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UNIQUE FIXED-POINT THEORIES IN VARIOUS MATHEMATICAL SPACES

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Abstract:
In a wide variety of mathematical fields, unique fixed-point theories are vital because they serve as fundamental instruments for the investigation of stability, convergence, and the presence of solutions in a variety of spaces. The purpose of this research is to analyse certain fixed-point theorems in a variety of mathematical situations. Metric spaces, Banach spaces, partially ordered sets, and topological spaces are all examples of settings that fall under this category. In the context of increasingly complex frameworks, such as fuzzy metric spaces and probabilistic spaces, traditional results such as Banach's and Brouwer's Fixed-Point Theorems are investigated in combination with their generalisations and extensions. As a means of ensuring that fixed points are one of a kind, the study places particular focus on the significance of compactness, monotonicity, and contractive mappings. Through considerations of their applications in optimisation, computer mathematics, functional analysis, and differential equations, the theoretical and practical significance of unique fixed-point solutions are brought to the forefront.

Keyword: Unique Fixed-Point, Fixed-Point Theorem, Metric Spaces, Banach Space.

INTRODUCTION

Fixed point theory is one of the famous and traditional theories in mathematics and has a broad set of applications. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction principle, which gives an answer on the existence and uniqueness of a solution of an operator equation Tx = x, is the most widely used fixed point theorem in all of analysis. This principle is constructive in nature and is one of the most useful tools in the study of nonlinear equations. There are many generalizations of Banach's contraction mapping principle in the literature. These generalizations were made either by using the contractive condition or by imposing some additional conditions on an ambient space. There have been a number of generalizations of metric spaces such as rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi-metric spaces, probabilistic metric spaces, D-metric spaces and cone metric spaces The basic topological properties of ordered sets were discussed the existence of fixed points in partially ordered metric spaces was considered. After this paper, published some new results. Recently, many papers have been reported on partially ordered metric.

The triple (X, d,) is called partially ordered metric spaces (POMS) if (X,) is a partially ordered set and (X, d) is a metric space. Further, if (X, d) is a complete metric space, the triple (X, d,≤) is called partially ordered complete metric spaces (POCMS). Throughout the manuscript, we assume that $X \neq \emptyset.$ A partially ordered metric space (X, d,) is called ordered complete (OC) if for each convergent sequence $\{x_n\}_{n=0}^{\infty} \subset X,$, the following condition holds: either

- if {xn} is a non-increasing sequence in X such that xn → x∗ implies x∗ xn ∀n ∈ N, that is, x∗ = inf{xn}, or

- if {xn} is a non-decreasing sequence in X such that xn → x∗ implies xn x∗ ∀n ∈ N, that is, x∗ = sup{xn}.

In this manuscript, we prove that an operator T satisfying certain rational contraction condition has a fixed point in a partially ordered metric space.

Various Abstract Spaces

The term "abstract space" refers to a collection of things that satisfy a set of axiom specifications. Axioms, as opposed to the qualities of the pieces that make up abstract space, are what comprise the essence of abstract space altogether. We may get a wide variety of abstract spaces by using a number of different sets of axioms. Metric Space was first presented by Fr'echet in his book Sur quelques points du Calcul fonctionnel, which was published in 1906. This was the work that began the idea of abstract spaces being used. This Fr'echet initiative has accomplished a great deal of success. It has recently come to light that the body of literature about abstract ideas has a great number of gaps. Over the course of the last two decades, there has been a dramatic growth in the number of surroundings that are considered to be abstract. On this page, we have defined a few abstract spaces as well as numerous essential notions that are related with them.

Metric Space

Definition 1.2.1. For "a non-empty set X, a metric on X is a function d defined on $X \times X$ such that for all $x,\ y,\ z \in X,$ we have:

The not-empty set X that has a metric d stated on it is referred to as (X, d), and this set is considered to be a metric space. The geometric representation of the distance between two points on the real line using the coordinates x and y is denoted by the symbol d (x, y). For example, the set R with d defined as usual distance $d(x,y) = |x-y|,$ for all $x, y \in X,$ is a metric space. In fact, it is so called as usual metric space.

