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Study on Infinite Series and Convergence of Series in Real Analysis

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Abstract:
In this paper, we studied about the Infinite Series and Convergence of Series. Discussed the convergence of series through examples and explained some basic properties of Infinite series. Also studied about Geometric Series, Auxiliary Series (or) p – Series Test, Comparison Test & Root Test to examine the character of the given series through examples.

Keywords: Sequence, Infinite series, Convergence of Series, Geometric Series, Auxiliary Series.

Introduction:
A **sequence** is a list of items/objects which have been arranged in a sequential way and a **series** can be highly generalized as the sum of all the terms in a sequence. However, there has to be a definite relationship between all the terms of the sequence.
A sequence is an arrangement of any objects or a set of numbers in a particular order followed by some rule. If $a_1, a_2, a_3, a_4, \ldots\ldots\ldots$ etc. denote the terms of a sequence, then 1,2,3,4,…..denotes the position of the term.
If $a_1, a_2, a_3, a_4, \ldots\ldots.$ is a sequence, then the corresponding series is given by
$S_n = a_1+a_2+a_3 + .. + a_n$
Definition of Infinite Series:
If $\{ u_n \}$ is a sequence of real numbers and $s_n = u_1 + u_2 + ... + u_n$, $n \epsilon z^+$, then the sequence $\{ s_n \}$ is called an infinite series.
The number u_n is called the nth term of the series and the number s_n is called the nth partial sum of the series.
Convergence of Series:
Definition:
Let $\sum_{n=1}^{\infty} u_n$ be a series of real numbers with partial sums $s_n = u_1 + u_2 + ... + u_n$, $n \epsilon z^+$ .If the sequence $\{ s_n \}$ converges to s, we say that the series $\sum_{n=1}^{\infty} u_n$ converges to s. The number s is called the sum of the series and we write $\sum_{n=1}^{\infty} u_n = s$.
If the limit of the sequence $\{ s_n \}$ does not exist we say that the series $\sum u_n$ diverges.

- If $\sum u_n$ converges to s, then $s_n \leq s$ for all n.
- If $\sum u_n$ diverges then for any real number $G > 0$ there exists $m \epsilon z^+$ such that $s_n > G$ for $n \geq m$.

Illustrations :

1. The series $\sum_{n=0}^{\infty}\frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2}$ +..... Converges.

Here, $u_n = \frac{1}{2^n}$ and $s_n = 1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^n}$, which is in Geometric Progression with a = 1 and r = $\frac{1}{2}$

$s_n = \frac{a(1-r^n)}{1-r}$

$\therefore s_n = \frac{1(1-\frac{1}{2^{n+1}})}{1-\frac{1}{2}}$

$= 2(1- \frac{1}{2^{n+1}})$

$= 2 - \frac{1}{2^n}$

⇨ $\lim s_n = \lim (2 - \frac{1}{2^n})$

= 2

The sequence { s_n } converges to 2 and hence by definition $\sum u_n$ converges to 2.

$\sum_{n=0}^{\infty}\frac{1}{2^n} = 2.$

2. The series $\sum_{n=1}^{\infty}\frac{1}{4^n} = 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} +$ ..... Converges to $\frac{1}{3}$

Here, $u_n = \frac{1}{4^n}$ and $s_n = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + ... + \frac{1}{4^n}$, which is in Geometric Progression with

a = $\frac{1}{4}$ and r = $\frac{1}{4}$

$s_n = \frac{a(1-r^n)}{1-r}$

$\therefore s_n = \frac{\frac{1}{4}(1-\frac{1}{4^n})}{1-\frac{1}{4}}$

$= \frac{1}{3} - \frac{1}{3} . \frac{1}{4^n}$

⇨ $\lim s_n = \lim (\frac{1}{3} - \frac{1}{3} . \frac{1}{4^n})$

$= \frac{1}{3}$

The sequence { s_n } converges to $\frac{1}{3}$ and hence by definition $\sum u_n$ converges to $\frac{1}{3}$.

