2{}^2 + P_1S_1{}^2\right) = (a^2 + b^2)^2 \qquad (26.7)$$

Eqn. (26.7) is mathematical expression of the theorem.

THEOREM- 27:

$$\boldsymbol{P_1T_1 \times P_2T_2 = P_1S_1 \times P_2S_2}$$

Derivation of equations for proof of the theorem

$$Eqn.\,(A.12) \Rightarrow P_1T_1 = \frac{sin(\theta°)\sqrt{a^2 sin^2(\theta°) + b^2\,cos^2(\theta°)}}{cos(\theta°)} \qquad (27.1)$$

$$Eqn.\,(B.12) \Rightarrow P_2T_2 = \frac{cos(\theta°)\sqrt{a^2 cos^2(\theta°) + b^2\,sin^2(\theta°)}}{sin(\theta°)} \qquad (27.2)$$

$$Eqn.\,(B.12) \Rightarrow P_1S_1 = \frac{cos(\theta°)\sqrt{a^2 sin^2(\theta°) + b^2\,cos^2(\theta°)}}{sin(\theta°)} \qquad (27.3)$$

$$Eqn.(B.13) \Longrightarrow P_2S_2 = \frac{sin(\theta°)\sqrt{a^2cos^2(\theta°) + b^2\,sin^2(\theta°)}}{cos(\theta°)} \quad (27.4)$$

Multiplying eqns. (27.1) & (27.2)

$$P_1T_1 \times P_2T_2 = \left(\frac{\cancel{sin(\theta°)}\sqrt{a^2sin^2(\theta°) + b^2\,cos^2(\theta°)}}{\cancel{cos(\theta°)}}\right) \times \left(\frac{\cancel{cos(\theta°)}\sqrt{a^2cos^2(\theta°) + b^2\,sin^2(\theta°)}}{\cancel{sin(\theta°)}}\right)$$

$$\therefore P_1T_1 \times P_2T_2 = \sqrt{a^2sin^2(\theta°) + b^2\,cos^2(\theta°)} \times \sqrt{a^2cos^2(\theta°) + b^2\,sin^2(\theta°)} \quad (27.5)$$

Multiplying eqns. (27.3) & (27.4)

$$P_1S_1 \times P_2S_2 = \left(\frac{\cancel{cos(\theta°)}\sqrt{a^2sin^2(\theta°) + b^2\,cos^2(\theta°)}}{\cancel{sin(\theta°)}}\right) \times \left(\frac{\cancel{sin(\theta°)}\sqrt{a^2cos^2(\theta°) + b^2\,sin^2(\theta°)}}{\cancel{cos(\theta°)}}\right)$$

$$\therefore P_1S_1 \times P_2S_2 = \sqrt{a^2sin^2(\theta°) + b^2\,cos^2(\theta°)} \times \sqrt{a^2cos^2(\theta°) + b^2\,sin^2(\theta°)} \quad (27.6)$$

Equating eqns. (27.5) & (27.6),

$$P_1T_1 \times P_2T_2 = P_1S_1 \times P_2S_2 \quad (27.7)$$

Eqn. (27.7) is mathematical expression of the theorem.

THEOREM- 28:

$$\boldsymbol{\frac{ON_1 + ON_2}{OM_1 + OM_2} = \frac{b}{a}}$$

Derivation of equations for proof of the theorem

$$Eqn.(A.17) \Longrightarrow ON_1 = \frac{(a^2 - b^2)cos(\theta°)}{a} \quad (28.1)$$

$$Eqn.(B.17) \Longrightarrow ON_2 = \frac{(a^2 - b^2)\,sin(\theta°)}{a} \quad (28.2)$$

$$Eqn.(A.19) \Longrightarrow OM_1 = \frac{(a^2 - b^2)sin(\theta°)}{b} \quad (28.3)$$

$$Eqn.(B.19) \Longrightarrow OM_2 = \frac{(a^2 - b^2)cos(\theta°)}{b} \quad (28.4)$$

Adding eqns. (28.1) & (28.2),

$$ON_1 + ON_2 = \left(\frac{(a^2 - b^2)cos(\theta°)}{a}\right) + \left(\frac{(a^2 - b^2)\,sin(\theta°)}{a}\right)$$

$$\therefore ON_1 + ON_2 = \frac{(a^2 - b^2) \times [sin(\theta°) + cos(\theta°)]}{a} \quad (28.5)$$

Adding eqns. (28.3) & (28.4),

$$OM_1 + OM_2 = \left(\frac{(a^2 - b^2)sin(\theta°)}{b}\right) + \left(\frac{(a^2 - b^2)cos(\theta°)}{b}\right)$$

$$\therefore OM_1 + OM_2 = \frac{(a^2 - b^2) \times [sin(\theta°) + cos(\theta°)]}{b} \qquad (28.6)$$