Definition 1.2.2. Let"(X,d) be a metric space. Then, for $x_0 \in X,\ r > 0,$ the open ball in (X,d) with center x0 and radius r is defined as:

$$B(x_0; r) = \{x \in X \mid d(x, x_0) < r\}.$$

A subset A of X is said to be an open set if, for each $a \in A$, there exists $r > 0$ such that $B(a; r) \subseteq A.$

Definition 1.2.3. Let "(X, d) be a metric space. Then:

(i) A sequence $\{x_n\}$ in X is said to be a convergent sequence if there is an $x \in X$ such that $\lim_{n\to+\infty} d(x_n, x) = 0.$. In this case, x is called the limit of $\{x_n\}$ and is denoted by $\lim_{n\to+\infty} x_n = x;$ or, simply, $x_n \to x.$

(ii) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if, for each $\epsilon > 0$ there is $an\ n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon,$ for every $m, n > n_0.$

(iii) If each and every Cauchy sequence in X converges, then the set (X, d) is said to be complete.

Lemma 1.2.1 Let "(X, d) be a metric space. Then:

(i) A convergent sequence in X is bounded, and its limit is unique.

(ii) $If\ x_n \to x\ and\ y_n \to y\ in\ X,\ then\ d(x_n, y_n) \to d(x, y).$

Theorem 1.2.2 Every “convergent sequence in a metric space is a Cauchy sequence.”
Definition 1.2.4. Let “(X, d) and (Y, d0) be two metric spaces. A mapping T: X → Y is said to be continuous at a point $x_0 \in X$ if, for each $\epsilon > 0$,

there is a $\delta > 0$ such that $d'(Tx, Tx_0) < \epsilon$, for all x satisfying $d(x, x_0) < \delta$.

OBJECTIVE

1. To investigate the idea of distinct fixed-point theorems in different areas of mathematics.
2. To examine traditional fixed-point outcomes, such the Brouwer and Banach fixed-point theorems.

PARTIAL METRIC SPACE

It was in the year 1992 that Matthews first presented the idea of partial metric space. Within this space, the conventional metric is replaced by a partial metric, which has the surprising characteristic of being unable to have a self-distance of zero at any location within the space.

Definition 1.2.9. Let X be a non-empty set. Then a mapping $d : X \times X \to [0, +\infty)$ is called a partial metric if, for all $x, y, z \in X$,

$(p1)\quad x = y \;\Leftrightarrow\; d(x,x) = d(x,y) = d(y,y);$

$(p2)\quad d(x,x) \leq d(x,y);$

$(p3)\quad d(x,y) = d(y,x);$

$(p4)\quad d(x,y) \leq d(x,z) + d(z,y) - d(z,z).$

Such a pair (X, d) is called a *partial metric space.*

Remark 1.2.10. The existence of partial metric spaces is shared by all "metric spaces," however the opposite is not necessarily the case. A good example of this would be when
if $X = [0, +\infty)$ and $d(x,y) = max\{x, y\}$, for all $x, y \in X$; then the pair (X,d) is a partial metric space, but not a metric space as $d(1,1) = 1 \neq 0.$.”
Convergent sequence, Cauchy sequence, and completion are some of the concepts that Matthews has presented in relation to partial metric space. Here are some examples of these concepts:
Definition 1.2.11. Let (X, d) be a partial metric space. Then:

(i) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if $\lim_{n,m\to+\infty} d(x_n, x_m)$ exists and is finite.

(ii) A sequence $\{x_n\}$ in X *converges to* $x \in X$ if $\lim_{n\to+\infty} d(x_n, x) = d(x,x)$.

(iii) (X, d) is said to be *complete* if every Cauchy sequence $\{x_n\}$ *in* X converges to some $x \in X$ with $\lim_{n,m\to+\infty} d(x_n, x_m) = d(x,x)$.

Remark 1.2.12. However, it is not essential for the limit of a sequence to be the only one of its kind in partial metric space. It is not necessary to have a Cauchy sequence in order to have a convergent sequence.