$\sum_{n=1}^{\infty}\frac{1}{4^n} = \frac{1}{3}.$

3. The series $\sum_{n=1}^{\infty} n = 1 + 2 + 3$ +..... Diverges.

Here, s_n = 1 + 2 + 3 +..... + n

$= \frac{n(n+1)}{2}$

$\lim s_n = \lim [\frac{n(n+1)}{2}]$

$\therefore$ $\lim s_n = \infty$

The sequence { s_n } diverges to ∞ and hence by definition $\sum_{n=1}^{\infty} n$ diverges to ∞ .

4. The series $\sum_{n=1}^{\infty} n^2 = 1^2 + 2^2 + 3^2$ +..... Diverges.

Here, $s_n = 1^2 + 2^2 + 3^2 + ... + n^2$

$= \frac{n(n+1)(2n+1)}{6}$

$\lim s_n = \lim [\frac{n(n+1)(2n+1)}{6}]$

$\therefore$ $\lim s_n = \infty$

The sequence { s_n } diverges to ∞ and hence by definition $\sum_{n=1}^{\infty} n^2$ diverges to ∞ .

5. The series $\sum_{n=1}^{\infty} k$, $k \in$ R$^+$ = $k + k + k +$ .... Diverges.

Here, u_n = k and $s_n = k + k + k +$........ (n terms) = $n\,k$

⇨ $s_n \to \infty$ as $n \to \infty$

∴ The series $\sum_{n=1}^{\infty} k$ diverges.

The constant series diverges.

6. The series $\sum_{n=1}^{\infty}(-1)^n$ = (-1) + 1 + (-1) + 1 + ... diverges.

Here, $s_n = -1$, when n is odd and $s_n = 0$, when n is even.

Therefore the sequence of partial sums = { s_n } = (-1, 0,-1, 0 ...) is not convergent.

Basic properties of Infinite series

Property 1:

If $\sum u_n$ converges to A and $\sum v_n$ converges to B, then $\sum(u_n + v_n)$ converges to A + B.

Proof :

Let $A_n = u_1 + u_2 + u_3 + ... + u_n$ and $B_n = v_1 + v_2 + v_3 + ... + v_n$, $n \in Z^+$

Here, $\sum u_n$ converges to A ⇨ lim A_n = A and $\sum v_n$ converges to B ⇨ lim B_n = B

Let S_n be the n th partial sum $\sum(u_n + v_n)$.

Then $S_n = (u_1 + v_1) + (u_2 + v_2) + ... + (u_n + v_n) = A_n + B_n$

∴ lim S_n = lim ($A_n + B_n$)

= lim A_n + lim B_n

= A + B

∴ $\sum(u_n + v_n)$ converges to (A + B).

Hence,

If $\sum u_n$ converges to A and $\sum v_n$ converges to B, then $\sum(u_n + v_n)$ converges to A + B.

Property 2 :

If $\sum u_n$ converges to A and $c \in$ R , then $\sum cu_n$ converges to c A.

Proof :

Let $A_n = u_1 + u_2 + u_3 + ... + u_n$, $n \in Z^+$

Here, $\sum u_n$ converges to A ⇨ lim A_n = A

Let B_n = n th partial sum of $\sum cu_n = cu_1 + cu_2 + cu_3 + ... + cu_n$ = c A_n

∴ lim B_n = lim (c A_n)

= c (lim A_n)

= c A

⇨ $\sum cu_n$ converges to c A.

Hence,

If $\sum u_n$ converges to A and $c \in$ R , then $\sum cu_n$ converges to c A.

Property 3 :

If $\sum u_n$ converges to A and $\sum v_n$ converges to B and $p,q \in$ R then $\sum(pu_n + qv_n)$ converges to p A + q B.

Proof :

$\sum u_n$ converges to A ⇨ $\sum pu_n$ converges to p A. (since, by the theorem 2) and

$\sum v_n$ converges to B ⇨ $\sum qv_n$ converges to q B.

Hence, $\sum(pu_n + qv_n)$ converges to p A + q B.