Dividing eqn. (28.5) by (28.6),

$$\frac{ON_1 + ON_2}{OM_1 + OM_2} = \left(\frac{(a^2 - b^2) \times [sin(\theta°) + cos(\theta°)]}{a}\right) \div \left(\frac{(a^2 - b^2) \times [sin(\theta°) + cos(\theta°)]}{b}\right)$$

$$\therefore \frac{ON_1 + ON_2}{OM_1 + OM_2} = \left(\frac{\cancel{(a^2 - b^2) \times [sin(\theta°) + cos(\theta°)]}}{a}\right) \div \left(\frac{b}{\cancel{(a^2 - b^2) \times [sin(\theta°) + cos(\theta°)]}}\right)$$

$$\therefore \frac{ON_1 + ON_2}{OM_1 + OM_2} = \frac{b}{a} \qquad (28.7)$$

Eqn. (28.7) is mathematical expression of the theorem.

THEOREM- 29:

$$\boldsymbol{\frac{OT_1 + OT_2}{OS_1 + OS_2} = \frac{a}{b}}$$

Derivation of equations for proof of the theorem

$$Eqn. (A.5) \Longrightarrow OT_1 = \frac{a}{cos(\theta°)} \qquad (29.1)$$

$$Eqn. (B.5) \Longrightarrow OT_2 = \frac{a}{sin(\theta°)} \qquad (29.2)$$

$$Eqn. (A.8) \Longrightarrow OS_1 = \frac{b}{sin(\theta°)} \qquad (29.3)$$

$$Eqn. (B.8) \Longrightarrow OS_2 = \frac{b}{cos(\theta°)} \qquad (29.4)$$

Adding eqns. (29.1) & (29.2),

$$OT_1 + OT_2 = \left(\frac{a}{cos(\theta°)}\right) + \left(\frac{a}{sin(\theta°)}\right)$$

$$\therefore OT_1 + OT_2 = \frac{a\,sin(\theta°) + a\,cos(\theta°)}{sin(\theta°)\,cos(\theta°)}$$

$$\therefore OT_1 + OT_2 = \frac{a[sin(\theta°) + cos(\theta°)]}{sin(\theta°)\,cos(\theta°)} \qquad (29.5)$$

Adding eqns. (29.3) & (29.4),

$$OS_1 + OS_2 = \left(\frac{b}{sin(\theta°)}\right) + \left(\frac{b}{cos(\theta°)}\right)$$

$$\therefore OS_1 + OS_2 = \frac{b\,cos(\theta°) + b\,sin(\theta°)}{sin(\theta°)\,cos(\theta°)}$$

$$\therefore OS_1 + OS_2 = \frac{b[sin(\theta°) + cos(\theta°)]}{sin(\theta°)\,cos(\theta°)} \qquad (29.6)$$

Dividing eqn. (29.5) by (29.6),

$$\frac{OT_1 + OT_2}{OS_1 + OS_2} = \left(\frac{a[sin(\theta°) + cos(\theta°)]}{sin(\theta°)\,cos(\theta°)}\right) \div \left(\frac{b[sin(\theta°) + cos(\theta°)]}{sin(\theta°)\,cos(\theta°)}\right)$$

$$\therefore \frac{OT_1 + OT_2}{OS_1 + OS_2} = \left(\frac{a[\cancel{sin(\theta°) + cos(\theta°)}]}{\cancel{sin(\theta°)\,cos(\theta°)}}\right) \times \left(\frac{\cancel{sin(\theta°)\,cos(\theta°)}}{b[\cancel{sin(\theta°) + cos(\theta°)}]}\right)$$

$$\therefore \frac{OT_1 + OT_2}{OS_1 + OS_2} = \frac{a}{b} \qquad (29.7)$$

Eqn. (29.7) is mathematical expression of the theorem.

CONCLUSION

This research presents a comprehensive mathematical investigation into the geometry of ellipses, with a particular focus on the properties and relationships of pair conjugate diameters. Through the application of parametric equations, the study introduces and rigorously proves 29 original theorems, each supplemented with detailed diagrams to facilitate visualization and understanding. These theorems extend classical knowledge by exploring the intricate interplay between tangent and normal associated with conjugate diameters, offering novel insights into the structural elegance of ellipses.

The implications of these findings are substantial, enriching the theoretical foundation of conic sections while also holding relevance for applied disciplines such as astronomy, optics, and physics fields in which elliptical forms naturally emerge. By integrating established geometric principles with newly derived results, this work contributes meaningfully to both pure and applied mathematics. It serves as a valuable reference for researchers and scholars engaged in advanced studies of geometry, celestial mechanics, and related mathematical frameworks. Future investigations may explore extensions into three-dimensional analogues or other conic-related curves, thereby broadening the impact and utility of elliptical geometry.

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