Example: "Let $X = [0, +\infty)$ and $d(x, y) = max\{x, y\}$, for all $x, y \in X$; then the pair (X,d) is a partial metric space. Define a sequence, $\{x_n\}$ in X by $x_n = \frac{1}{n+1}$,

$\lim_{n\to+\infty} d(x_n, 1) = d(1, 1)$ and $\lim_{n\to+\infty} d(x_n, 2) = d(2, 2)$. Thus, the limit of a sequence in partial metric space is not necessarily unique. Now, define another sequence $\{y_n\}$ in X by $y_n = 1$ and 2 according as n is odd and even, respectively. We see that $\lim_{n\to+\infty} d(y_n, 3) = d(3, 3)$, i.e., sequence $\{y_n\}$ is convergent, but $\{y_n\}$ is not a Cauchy sequence.

Definition 1.2.13. Every "partial metric p on X generates a T_0-topology τ_p on X with a base of the family of open p-balls $\{B_p(x, r) \mid x \in X, \ r > 0\}$, where

$B_p(x, r) = \{y \in X \mid d(x, y) < d(x, x) + r\}$, for all $x \in X$ and $r > 0$.

Lemma 1.2.4 Let "(X, d) be a partial metric space and let {x n} be a sequence in

in X such that $x_n \to x$ and $d(x, x) = 0$. Then, for every $y \in X$, $\lim_{n\to+\infty} d(x_n, y) = d(x, y)$.

METRIC-LIKE SPACE

The concept of partial metric was developed upon by Harandi in the year 2012 via the creation of a new space under the name metric-like space. It is possible that in a space that is similar to the metric, the self-distance of a point might be greater than its distance from any other point. This is something that has been seen. analogous to a metric space

Definition 1.2.14. Let X be a non-empty set. Then a mapping $d : X \times X \to [0, +\infty)$ is called a metric-like if, for all x, y, z ∈ X,

(m_l1) $d(x, y) = 0 \ \Rightarrow \ x = y$;

(m_l2) $d(x, y) = d(y, x)$;

(m_l3) $d(x, y) \leq d(x, z) + d(z, y)$.

The space (X, d) is an example of a metric-like space. With the exception of the fact that d (x, x) may be positive for each and every x in X, a metric-like function on X satisfies all of the characteristics that are normally associated with a metric. It is not always the case that the opposite is true, although all partial metric spaces seem to be metric-like.

Example 1.2.15.

Let $X = \{0,1\}$ and define $d : X \times X \to [0,+\infty)$ by

$$d(x,y) = \begin{cases} 2, & \text{if } x = y = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Then, the pair (X,d) is a metric-like space, but not a partial metric space as $d(0,0) \nleq$d(0,1)."
Within a framework of metric-like space that is comparable to partial metric space, Harandi also provided a description of the concepts of convergent sequence, Cauchy sequence, and completion. In addition, the following is the definition of an open set when it is considered in the context of metric-like space:

Definition 1.2.16. Let "(X, d) be a metric-like space. Then, for $x_0 \in X, \ r > 0,$, the open ball in (X, d) with center x0 and radius r is

$$B(x_0, r) = \{y \in X \mid |d(x_0, y) - d(x_0, x_0)| < r\}.$$

Also, a subset A of X is said to be an *open set* in X if, for each $a \in A$, there exists $r > 0$ such that $B(a, r) \subseteq A$. The family of all open subsets of X is denoted by τ_{ml} which is a T_0-topology on X.

Remark 1.2.17. In a metric-like space, limit of a convergent sequence need not be unique.

Example: "Let $X = \{0,1\}$ and $d(x,y) = 1$ for all $x, y \in X$, then (X,d) is a metric like space, Define a sequence $\{x_n\}$ in X by $x_n = 1$, then $\lim_{n\to+\infty} d(x_n, 1) = d(1,1)$ and $\lim_{n\to+\infty} d(x_n, 0) = d(0,0)$. Thus, limit of a sequence in metric-like space is not n→+∞ unique."

On the other hand, it is not essential for a convergent sequence to be a Cauchy sequence in a space that is similar to metric space. Consider the concept of metric space using the example of 1.2.15. When we assume that, each $n \in \mathbb{N}$, ,

$$x_n = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{otherwise,} \end{cases}$$

then $\lim_{n\to+\infty} d(x_n, 1) = d(1,1)$, but $\lim_{n,m\to+\infty} d(x_n, x_m)$ does not exist.