Property 4 :
If $\sum u_n$ diverges and c ∈ R , c ≠ 0 then $\sum cu_n$ diverges.
Proof :
Given that $\sum u_n$ diverges
To prove that $\sum cu_n$ diverges
Suppose that $\sum cu_n$ converges.
Since c ≠ 0, $\frac{1}{c}$ ∈ R
We know that, If $\sum u_n$ converges , then $\sum cu_n$ converges .
Now, $\sum cu_n$ converges ⇨ $\sum\frac{1}{c}(cu_n) = \sum u_n$ converges . This is a contradiction.
Our assumption is wrong.
∴ $\sum cu_n$ diverges
Hence,
If $\sum u_n$ diverges and c ∈ R , c ≠ 0 then $\sum cu_n$ diverges.

Property 5 :
If $\sum u_n$ diverges and $\sum v_n$ diverges , then $\sum(u_n + v_n)$ diverges .
Proof :
Let $A_n = u_1 + u_2 + u_3 + \dots + u_n$, and $B_n = v_1 + v_2 + v_3 + \dots + v_n$, $n \in Z^+$
Here, $\sum u_n$ diverges ⇨ $\lim A_n \to \infty$
And $\sum v_n$ diverges ⇨ $\lim B_n \to \infty$
Let S_n be the n th partial sum $\sum(u_n + v_n)$.
Then $S_n = (u_1+v_1) + (u_2+v_2) + \dots + (u_n+v_n) = A_n + B_n$
∴ $\lim S_n = \lim (A_n + B_n)$
$= \lim A_n + \lim B_n \to \infty + \infty$
$\to \infty$
∴ $\lim S_n \to \infty$
Hence, $\sum(u_n + v_n)$ diverges
If $\sum u_n$ diverges and $\sum v_n$ diverges , then $\sum(u_n + v_n)$ diverges .

Property 6 :
Necessary condition for convergence (or) n^{th} term test
If $\sum u_n$ converges, then $\lim u_n = 0$
Proof :
Here, $\sum u_n$ converges to A and $s_n = u_1 + u_2 + u_3 + \dots + u_n$, $n \in Z^+$ be the n^{th} partial sum .
Then, $\lim s_n$ = A and $\lim s_{n-1}$ = A.
Hence, $\lim u_n = \lim (s_n - s_{n-1})$
$= \lim s_n - \lim s_{n-1}$
= A - A = 0
∴ If $\sum u_n$ converges, then $\lim u_n = 0$
Note :

1. The converse of this theorem may not be true.
That is if $\lim u_n = 0$, then $\sum u_n$ may not converge.
Ex: Consider the series $\sum \frac{1}{\sqrt{n}}$.
Here $u_n = \frac{1}{\sqrt{n}} \to 0$ as n → ∞
But $s_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + ...\ n$ terms $> \frac{n}{\sqrt{n}} = \sqrt{n}$
Since, $\sqrt{n} \to \infty$ as $n \to \infty$, $s_n \to \infty$ as $n \to \infty$
Therefore, $\{ s_n\}$ diverges and hence $\sum u_n$ diverges.
Hence, $\lim u_n = 0$ is only a necessary but not sufficient condition for the series $\sum u_n$ to be convergent.
2. If $\lim u_n \neq 0$ then $\sum u_n$ diverges.
Ex: The series $\sum \frac{n}{n+1}$ is divergent.
Here, $u_n = \frac{n}{n+1}$
$= \frac{1}{1+\frac{1}{n}}$
⇨ $\lim u_n = \lim \frac{1}{1+\frac{1}{n}}$
$= 1 \neq 0$
Hence, if $\lim u_n \neq 0$ then $\sum u_n$ diverges.

Discussion of some important tests:

(i) Geometric Series:

If $|r| < 1$ then the series $\sum_{n=0}^{\infty} r^n$ ($r \in$ R) converges to $\frac{1}{1-r}$ and if $|r| \geq 1$, the series $\sum_{n=0}^{\infty} r^n$ diverges.