Proposition 1.2.5 Let " $\{x_n\}$ be a sequence in a metric-like space (X,d) such that *that* $x_n \to x$ *and* $d(x,x) = 0$. *Then, for every* $y \in X$, $\lim_{n\to+\infty} d(x_n, y) = d(x, y)$.

PARTIAL B-METRIC SPACE

After introducing the concepts of partial metric and b-metric, Shukla presented the idea of partial b-metric in the year 2014. This was a logical step.

Definition 1.2.18. Let X be a non-empty set. Then, a mapping $d : X \times X \to [0, +\infty)$ is If there is a positive integer s such that for every member x, y, and z in the set X, the metric is considered to be a partial b-metric, then it is as follows:

$(p_b1)\quad x = y \Leftrightarrow d(x,x) = d(x,y) = d(y,y);$

$(p_b2)\quad d(x,x) \le d(x,y);$

$(p_b3)\quad d(x,y) = d(y,x);$

$(p_b4)\quad d(x,y) \le s(d(x,z) + d(z,y)) - d(z,z).$

The "pair (X, d) is called a partial b-metric space."

It is not necessarily the case that the opposite is true, but regardless of whether it is a "partial metric space and b-metric space" or not, it is a partial b-metric space. Here is an illustration of what I mean:

Example: "Let $X = [0, +\infty)$ and $P_b : X \times X \to [0, +\infty)$ is such that $d(x,y) = [max\{x,y\}]^2 + |x-y|^2$, for all $x, y \in X$. Then (X,d) is a partial b-metric space on X with $s = 2 > 1$. But, d is not a b-metric as $d(1,1) = 1 \neq 0$. Also, d is not a partial metric on X as for $x = 1,\ y = 2$ and $z = 3$, (p4) does not hold."

Within the setting of partial b-metric space, which is analogous to partial metric space, Shukla also presented the concepts of completeness, convergent sequence, and Cauchy sequence. These concepts were proposed by Shukla. In addition, the following is the definition of an open set when it is considered in the context of partial b-metric space:

Definition 1.2.19. Every "partial b-metric d on X generates a topology τ_d on X with a base of the family of open d-balls $\{B_d(x,r) \mid x \in X,\ r > 0\}$, where $B_d(x,r) = \{y \in X \mid d(x,y) < d(x,x) + r\}$. Obviously, τ_d on partial b-metric space X is T_0, but need not be T_1.

B-METRIC-LIKE SPACE

"A generalisation of the ideas of metric-like space and partial b-metric space was introduced in 2013 with the notion of b-metric-like spaces."

Definition 1.2.20. Let X be a non-empty set. Then, a mapping $d : X \times X \to [0, +\infty)$ is called a b-metric-like if there exists a number s ≥ 1 such that for all x,y,z ∈ X,

$(b_{ml}1)\quad d(x,y) = 0 \Rightarrow x = y;$

$(b_{ml}2)\quad d(x,y) = d(y,x);$

$(b_{ml}3)\quad d(x,y) \le s(d(x,z) + d(z,y)).$

In this case, the pair (X, d) is called a b-metric-like space.

Remark 1.2.21. Although it is not always the case, every space that is metric-like or partial b-metric is also a b-metric-like space. This being said, the opposite is not always true.

Example: Let $X = [0, +\infty)$ and let the function $d : X \times X \to [0, +\infty)$ be defined by $d(x, y) = (x + y)^2$. Then (X, d) is a b-metric-like space with constant $s = 2$. For, $(b_{ml}3)$, let $x, y, z \in X$, then

$$\begin{aligned} d(x,y) &= (x+y)^2 \leq (x+z+z+y)^2 \\ &= (x+z)^2 + (z+y)^2 + 2(x+z)(z+y) \\ &\leq 2[(x+z)^2 + (z+y)^2] = 2[d(x,z) + d(z,y)]. \end{aligned}$$

Clearly, (X, d) is not a metric-like space as $d(1, 1) = 4 \nleq 2 = d(1, 0) + d(0, 1)$, i.e., (m_l3) does not hold. Also, (X, d) is not a partial b-metric space as $d(2, 2) = 16 \nleq 9 = d(2, 1)$, i.e., (p_b2) does not hold.