Examples:

- The series $\sum \frac{1}{4^n}$ converges to $\frac{4}{3}$, where $r = \frac{1}{4} < 1$
- The series $\sum 5^n$ diverges , where $r = 5 > 1$

(ii) Auxiliary Series (or) *p*- Series test:

The series $\sum \frac{1}{n^p}$ converges if $p>1$, diverges if $0 < p \leq 1$ and Diverges if $p \leq 0$.

Examples:

- The series $\sum \frac{1}{n}$ diverges since $p = 1$
- The series $\sum \frac{1}{n^4}$ converges since $p = 4 > 1$
- The series $\sum \frac{1}{n^{\frac{3}{2}}}$ converges since $p = \frac{3}{2} > 1$
- The series $\sum n$ diverges since $p = -1$

Comparison test:

$\sum u_n$ and $\sum v_n$ be the two series of +ve terms such that $\lim \frac{u_n}{v_n} = l \in$ R.

(a) The series $\sum u_n$ and $\sum v_n$ converge or diverge together when $l \neq 0$

(b) $\sum u_n$ is convergent if $\sum v_n$ is convergent when $l = 0$.

Examples:

The series $\sum \frac{n}{(n+1)^2}$ diverges.

Explanation: Let $u_n = \frac{n}{(n+1)^2}$

Take $v_n = \frac{1}{n}$

Now $\lim \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{n}{(n+1)^2}\right)\left(\frac{n}{1}\right)$

$= \lim_{n\to\infty} \left[\frac{1}{(1+\frac{1}{n})^2}\right]$

$= 1$

$\lim \frac{u_n}{v_n} = 1 \neq 0$

Hence by Comparison test both $\sum u_n$ and $\sum v_n$ will converge or diverge together.

Here, $\sum v_n = \sum \frac{1}{n}$, which is divergent by *p*- Test (p = 1).

So, $\sum u_n$ is divergent.

Hence given series is divergent.

The series $\sum_{n=1}^{\infty} \frac{1}{n} \tan\frac{1}{n}$ is convergent.

Explanation: Let $u_n = \frac{1}{n} \tan\frac{1}{n}$

Take $v_n = \frac{1}{n^2}$

Now $\lim \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{1}{n} tan\frac{1}{n}\right)\left(\frac{n^2}{1}\right)$

$= \lim_{n\to\infty} \left(tan\frac{1}{n}\right)\left(\frac{n}{1}\right)$

$= \lim_{n\to\infty} \frac{tan\frac{1}{n}}{\frac{1}{n}}$

$= \lim_{\frac{1}{n}\to 0} \frac{tan\frac{1}{n}}{\frac{1}{n}}$

$= 1$

$\lim \frac{u_n}{v_n} = 1 \neq 0$

Hence by Comparison test both $\sum u_n$ and $\sum v_n$ will converge or diverge together.

Here, $\sum v_n = \sum \frac{1}{n^2}$, which is convergent by *p*- Test (p = 2).

So, $\sum u_n$ is convergent.

Hence given series is convergent.

The series $\sum_{n=1}^{\infty}(\sqrt{n^3+1}$ - $\sqrt{n^3-1}$) is convergent.

Explanation:

Let $u_n = \sqrt{n^3+1} - \sqrt{n^3-1}$

$u_n = \frac{\left(\sqrt{n^3+1} - \sqrt{n^3-1}\right)\left(\sqrt{n^3+1} + \sqrt{n^3-1}\right)}{\sqrt{n^3+1} + \sqrt{n^3-1}}$

$= \frac{2}{\sqrt{n^3+1} + \sqrt{n^3-1}}$

Take $v_n = \frac{1}{n^{\frac{3}{2}}}$

Now $\lim \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{2}{\sqrt{n^3+1}+\sqrt{n^3-1}}\right)\left(\frac{n^{\frac{3}{2}}}{1}\right)$

$= \lim_{n\to\infty} \left(\frac{2}{\sqrt{1+\frac{1}{n^3}}+\sqrt{1-\frac{1}{n^3}}}\right)$

$= 1$

$\lim \frac{u_n}{v_n} = 1 \neq 0$

Hence by Comparison test both $\sum u_n$ and $\sum v_n$ will converge or diverge together.