"The following ideas within the context of b-metric-like space: convergent sequence, Cauchy sequence, and completeness:"

Definition 1.2.22. Let (X, d) be a b-metric-like space and let $\{x_n\}$ be a sequence of points of X. A point x ∈ X is said to be the limit of sequence $\{x_n\}$ if $\lim_{n\to+\infty} d(x, x_n) = d(x, x)$, and we say that the sequence $\{x_n\}$ is convergent to x, and denote it by $x_n \to x$ as $n \to +\infty$.

Definition 1.2.23. Let (X, d) be a b-metric-like space.

(i) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if $\lim_{n,m\to+\infty} d(x_n, x_m)$ exists and is finite.

(ii) (X, d) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$ so that $\lim_{n,m\to+\infty} d(x_n, x_m) = d(x, x) = \lim_{n\to+\infty} d(x_n, x)$.

In the following, we will discuss this idea on the singularity of the limit of a sequence.

Proposition 1.2.6 Let (X,d) be a b-metric-like space with s ≥ 1, and {xn} be a sequence in X such that for some $x \in X$, $\lim_{n\to+\infty} d(x_n, x) = 0$. Then

(i) x *is unique.*

(ii) $\frac{1}{s}d(x,y) \leq \lim_{n\to+\infty} d(x_n,y) \leq sd(x,y), \quad for\ all\ y \in X.$

The following figure provides a visual representation of the many abstract spaces that have been discussed up to this point.

$$\begin{array}{ccccc} \text{Metric space} & \Rightarrow & \text{Partial Metric space} & \Rightarrow & \text{Metric-like space} \\ \Downarrow & & \Downarrow & & \Downarrow \\ b\text{-Metric space} & \Rightarrow & \text{Partial } b\text{-Metric space} & \Rightarrow & b\text{-Metric-like space} \end{array}$$

GENERALIZED METRIC SPACE

Following the expansions that came before it, Jleli and Samet provided a further fascinating extension of metric space in the year 2015, as can be seen in the following:

Definition 1.2.24. Let "X be a non-empty set and $d : X \times X \to [0, +\infty]$ be a given mapping. For every x ∈ X, define the set

$$\mathcal{C}(d, X, x) = \left\{ \{x_n\} \subset X \mid \lim_{n\to+\infty} d(x_n, x) = 0 \right\}.$$

A generalised metric on X is denoted by the letter d if there is a positive constant C that gives rise to the following statement.

$(D1)$ $(x, y) \in X \times X,\ d(x, y) = 0$ implies $x = y$;

$(D2)$ $d(x, y) = d(y, x)$, for all $(x, y) \in X \times X$;

$(D3)$ if $\{x_n\} \in \mathcal{C}(d, X, x)$, then $d(x, y) \leq C \limsup_{n\to+\infty} d(x_n, y)$.

In this case, the pair (X, d) is a generalized metric space."

Example 1.2.25. Let $X = \{0, 1\}$ and $d : X \times X \to [0, +\infty]$ be a mapping defined by $d(0, 0) = 0$ and $d(0, 1) = d(1, 0) = d(1, 1) = +\infty.$ Then (X, d) is a generalized metric space."

Definition 1.2.26. Let "(X, d) be a generalized metric space. Then:

(i) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if $\lim_{n,m\to+\infty} d(x_n, x_m) = 0$.

(ii) A sequence $\{x_n\}$ in X *converges to* $x \in X$ if $\{x_n\} \in \mathcal{C}(d, X, x)$.

(iii) (X, d) is said to be *complete* if every Cauchy sequence in X is convergent to some element in X.

Proposition 1.2.7 Let "(X, d) be a generalized metric space. Let $\{x_n\}$ be a sequence in X such that $\{x_n\}$ converges to some x in X and $\{x_n\}$ converges to some y *in* X, *then* $x = y$.

CONCLUSION

One-of-a-kind fixed-point theories provide a solid framework for the investigation of stability, existence, and convergence in a variety of mathematical spaces. The focus of this study is on classical and extended fixed-point theorems, with particular attention paid to metric spaces, Banach spaces, topological spaces, fuzzy and probabilistic circumstances. Investigations into the role that contractive mapping, order structures, and compactness requirements play in ensuring uniqueness have shown that the major value of these concepts lies in the field of mathematical analysis. In addition, the comprehensive applications of fixed-point results in the fields of optimisation, computer mathematics, functional analysis, and differential equations demonstrate the practical value of these findings. It is important to note that the findings highlight the significance of fixed-point theorems as key tools in both theoretical and practical mathematics. They open the way to further research and advancement in a wide range of fields.

REFERENCES:

1. Bucur, Amelia. (2022). About Applications of the Fixed Point Theory. Scientific Bulletin. 22. 10.1515/bsaft-2017-0002.
2. Kirk, W. & Shahzad, Naseer. (2021). Fixed Point Theory in Distance Spaces. 10.1007/978-3-319-10927-5.
3. Yuan, George. (2020). Fixed Point Theory in p -Vector Spaces. 10.1142/13456.
4. Dolhare, Uttam. (2019). Review of Fixed Point Theory.
5. Dzitac, Simona & Oros, Horea & Deac, Dan & Sorin, Nadaban. (2018). Fixed Point Theory in Fuzzy Normed Linear Spaces: A General View. INTERNATIONAL JOURNAL OF COMPUTERS COMMUNICATIONS & CONTROL. 16. 10.15837/ijccc.2021.6.4587.
6. Debnath, Pradip & Konwar, Nabanita & Radenović, Stojan. (2017). Metric Fixed Point Theory: Applications in Science, Engineering and Behavioural Sciences. 10.1007/978-981-16-4896-0.
7. Tomar, Anita & Upadhyay, Shivangi & Sharma, Ritu. (2016). EXISTENCE OF COMMON FIXED POINT IN SYMMETRIC SPACE WITH APPLICATION.
8. Kumar, Sanjay & Chugh, Renu & Vats, Ramesh. (2015). Fixed point theorems for expansion mappings theorems in various spaces. Studii şi Cercetări Ştiinţifice. Seria: Matematică.
9. Lili Chen et al., “ Several fixed point theorems in convex-b metric space and applications “. Journal of mathematics. 8:242.(2020).
10. Eliyas Zinab et al., “ A fixed point theorem for generalized weakly contractive mappings in bmetric spaces”. Open journal of mathematical sciences. 4:1-8.(2020).
11. Karim Chaira et al., “Fisher fixed point results in generalized metric spaces with a graph”. International journal of mathematics and mathematical sciences. 2020:8.(2020).
12. Khaled Berrah et al., “ Applications and theorem on common fixed point theorem in complex valued b-metric space”. AIMS mathematics. 4(3): 1019-1033.(2019).
13. Duran Turkoglu et al., “Fixed point theorems in a new type of modular metric spaces”. Turkoglu and Manav fixed point theory and applications. 3:10. (2018).
14. Puspendra Semwal and Komal, “ Common fixed point theorems in metric spaces with applications”. European journal of pure and applied mathematics. 11.4:1177-1190.(2018).
15. Kanayo Stella Eke and Jimevwo Godwin Oghonyon’ “ Some fixed point theorems in ordered partial metric spaces with applications”. Cogent mathematics and statistics. 5: 2574- 2558.(2018).
16. Rajendra Pant et al., “Some new fixed point theorems in partial metric spaces with applications”. Journal of function spaces. 2017: 13 .(2017).
17. Amelia BUCUR, “ About applications of the fixed point theory”. Scientific bulletin. 43:1.(2017).
18. N. Hussain et al., “Fixed point theory in α-complete metric spaces with applications”. Abstract and applied analysis. 2014:11.(2014).
19. Farshid Khojasteh et al., “θ-Metric Space: A Generalization”. Mathematical problems in engineering. 2013: 7.(2013).

20. Shyam Lal Singh et al., "General common fixed point theorems and applications". Journal of applied mathematics. 2012: 14.(2012).