Here, $\sum v_n = \sum \frac{1}{n^{\frac{3}{2}}}$, which is convergent by p- Test ($p = \frac{3}{2}$).

So, $\sum u_n$ is convergent.

Hence given series is convergent.

(iii) **Root Test:**

If $\sum u_n$ is a series of +ve terms such that $\lim u_n^{\frac{1}{n}} = l$, then

(a) $\sum u_n$ converges for $l < 1$

(b) $\sum u_n$ diverges for $l > 1$

(c) For $l = 1$ test fails. i.e series may converge or diverge.

Examples:

The series $\sum n(\frac{3}{2})^n$ diverges.

Explanation:

Let $u_n = n(\frac{3}{2})^n$

Now, $\lim u_n^{\frac{1}{n}} = \lim_{n\to\infty} [n(\frac{3}{2})^n]^{\frac{1}{n}}$

$= \lim_{n\to\infty} (n^{\frac{1}{n}})(\frac{3}{2})$

$= \frac{3}{2}$, since $\lim_{n\to\infty} (n^{\frac{1}{n}}) = 1$

$\lim u_n^{\frac{1}{n}} = \frac{3}{2} > 1$

Hence, by Root test given series is divergent.

The series $\sum \frac{x^n}{n^2}$ converges for $x \leq 1$ and diverges for $x > 1$.

Explanation:

Let $u_n = \frac{x^n}{n^2}$

Now, $\lim u_n^{\frac{1}{n}} = \lim_{n\to\infty} (\frac{x^n}{n^2})^{\frac{1}{n}}$

$= \lim_{n\to\infty} (\frac{x}{n^{\frac{2}{n}}})$

$= \boldsymbol{x}$, since $\lim_{n\to\infty} (n^{\frac{1}{n}}) = 1$

$\lim u_n^{\frac{1}{n}} = x.$

Hence, by Root test given series is convergent for $x < 1$ and divergent for $x > 1$

Now for $x = 1$, $\sum u_n = \sum \frac{1}{n^2}$, which is convergent.

So, given series converges for $x \leq 1$ and diverges for $x > 1$.

The series $\sum \frac{n^2}{2^n}$ converges.

Explanation:

Let $u_n = \frac{n^2}{2^n}$

Now, $\lim u_n^{\frac{1}{n}} = \lim_{n\to\infty} (\frac{n^2}{2^n})^{\frac{1}{n}}$

$= \frac{1}{2}$, since $\lim_{n\to\infty} (n^{\frac{1}{n}}) = 1$

$\lim u_n^{\frac{1}{n}} = \frac{1}{2} < 1$

Hence, by Root test given series is convergent.

Conclusion:

In this paper we studied about the Infinite Series and Convergence of Series. Discussed the convergence of series through examples and explained some basic properties of Infinite series. Also studied about Geometric Series, Auxiliary Series (or) p – Series Test, Comparison Test & Root Test and we concluded that,

For **Geometric series**, If $|r| < 1$ then the series $\sum_{n=0}^{\infty} r^n$ ($r \in$ R) converges to $\frac{1}{1-r}$ and if $|r| \geq 1$, the series $\sum_{n=0}^{\infty} r^n$ diverges. For **Auxiliary Series (or) *p* – Series Test**, $\sum \frac{1}{n^p}$ converges if p>1, diverges if 0 < $p \leq 1$ and Diverges if $p \leq 0$. For **Comparison Test**, if $\sum u_n$ and $\sum v_n$ be the two series of +ve terms such that $\lim \frac{u_n}{v_n} = l \in$ Rthen the series $\sum u_n$ and $\sum v_n$ converge or diverge together when $l \neq 0$ and $\sum u_n$ is convergent if $\sum v_n$ is convergent when $l = 0$. For **Root Test**, if $\sum u_n$ is a series of +ve terms such that $\lim u_n^{\frac{1}{n}} = l$, then $\sum u_n$ converges for $l < 1$, $\sum u_n$ diverges for $l > 1$and for $l = 1$ test fails. i.e series may converge or diverge.

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